Ordering Knot Groups

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Ordered groups

A group is left-ordered if there is a strict total ordering $<$ of its elements such that

$$g < h \quad \text{implies} \quad fg < fh.$$

If P denotes the set of elements greater than the identity, then

1. $P \cdot P \subset P$, and
2. $P \sqcup P^{-1} \sqcup \{1\} = G$.

Conversely, if a group has a subset P satisfying the above, then it is left-ordered by the formula

$$f < g \iff f^{-1}g \in P$$

A group is left-orderable if and only if it is right-orderable.

Ordered groups

If a group has a strict total ordering $<$ which is both right- and left-invariant, we call it bi-orderable. Equivalently, the positive cone P is invariant under conjugation.

A group is indicable if it has the integers as a quotient, and locally indicable if every nontrivial f. g. subgroup is indicable.

Theorem

Bi-orderable $\Longrightarrow$ *Locally indicable* $\Longrightarrow$ *Left-orderable*

Neither of these implications is reversible.

Examples

- Abelian groups are bi-orderable iff torsion-free.
- Free groups are bi-orderable.
- Braid groups are left-orderable (Dehornoy) but not bi-orderable
- Pure braid groups are bi-orderable.
- Surface groups are bi-orderable, except the Klein bottle group

$$\langle a, b | \ a^2 = b^2 \rangle$$

which is only left-orderable, and the group of the projective plane.

Properties of orderable groups

- Left-ordered groups G are torsion-free and satisfy the zero-divisor conjecture, that is, $\mathbb{Z}G$ has no zero divisors.
- (LaGrange, Rhemtulla) If G is left-orderable and H is any group, then $\mathbb{Z}G \cong \mathbb{Z}H \implies G \cong H$.
- Bi-ordered groups have unique roots: $g^n = h^n,\ n > 0 \implies g = h$
- In a bi-ordered group, if g commutes with $h^n, n \neq 0$, then g commutes with h.
- If G is bi-ordered, then $\mathbb{Z}G$ embeds in a division ring.

Knot groups

If K is a knot in $\mathbb{S}^3$, its knot group is $\pi_1(\mathbb{S}^3 \setminus K)$. Another classical knot invariant is the Alexander polynomial $\Delta_K(t)$. A knot is said to be fibred if there is a fibre bundle map $\mathbb{S}^3 \setminus K \to \mathbb{S}^1$ with fibres being open surfaces which have K as boundary in $\mathbb{S}^3$.

Theorem

1. *(Howie-Short) All knot groups are locally indicable, hence left-orderable.*
2. *(Perron - R.) If K is fibred and $\Delta_K(t)$ has all roots real and positive, then its group is bi-orderable.*
3. *(Clay-R.) If K is fibred and its group is bi-orderable, then $\Delta_K(t)$ has some real positive roots.*

Examples

Torus knots: curves which can be inscribed on the surface of an unknotted torus in $\mathbb{S}^3$. For relatively prime integers p, q the torus knot $T_{p,q}$ has group

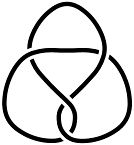
$$\langle a, b | a^p = b^q \rangle.$$

Note that a commutes with b^q but not with b (unless the group is abelian, and the knot unknotted). Therefore:

Proposition

The group of a torus knot is not bi-orderable.

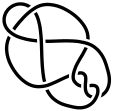
Examples

The figure-eight knot 4_1  is fibred and has Alexander polynomial $\Delta_{4_1} = t^2 - 3t + 1$ with roots $\frac{3 \pm \sqrt{5}}{2}$, both real and positive. From Theorem 2 we conclude

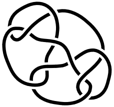
Proposition

The group of the knot 4_1 is bi-orderable.

More bi-orderable knot groups

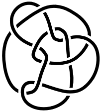


8_{12} $1 - 7t + 13t^2 - 7t^3 + t^4$



10_{137} $1 - 6t + 11t^2 - 6t^3 + t^4$

$11a_5$ $1 - 9t + 30t^2 - 45t^3 + 30t^4 - 9t^5 + t^6$

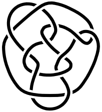


$11n_{142}$ $1 - 8t + 15t^2 - 8t^3 + t^4$

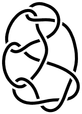
More bi-orderable knot groups

$12a_{0125}$ $1 - 12t + 44t^2 - 67t^3 + 44t^4 - 12t^5 + t^6$

$12a_{0181}$ $1 - 11t + 40t^2 - 61t^3 + 40t^4 - 11t^5 + t^6$

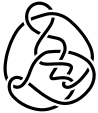


$12a_{1124}$ $1 - 13t + 50t^2 - 77t^3 + 50t^4 - 13t^5 + t^6$

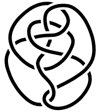


$12n_{0013}$ $1 - 7t + 13t^2 - 7t^3 + t^4$

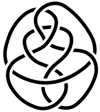
More bi-orderable knot groups



$12n_{0145}$ $1 - 6t + 11t^2 - 6t^3 + t^4$



$12n_{0462}$ $1 - 6t + 11t^2 - 6t^3 + t^4$



$12n_{0838}$ $1 - 6t + 11t^2 - 6t^3 + t^4$

More non bi-orderable knot groups

Recall Theorem 3: fibred and bi-orderable $\implies$ Δ has positive roots.
This can be used for an alternative proof that torus knots $T_{p,q}$, which are fibred, have non-bi-orderable group, because

$$\Delta_{T(p,q)} = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$$

whose roots are on the unit circle and not real.
There are many other fibred knots which have non-biorderable group for similar reasons ....

More non bi-orderable knot groups

The prime knots with 12 or fewer crossings which are known to have non-bi-orderable group, because they are fibred and have Alexander polynomials without positive real roots, are as follows:

3_1, 5_1, 6_3, 7_1, 7_7, 8_7, 8_{10}, 8_{16}, 8_{19}, 8_{20}, 9_1, 9_{17}, 9_{22}, 9_{26}, 9_{28}, 9_{29}, 9_{31}, 9_{32}, 9_{44}, 9_{47}, 10_5, 10_{17}, 10_{44}, 10_{47}, 10_{48}, 10_{62}, 10_{69}, 10_{73}, 10_{79}, 10_{85}, 10_{89}, 10_{91}, 10_{99}, 10_{100}, 10_{104}, 10_{109}, 10_{118}, 10_{124}, 10_{125}, 10_{126}, 10_{132}, 10_{139}, 10_{140}, 10_{143}, 10_{145}, 10_{148}, 10_{151}, 10_{152}, 10_{153}, 10_{154}, 10_{156}, 10_{159}, 10_{161}, 10_{163}, $11a_9$, $11a_{14}$, $11a_{22}$, $11a_{24}$, $11a_{26}$, $11a_{35}$, $11a_{40}$, $11a_{44}$, $11a_{47}$, $11a_{53}$, $11a_{72}$, $11a_{73}$, $11a_{74}$, $11a_{76}$, $11a_{80}$, $11a_{83}$, $11a_{88}$, $11a_{106}$, $11a_{109}$, $11a_{113}$, $11a_{121}$, $11a_{126}$, $11a_{127}$, $11a_{129}$, $11a_{160}$, $11a_{170}$, $11a_{175}$, $11a_{177}$, $11a_{179}$, $11a_{180}$, $11a_{182}$, $11a_{189}$, $11a_{194}$, $11a_{215}$, $11a_{233}$, $11a_{250}$, $11a_{251}$, $11a_{253}$, $11a_{257}$, $11a_{261}$, $11a_{266}$, $11a_{274}$, $11a_{287}$, $11a_{288}$, $11a_{289}$, $11a_{293}$, $11a_{300}$, $11a_{302}$, $11a_{306}$, $11a_{315}$, $11a_{316}$,

More non bi-orderable knot groups

$11a_{326}$, $11a_{330}$, $11a_{332}$, $11a_{346}$, $11a_{367}$, $11n_{7}$, $11n_{11}$, $11n_{12}$, $11n_{15}$, $11n_{22}$, $11n_{23}$, $11n_{24}$, $11n_{25}$, $11n_{28}$, $11n_{41}$, $11n_{47}$, $11n_{52}$, $11n_{54}$, $11n_{56}$, $11n_{58}$, $11n_{61}$, $11n_{74}$, $11n_{76}$, $11n_{77}$, $11n_{78}$, $11n_{82}$, $11n_{87}$, $11n_{92}$, $11n_{96}$, $11n_{106}$, $11n_{107}$, $11n_{112}$, $11n_{124}$, $11n_{125}$, $11n_{127}$, $11n_{128}$, $11n_{129}$, $11n_{131}$, $11n_{133}$, $11n_{145}$, $11n_{146}$, $11n_{147}$, $11n_{149}$, $11n_{153}$, $11n_{154}$, $11n_{158}$, $11n_{159}$, $11n_{160}$, $11n_{167}$, $11n_{168}$, $11n_{173}$, $11n_{176}$, $11n_{182}$, $11n_{183}$, $12a_{0001}$, $12a_{0008}$, $12a_{0011}$, $12a_{0013}$, $12a_{0015}$, $12a_{0016}$, $12a_{0020}$, $12a_{0024}$, $12a_{0026}$, $12a_{0030}$, $12a_{0033}$, $12a_{0048}$, $12a_{0058}$, $12a_{0060}$, $12a_{0066}$, $12a_{0070}$, $12a_{0077}$, $12a_{0079}$, $12a_{0080}$, $12a_{0091}$, $12a_{0099}$, $12a_{0101}$, $12a_{0111}$, $12a_{0115}$, $12a_{0119}$, $12a_{0134}$, $12a_{0139}$, $12a_{0141}$, $12a_{0142}$, $12a_{0146}$, $12a_{0157}$, $12a_{0184}$, $12a_{0186}$, $12a_{0188}$, $12a_{0190}$, $12a_{0209}$, $12a_{0214}$, $12a_{0217}$, $12a_{0219}$, $12a_{0222}$, $12a_{0245}$, $12a_{0246}$, $12a_{0250}$, $12a_{0261}$, $12a_{0265}$, $12a_{0268}$, $12a_{0271}$, $12a_{0281}$, $12a_{0299}$, $12a_{0316}$, $12a_{0323}$, $12a_{0331}$, $12a_{0333}$, $12a_{0334}$, $12a_{0349}$,

More non bi-orderable knot groups

$12a_{0351}$, $12a_{0362}$, $12a_{0363}$, $12a_{0369}$, $12a_{0374}$, $12a_{0386}$, $12a_{0396}$, $12a_{0398}$, $12a_{0426}$, $12a_{0439}$, $12a_{0452}$, $12a_{0464}$, $12a_{0466}$, $12a_{0469}$, $12a_{0473}$, $12a_{0476}$, $12a_{0477}$, $12a_{0479}$, $12a_{0497}$, $12a_{0499}$, $12a_{0515}$, $12a_{0536}$, $12a_{0561}$, $12a_{0565}$, $12a_{0569}$, $12a_{0576}$, $12a_{0579}$, $12a_{0629}$, $12a_{0662}$, $12a_{0696}$, $12a_{0697}$, $12a_{0699}$, $12a_{0700}$, $12a_{0706}$, $12a_{0707}$, $12a_{0716}$, $12a_{0815}$, $12a_{0824}$, $12a_{0835}$, $12a_{0859}$, $12a_{0864}$, $12a_{0867}$, $12a_{0878}$, $12a_{0898}$, $12a_{0916}$, $12a_{0928}$, $12a_{0935}$, $12a_{0981}$, $12a_{0984}$, $12a_{0999}$, $12a_{1002}$, $12a_{1013}$, $12a_{1027}$, $12a_{1047}$, $12a_{1065}$, $12a_{1076}$, $12a_{1105}$, $12a_{1114}$, $12a_{1120}$, $12a_{1122}$, $12a_{1128}$, $12a_{1168}$, $12a_{1176}$, $12a_{1188}$, $12a_{1203}$, $12a_{1219}$, $12a_{1220}$, $12a_{1221}$, $12a_{1226}$, $12a_{1227}$, $12a_{1230}$, $12a_{1238}$, $12a_{1246}$, $12a_{1248}$, $12a_{1253}$, $12n_{0005}$, $12n_{0006}$, $12n_{0007}$, $12n_{0010}$, $12n_{0016}$, $12n_{0019}$, $12n_{0020}$, $12n_{0038}$, $12n_{0041}$, $12n_{0042}$, $12n_{0052}$, $12n_{0064}$, $12n_{0070}$, $12n_{0073}$, $12n_{0090}$, $12n_{0091}$, $12n_{0092}$, $12n_{0098}$, $12n_{0104}$, $12n_{0105}$, $12n_{0106}$, $12n_{0113}$, $12n_{0115}$, $12n_{0120}$, $12n_{0121}$, $12n_{0125}$, $12n_{0135}$,

More non bi-orderable knot groups

$12n_{0136}$, $12n_{0137}$, $12n_{0139}$, $12n_{0142}$, $12n_{0148}$, $12n_{0150}$, $12n_{0151}$, $12n_{0156}$, $12n_{0157}$, $12n_{0165}$, $12n_{0174}$, $12n_{0175}$, $12n_{0184}$, $12n_{0186}$, $12n_{0187}$, $12n_{0188}$, $12n_{0190}$, $12n_{0192}$, $12n_{0198}$, $12n_{0199}$, $12n_{0205}$, $12n_{0226}$, $12n_{0230}$, $12n_{0233}$, $12n_{0235}$, $12n_{0242}$, $12n_{0261}$, $12n_{0272}$, $12n_{0276}$, $12n_{0282}$, $12n_{0285}$, $12n_{0296}$, $12n_{0309}$, $12n_{0318}$, $12n_{0326}$, $12n_{0327}$, $12n_{0328}$, $12n_{0329}$, $12n_{0344}$, $12n_{0346}$, $12n_{0347}$, $12n_{0348}$, $12n_{0350}$, $12n_{0352}$, $12n_{0354}$, $12n_{0355}$, $12n_{0362}$, $12n_{0366}$, $12n_{0371}$, $12n_{0372}$, $12n_{0377}$, $12n_{0390}$, $12n_{0392}$, $12n_{0401}$, $12n_{0402}$, $12n_{0405}$, $12n_{0409}$, $12n_{0416}$, $12n_{0417}$, $12n_{0423}$, $12n_{0425}$, $12n_{0426}$, $12n_{0427}$, $12n_{0437}$, $12n_{0439}$, $12n_{0449}$, $12n_{0451}$, $12n_{0454}$, $12n_{0456}$, $12n_{0458}$, $12n_{0459}$, $12n_{0460}$, $12n_{0466}$, $12n_{0468}$, $12n_{0472}$, $12n_{0475}$, $12n_{0484}$, $12n_{0488}$, $12n_{0495}$, $12n_{0505}$, $12n_{0506}$, $12n_{0508}$, $12n_{0514}$, $12n_{0517}$, $12n_{0518}$, $12n_{0522}$, $12n_{0526}$, $12n_{0528}$, $12n_{0531}$, $12n_{0538}$,

More non bi-orderable knot groups

$12n_{0543}$, $12n_{0549}$, $12n_{0555}$, $12n_{0558}$, $12n_{0570}$, $12n_{0574}$, $12n_{0577}$, $12n_{0579}$, $12n_{0582}$, $12n_{0591}$, $12n_{0592}$, $12n_{0598}$, $12n_{0601}$, $12n_{0604}$, $12n_{0609}$, $12n_{0610}$, $12n_{0613}$, $12n_{0619}$, $12n_{0621}$, $12n_{0623}$, $12n_{0627}$, $12n_{0629}$, $12n_{0634}$, $12n_{0640}$, $12n_{0641}$, $12n_{0642}$, $12n_{0647}$, $12n_{0649}$, $12n_{0657}$, $12n_{0658}$, $12n_{0660}$, $12n_{0666}$, $12n_{0668}$, $12n_{0670}$, $12n_{0672}$, $12n_{0673}$, $12n_{0675}$, $12n_{0679}$, $12n_{0681}$, $12n_{0683}$, $12n_{0684}$, $12n_{0686}$, $12n_{0688}$, $12n_{0690}$, $12n_{0694}$, $12n_{0695}$, $12n_{0697}$, $12n_{0703}$, $12n_{0707}$, $12n_{0708}$, $12n_{0709}$, $12n_{0711}$, $12n_{0717}$, $12n_{0719}$, $12n_{0721}$, $12n_{0725}$, $12n_{0730}$, $12n_{0739}$, $12n_{0747}$, $12n_{0749}$, $12n_{0751}$, $12n_{0754}$, $12n_{0761}$, $12n_{0762}$, $12n_{0781}$, $12n_{0790}$, $12n_{0791}$, $12n_{0798}$, $12n_{0802}$, $12n_{0803}$, $12n_{0835}$, $12n_{0837}$, $12n_{0839}$, $12n_{0842}$, $12n_{0848}$, $12n_{0850}$, $12n_{0852}$, $12n_{0866}$, $12n_{0871}$, $12n_{0887}$, $12n_{0888}$.

Proof sketches

Let us now turn to the proofs of the three theorems:

Theorem

1. *(Howie-Short) All knot groups are locally indicable, hence left-orderable.*
2. *(Perron - R.) If K is fibred and $\Delta_K(t)$ has all roots real and positive, then its group is bi-orderable.*
3. *(Clay-R.) If K is fibred and its group is bi-orderable, then $\Delta_K(t)$ has some real positive roots.*

First, the local indicability of knot groups.

Knots groups are locally indicable

Consider a knot complement $X = \mathbb{S}^3 \setminus K$. And the knot group $\pi_1(X)$. $\pi_1(X)$ is indicable, using the (surjective) Hurewicz homomorphism

$$\pi_1(X) \to H_1(X) \cong \mathbb{Z}$$

To show $\pi_1(X)$ is locally indicable, consider a finitely generated nontrivial subgroup $H < \pi_1(X)$. We need to find a surjection $H \to \mathbb{Z}$.

Case 1: H has finite index. This is easy; the Hurewicz map takes H to a finite index subgroup of $\mathbb{Z}$, which is therefore a copy of $\mathbb{Z}$.

Knots groups are locally indicable

Case 2: H has infinite index. Then there is a covering $p : \tilde{X} \to X$ with $p_* \pi_1(\tilde{X}) = H$. $\tilde{X}$ is noncompact, but its fundamental group is f. g. so, by a theorem of Scott, there is a compact submanifold $C \subset \tilde{X}$ with inclusion inducing an isomorphism $\pi_1(C) \cong \pi_1(\tilde{X}) \cong H$.

C necessarily has nonempty boundary. If $B \subset \partial C$ is a boundary component which is a sphere, then irreducibility implies that B bounds a 3-ball in $\tilde{X}$. That 3-ball either contains C or its interior is disjoint from C, and the former can't happen because that would imply the inclusion map $\pi_1(C) \to \pi_1(\tilde{X})$ is trivial. Therefore, we can adjoint that 3-ball to C removing B as a boundary component and not changing $\pi_1(C)$. This process allows us to assume that ∂C is nonempty and has infinite homology groups. By an Euler characteristic argument, we conclude that C also has infinite homology. Then we have surjections $H \cong \pi_1(C) \to H_1(C) \to \mathbb{Z}$ as required.

Theorem 2: Positive roots implies bi-orderable

As motivation, consider a matrix in Jordan normal form multiplied by a vector, such as:

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 + x_2 \\ \lambda_1 x_2 \\ \lambda_2 x_3 \end{pmatrix}$$

Now, declaring a vector (in $\mathbb{R}^3$) to be "positive" if its last nonzero entry is greater than zero, we see that, if also the eigenvectors λ_i are positive, then multiplication by such a matrix preserves that positive cone of $\mathbb{R}^3$, considered as an additive group. So we see

Proposition

If all the eigenvalues of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ are real and positive, then there is a bi-ordering of $\mathbb{R}^n$ which is preserved by L.

Theorem 2: Positive roots implies bi-orderable

A fibration $X \to S^1$ with fibre S can be considered as the mapping cylinder of a (monodromy) homeomorphism $h : S \to S$:

$$X \cong \frac{S \times [0,1]}{(x,1) \sim (h(x),0)}$$

For a fibred knot with $X = \mathbb{S}^3 \setminus K$ the Alexander polynomial is just the characteristic polynomial of the homology monodromy $H_1(S) \to H_1(S)$. Also, the knot group $\pi_1(X)$ is an HNN extension of the free group $\pi_1(S)$, corresponding to the homotopy monodromy $h_* : \pi_1(S) \to \pi_1(S)$, where $\pi_1(S) \cong \langle x_1, \ldots, x_{2g} \rangle$ is a free group.

$$\pi_1(X) \cong \langle x_1, \ldots, x_{2g}, t | h_*(x_i) = tx_i t^{-1} \rangle$$

This group is bi-orderable if and only if there is a bi-ordering of $\pi_1(S)$ which is preserved by h_*.

Theorem 2: Positive roots implies bi-orderable

So our problem reduces to showing:

Proposition

Let F be a finitely generated free group and $h : F \to F$ an automorphism. If all the eigenvalues of $h_* : H_1(F; \mathbb{Q}) \to H_1(F; \mathbb{Q})$ are real and positive, then there is a bi-ordering of F preserved by h.

One way to order a free group F is to use the lower central series $F_1 \supset F_2 \supset \cdots$ defined by

$$F_1 = F, \quad F_{i+1} = [F, F_i]$$

Which has the properties that $\bigcap F_i = \{1\}$ and F_i/F_{i+1} is free abelian. Choose an arbitrary bi-ordering of F_i/F_{i+1}, and define a positive cone of F by declaring $1 \neq x \in F$ positive if its class in F_i/F_{i+1} is positive in the chosen ordering, where i is the last subscript such that $x \in F_i$.

Theorem 2: Positive roots implies bi-orderable

If $h : F \to F$ is an automorphism it preserves the lower central series and induces maps of the lower central quotients: $h_i : F_i/F_{i+1} \to F_i/F_{i+1}$. With this notation, h_1 is just the abelianization h_{ab}. In a sense, all the h_i are determined by h_1. That is, there is an embedding of F_i/F_{i+1} in the tensor power $F_{ab}^{\otimes k}$, and the map h_i is just the restriction of $h_{ab}^{\otimes k}$.
The assumption that all eigenvalues of h_{ab} are real and positive implies that the same is true of all its tensor powers.
This allows us to find bi-orderings of the free abelian groups F_i/F_{i+1} which are invariant under h_i. Using these to bi-order F, we get invariance under h, which proves the proposition.

Theorem 3: Bi-orderable implies some positive roots

We now turn to the proof of our third theorem: If K is fibred and its group is bi-orderable, then $\Delta_K(t)$ has some real positive roots. Recall that the knot group is an HNN extension of a free group, and is bi-orderable if and only if the homotopy monodromy map preserves a bi-ordering of the free group. Moreover, $\Delta_K(t)$ is the characteristic polynomial of the homology monodromy.

Our third theorem will follow from a more general result. If $\phi : G \to G$ is an automorphism, we can define its eigenvalues to be the eigenvalues of its induced map on the rational vector space $H_1(G, \mathbb{Q}) \cong G_{ab} \otimes \mathbb{Q}$.

Theorem

Suppose G is a nontrivial finitely generated bi-orderable group and that $\phi : G \to G$ preserves a bi-ordering of G. Then ϕ has a positive eigenvalue.

Theorem 3: Bi-orderable implies some positive roots

The idea is to consider the induced map ϕ_*, which can be considered as a linear transformation of $\mathbb{Q}^n$, which we regard as a subset of $\mathbb{R}^n$. If ϕ_* preserves an ordering of $\mathbb{Q}^n$, there is a hyperplane $H \subset \mathbb{R}^n$ defined by
$H = \{x \in \mathbb{R}^n |$ every nbhd. of x contains positive and negative points$\}$
H separates $\mathbb{R}^n$ and is invariant under ϕ_*.
Consider the unit sphere $\mathbb{S}^{n-1}$ of R^n, and let D denote the closed hemisphere of $\mathbb{S}^{n-1}$ which lies on the "positive" side of H. There is a mapping $D \to D$ defined by

$$x \to \phi_*(x)/|\phi_*(x)|$$

Since D is an $(n-1)$-ball, this map has a fixed point (Brouwer). This fixed point corresponds to an eigenvector of ϕ_*, which has a positive real eigenvalue. □

Surgery

We conclude with some applications to surgery on a knot K in $\mathbb{S}^3$. One removes a tubular neighborhood of K and attaches a solid torus $\mathbb{S}^1 \times D^2$ so that the meridian $\{*\} \times S^1$ is attached to a specified "framing" curve on the boundary of the neighborhood.

Theorem

Suppose K is a fibred knot in S^3 and nontrivial surgery on K produces a 3-manifold M whose fundamental group is bi-orderable. Then the surgery must be longitudinal (that is, 0-framed) and $\Delta_K(t)$ has a positive real root. Moreover, M fibres over S^1.

Surgery

Ozsváth and Szabó define an *L-space* to be a closed 3-manifold M such that $H_1(M;\mathbb{Q}) = 0$ and its Heegaard-Floer homology $\widehat{HF}(M)$ is a free abelian group of rank equal to $|H_1(M;\mathbb{Z})|$. Lens spaces, and more generally 3-manifolds with finite fundamental group are examples of L-spaces.

Theorem

If surgery on a knot K in S^3 results in an L-space, then the knot group $\pi_1(S^3 \setminus K)$ is not bi-orderable.