The Dehornoy floor and the Markov Theorem without Stabilization

featuring work of Ito & Malyutin-Netsvetaev

joint work with Doug Lafountain & Hiroshi Matsuda

Banff, Feb. 16, 2012

Braided block-strand (BBS) diagrams

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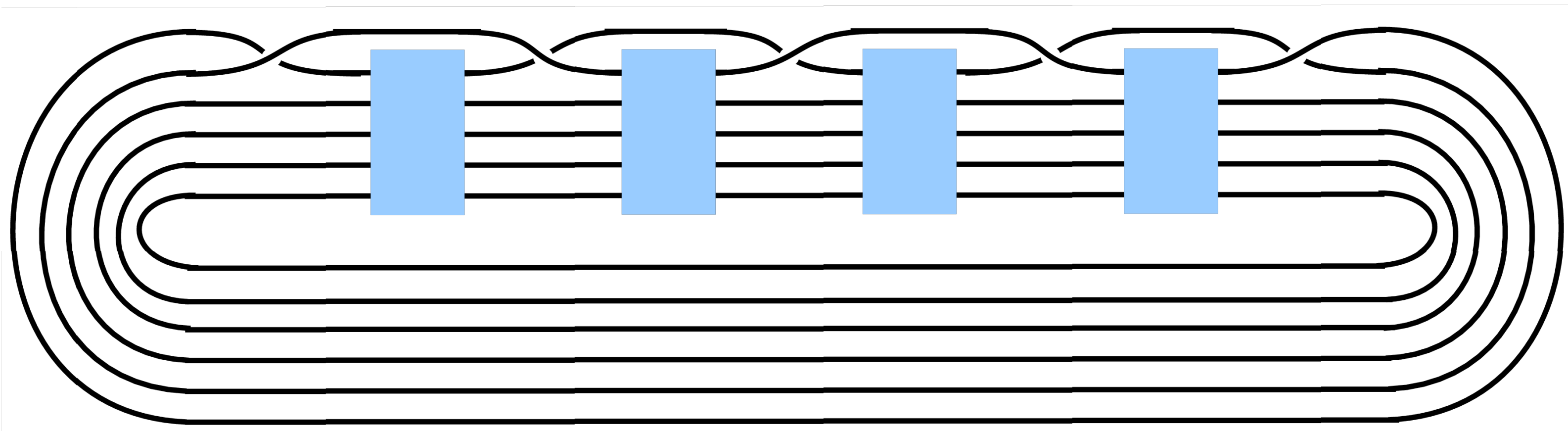

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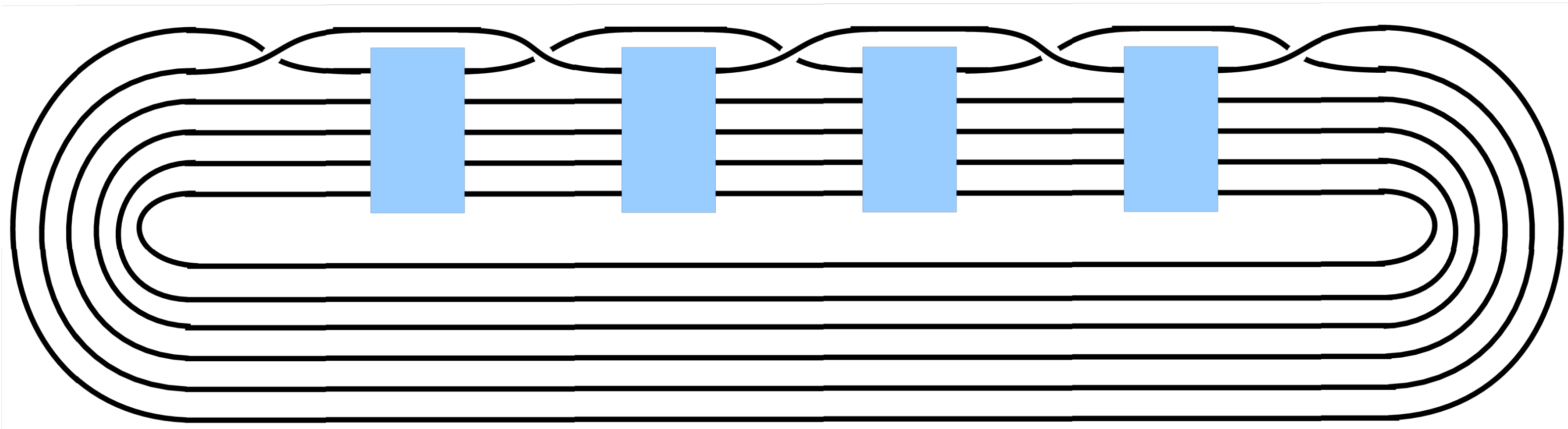

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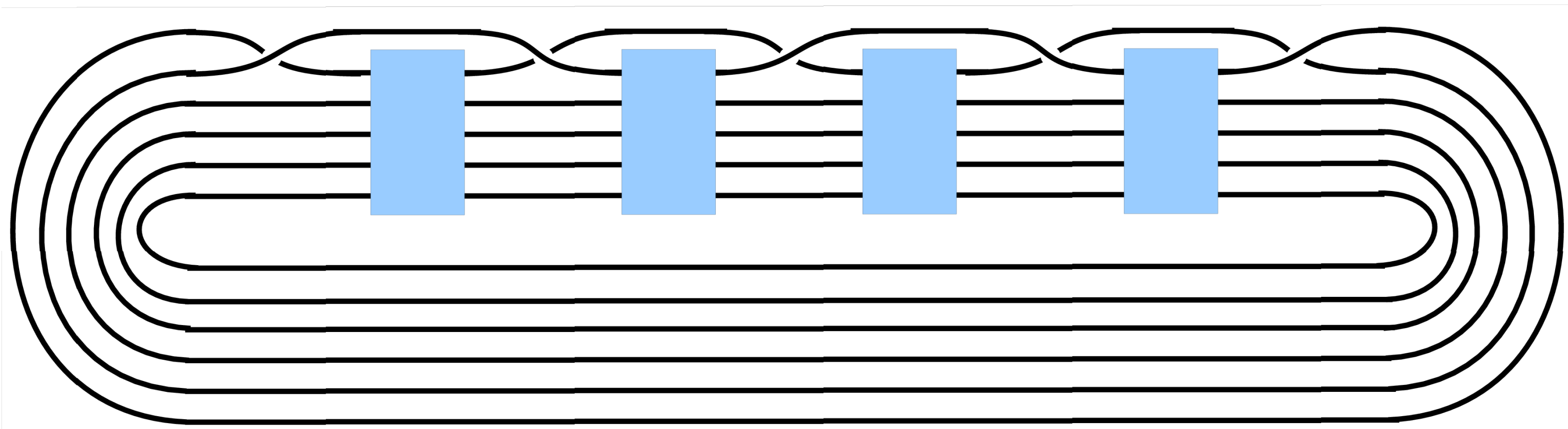

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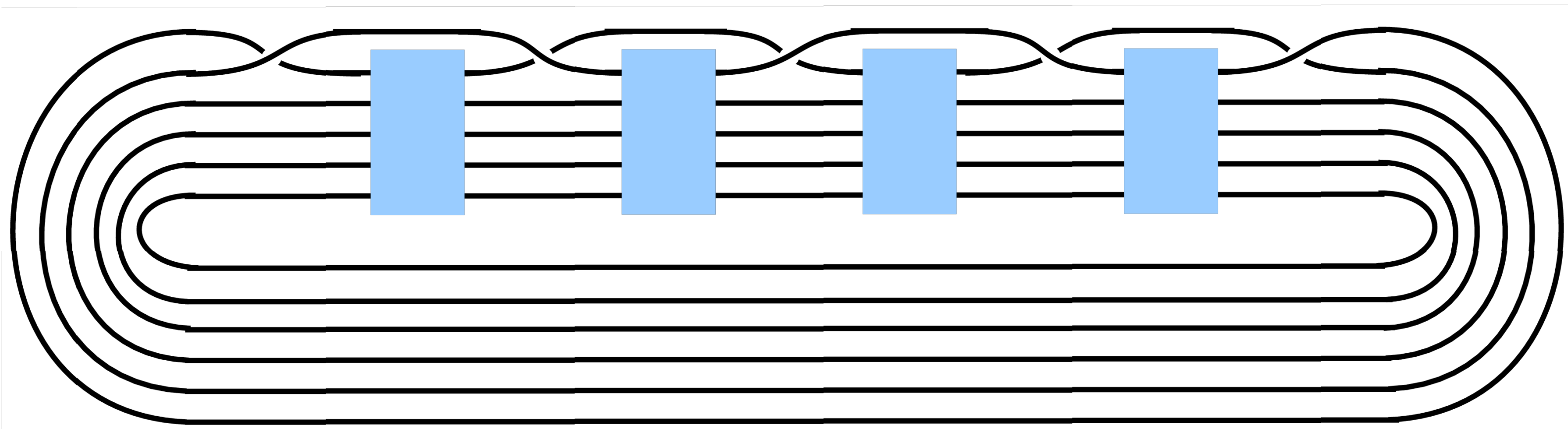

- We can talk about the braid index of a BBS.
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- We can talk about braid isotopies between two BBS.

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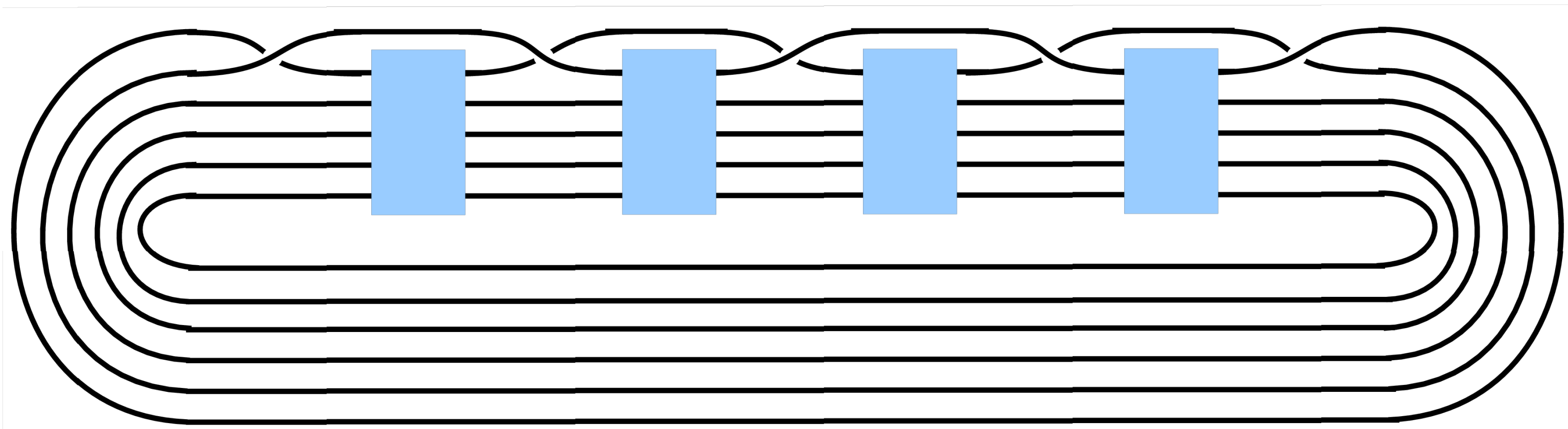

<u>Theorem</u>- Let D be a BBS diagram of braid index n. Then there exists infinitely many closed n-braids that D does not carry.
(Birman-M 2006)

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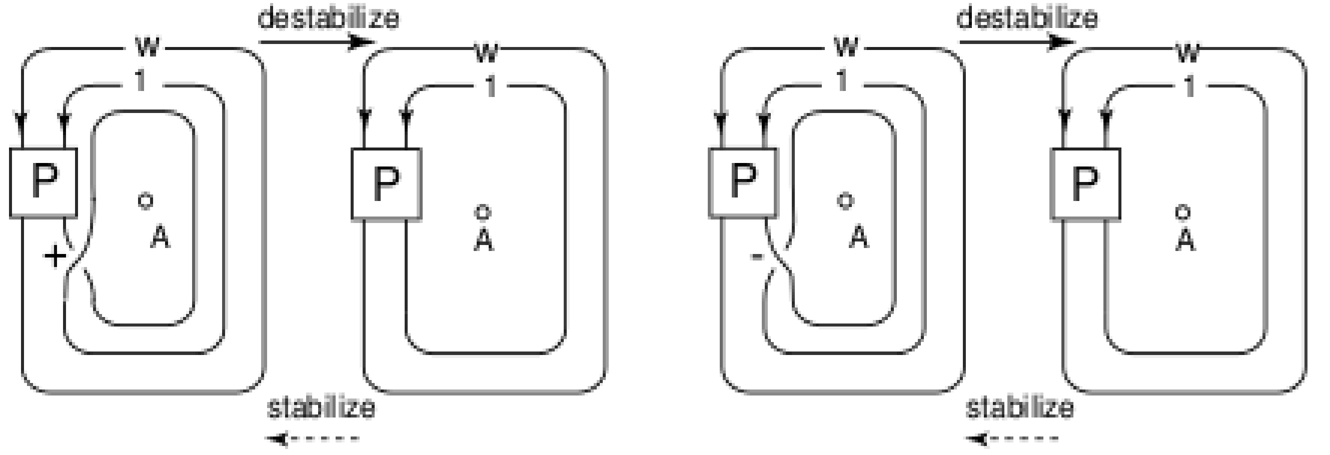

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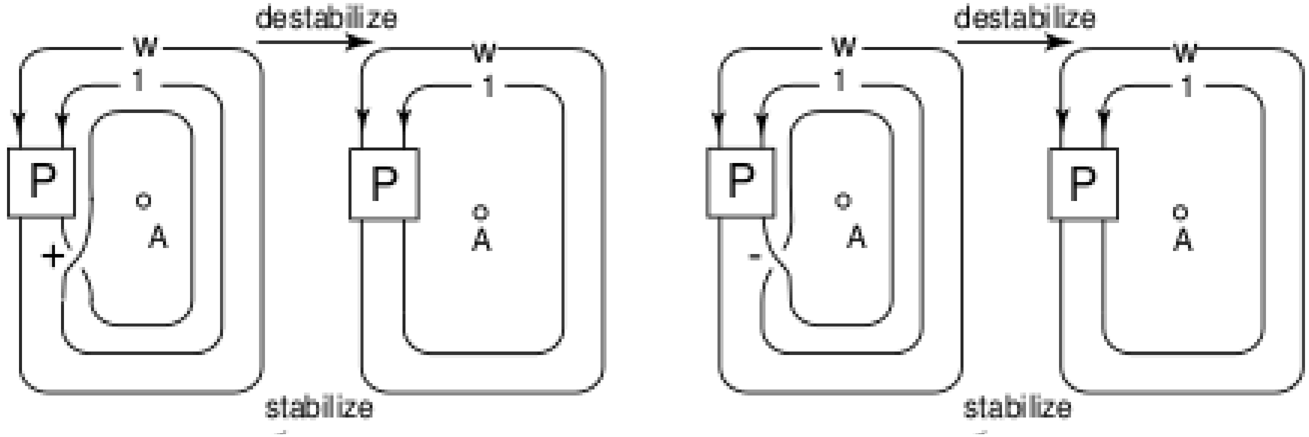

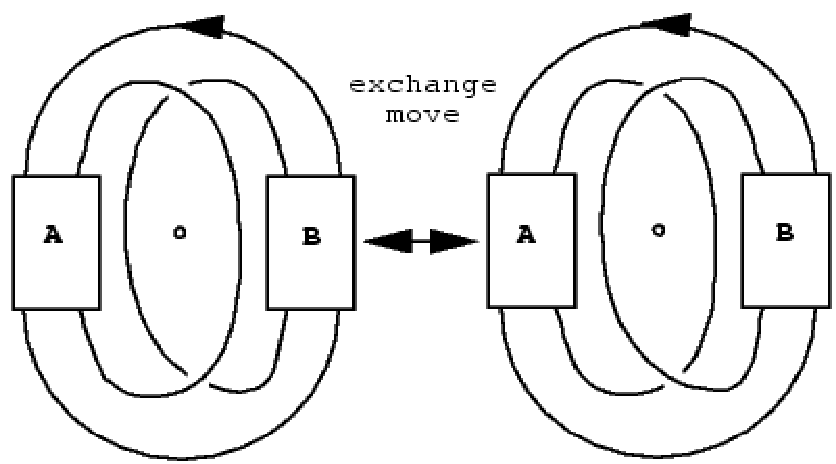

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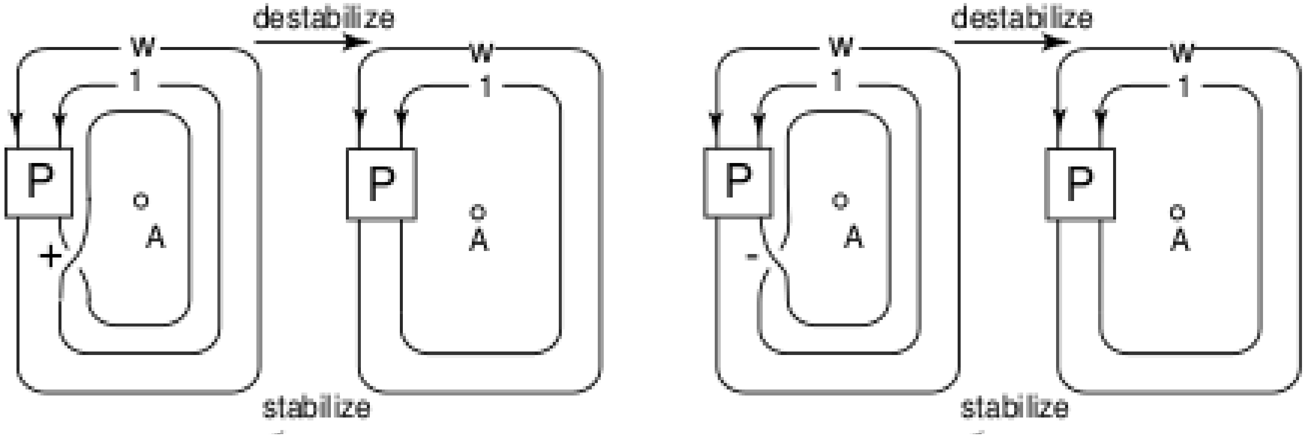

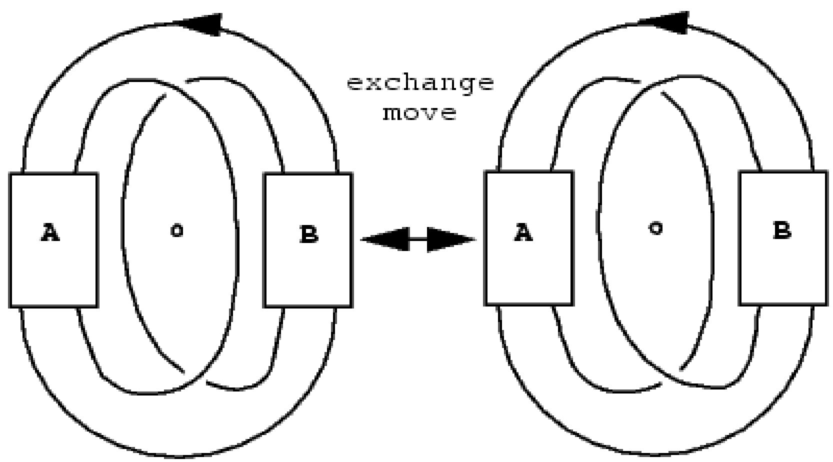

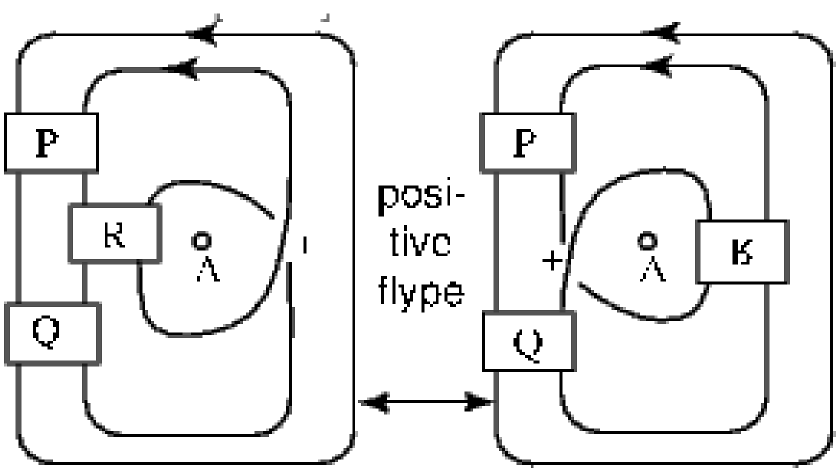

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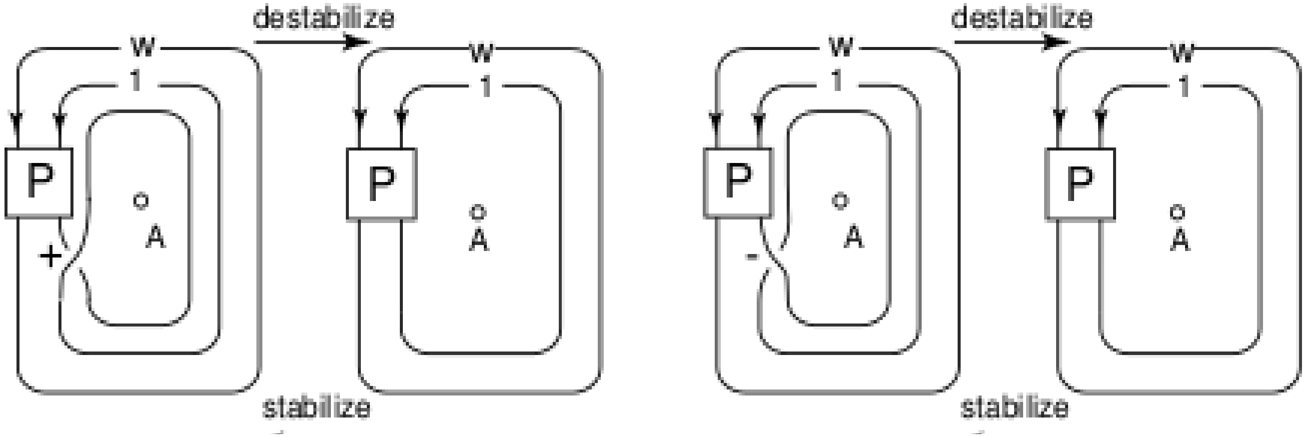

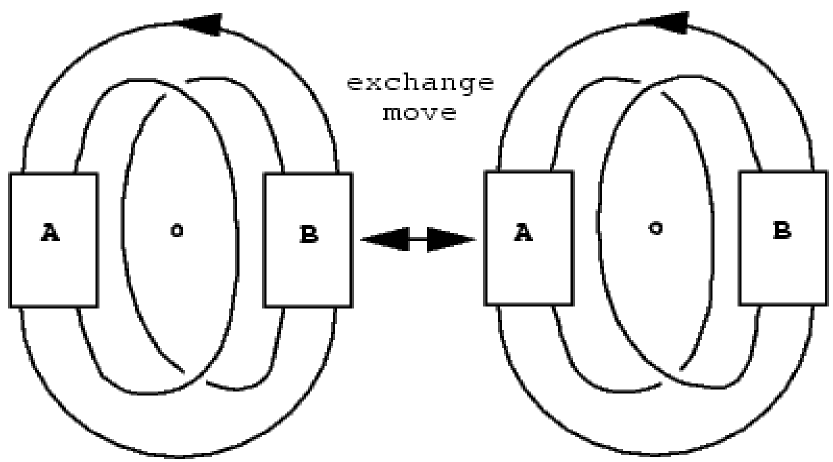

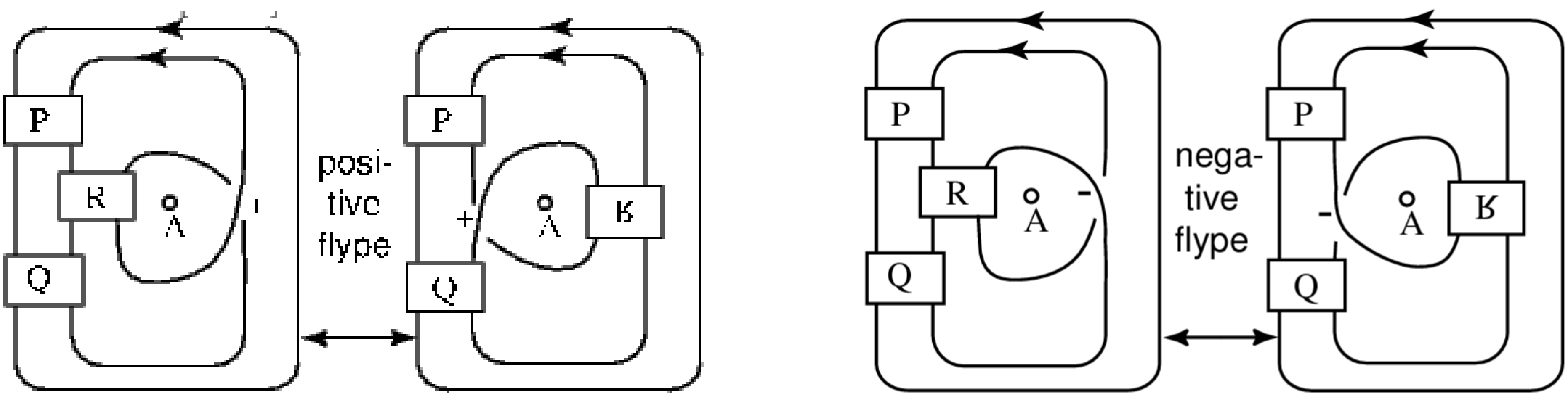

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  where $\Delta = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)(\sigma_1)$ is the Garside fundamental n-braid.

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- For each BBS diagram $\mathcal{D}$, there exists an integer $m(\mathcal{D})$ such that if $[\beta]_D \geq m(\mathcal{D})$ then β is not carried by $\mathcal{D}$ then . (T. Ito)

Back to BBS diagrams. Their initial use comes from the MTWS.

Theorem 2 (The Markov Theorem without Stabilization, Birman-M 2003) *Given a pair of positive integers (n^b, n^a), where $n^b \geq n^a > 1$, there exists a finite collection of associated templates $\{(\mathcal{D}_1^b, \mathcal{D}_1^a), \cdots, (\mathcal{D}_l^b, \mathcal{D}_l^a)\}$ such that for any oriented link type $\mathcal{L}$ having braid index $b(\mathcal{X}) = n^a$ and any closed braid representatives $L_1, L_2 \in \mathcal{L}$ with $b(L_1) = n_b$ and $b(L_2) = n_a$ there exists $L_1', L_2' \in \mathcal{L}$ with $b(L_1') = n^{b'}$, $n^b \geq n^{b'}$, and $b(L_2') = n^a$ such that:*

i) L_1' is obtained from L_1 via destabilizations and exchange moves; and, L_2' is obtained from X_1 via exchange moves, i.e. exchange equivalent.

i) L_1' & L_2' are carried by some $\mathcal{D}_j^b$ & $\mathcal{D}_j^a$, respectively. Moreover, the braiding assignment L_1' imposes on the blocks of $\mathcal{D}_j^b$ coincides with the braiding assignment L_2' imposes on the corresponding blocks of $\mathcal{D}_j^a$.

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4. *For each n, there exists a positive integer $r(n)$ with the following property: For $\alpha, \beta \in B_n$, if $[\alpha]_D \geq r(n)$ then the closed braids $\hat{\alpha}$ and $\hat{\beta}$ represent the same oriented link if and only if α and β are conjugate.*

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5. *The minimal value of $r(3)$ is 2.*

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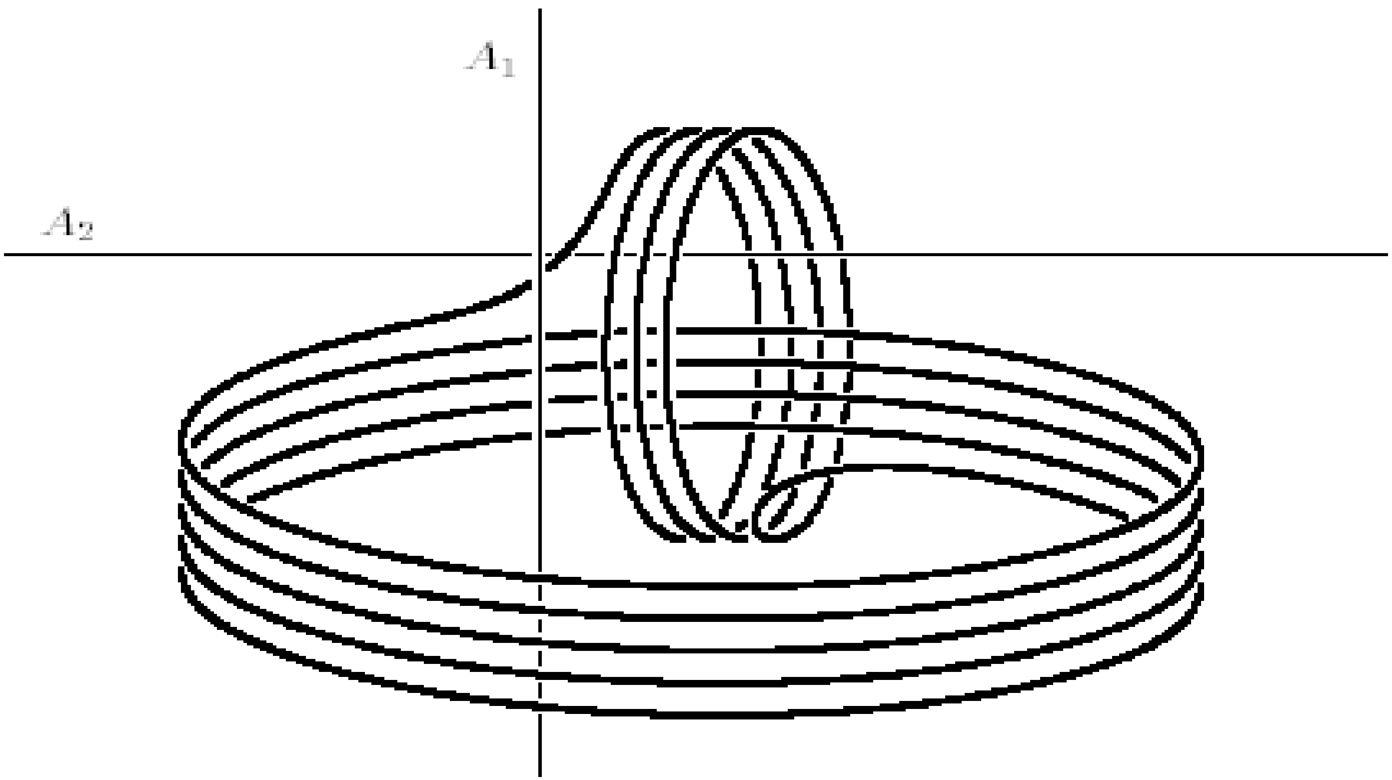

Dehornoy's Ordering on the braid group and braid moves, St. Petersbug Math. J. 2004. Figure 2.

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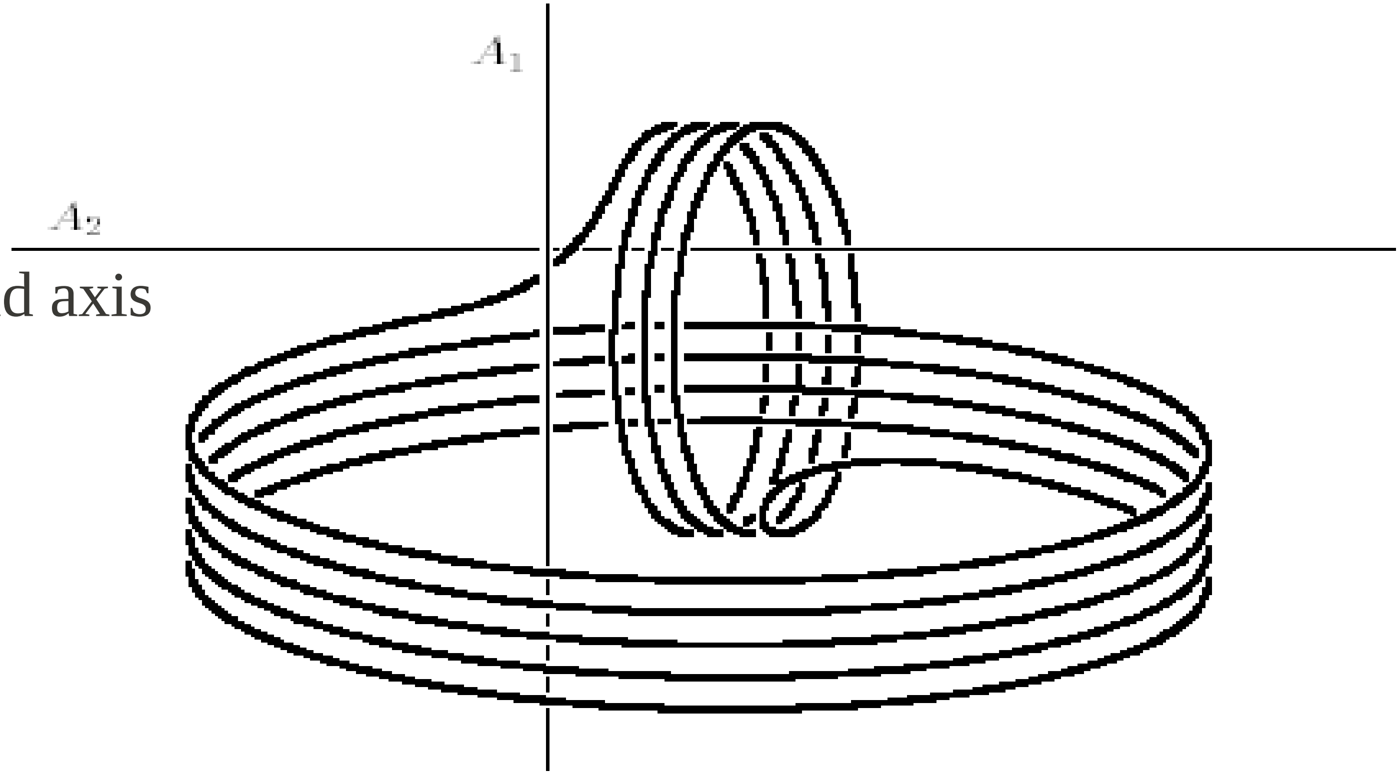
4-braid axis



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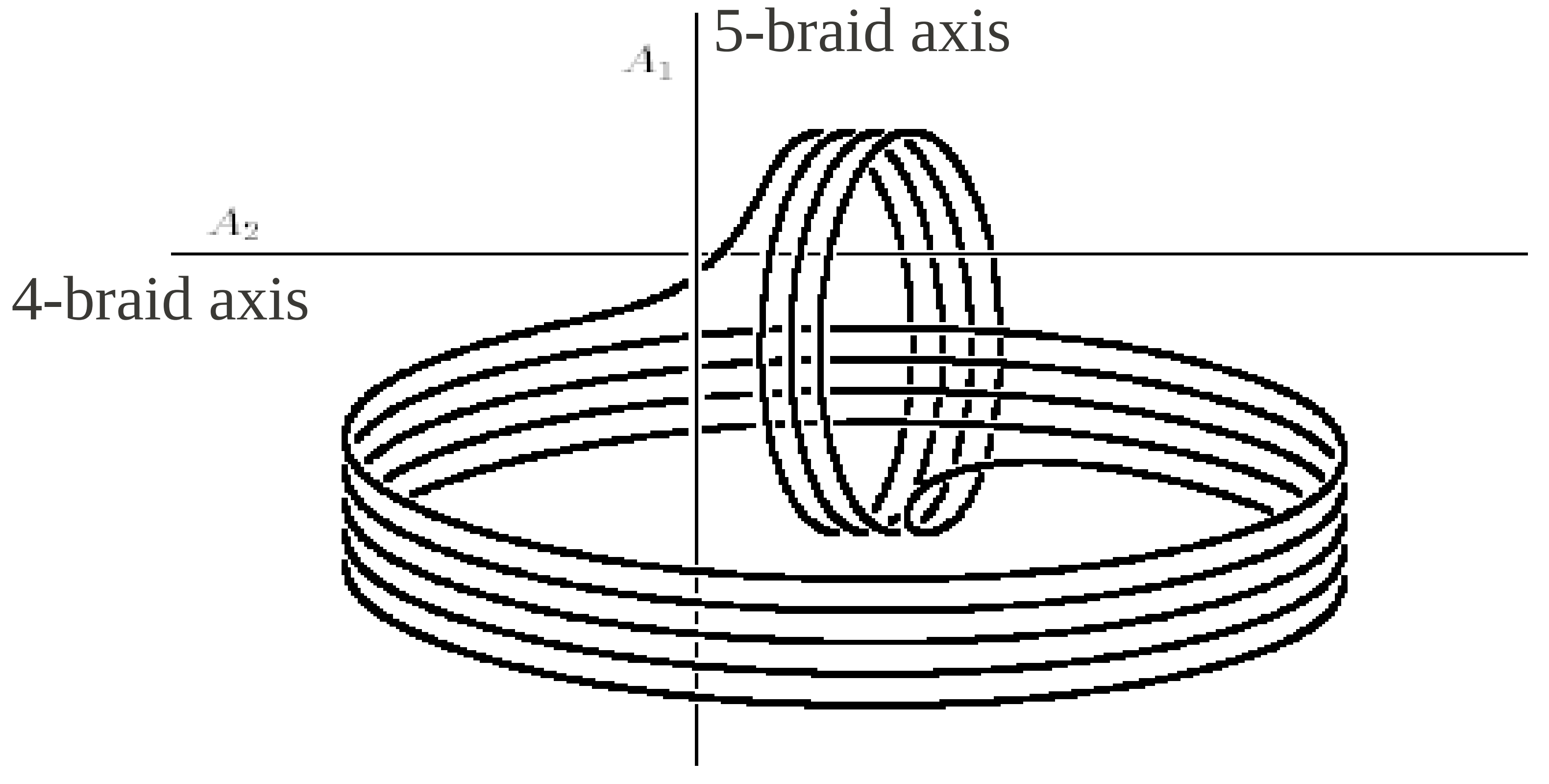

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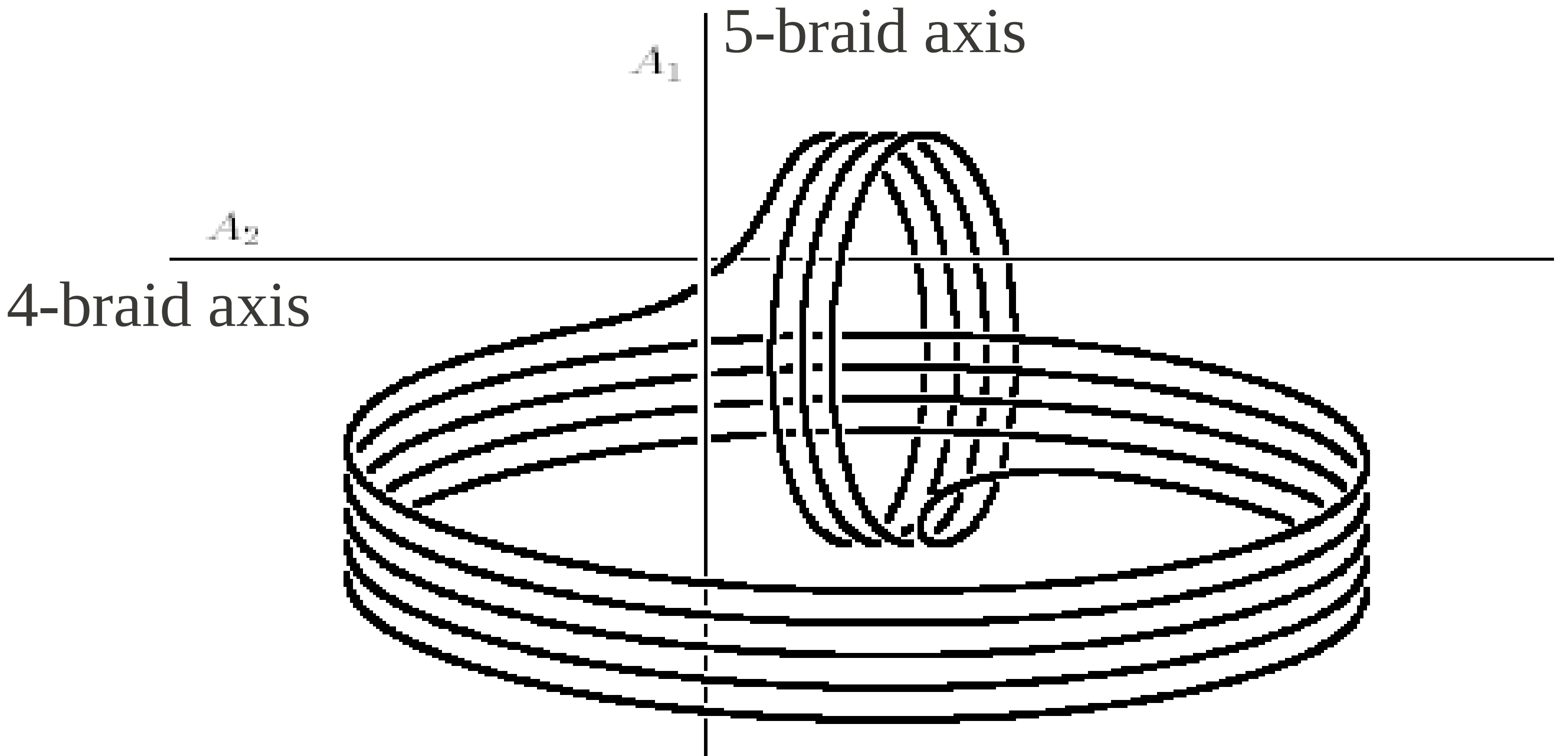

5-braid does not admit a destabilization, exchange move or flype

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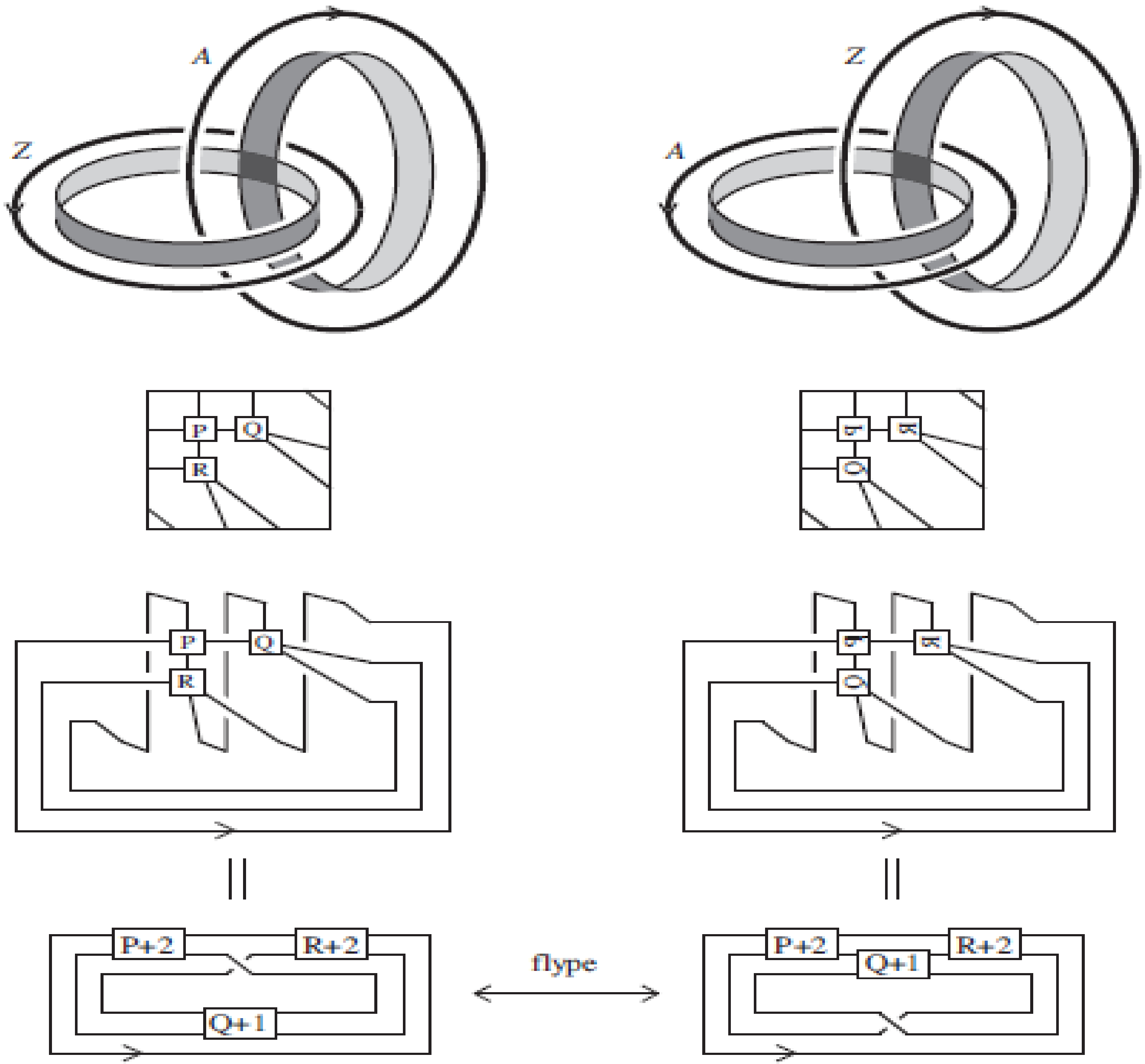

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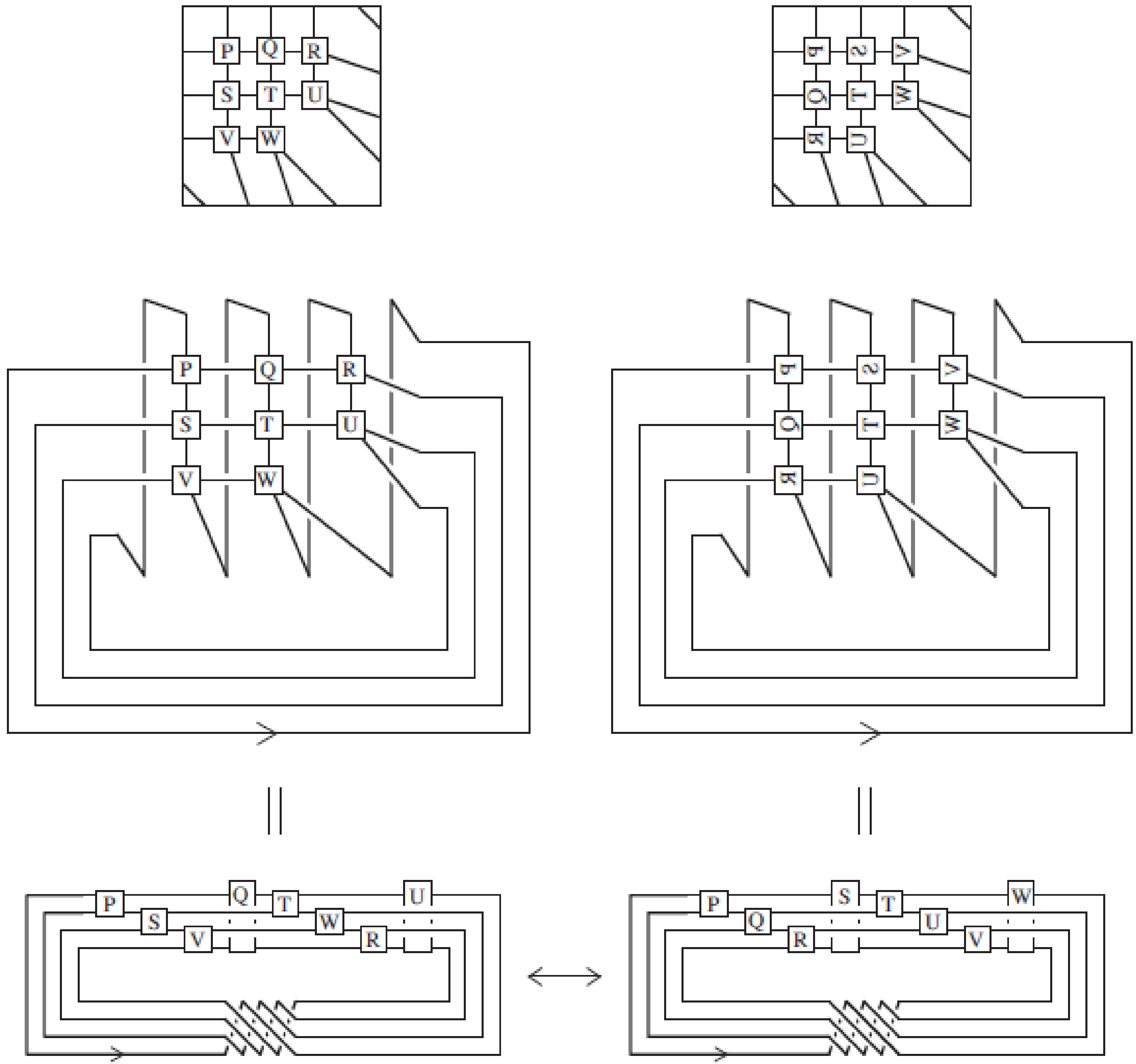

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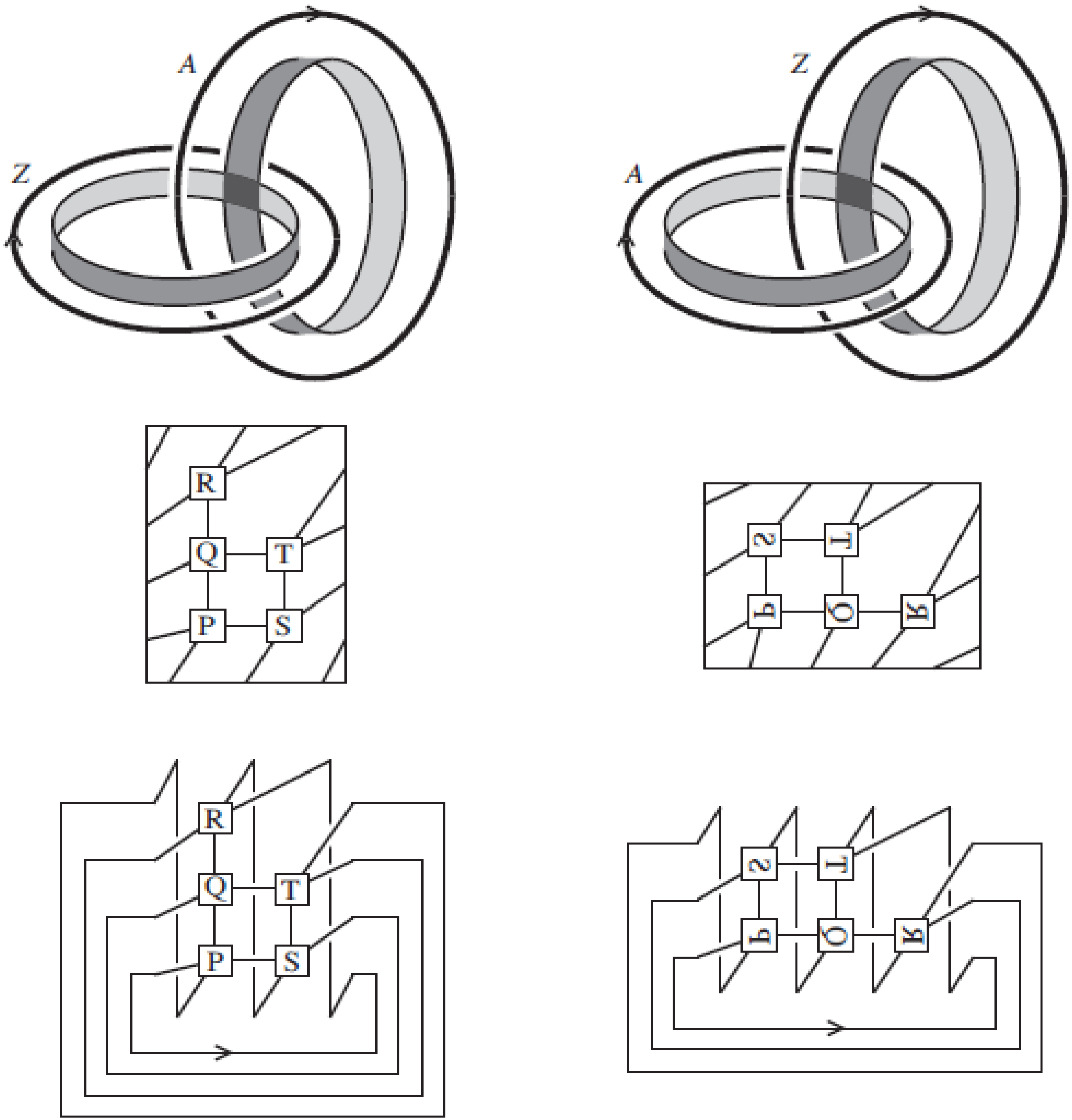

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- Over all closed braid representatives of a fixed knot, the maximal Dehornoy floor value is only realized by minimal index representatives.
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Iterated doubling pairs
&
Dual-Foliated Branched Surfaces

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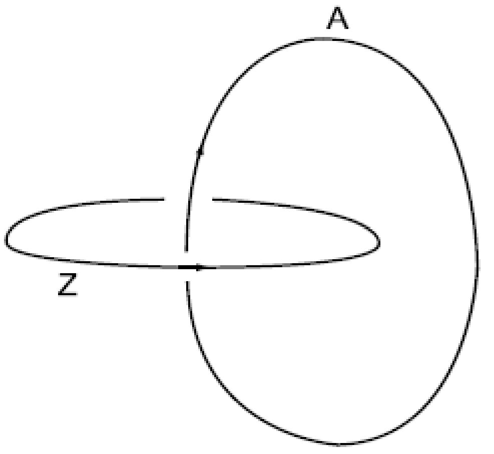
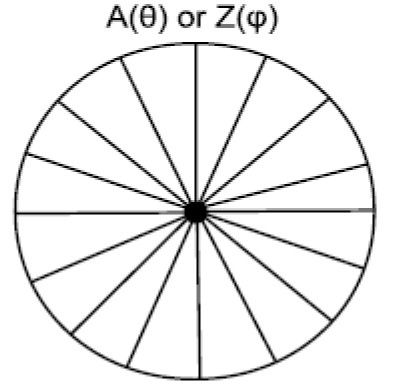
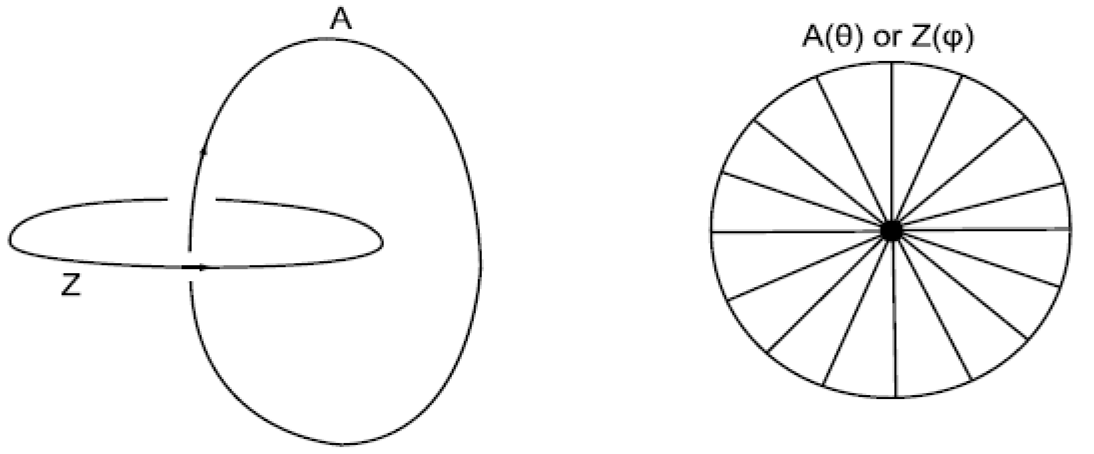
- A *and* Z *are mutually braided.* Each disc fiber $A(\theta)$ has a singular foliation induced by the fibration $\{Z(\phi)\}$, consisting of m transverse intersections with Z and $m - 1$ saddle singularities. We have a similar arrangement for $Z(\phi)$ and $\{A(\theta)\}$.





The Hopf link



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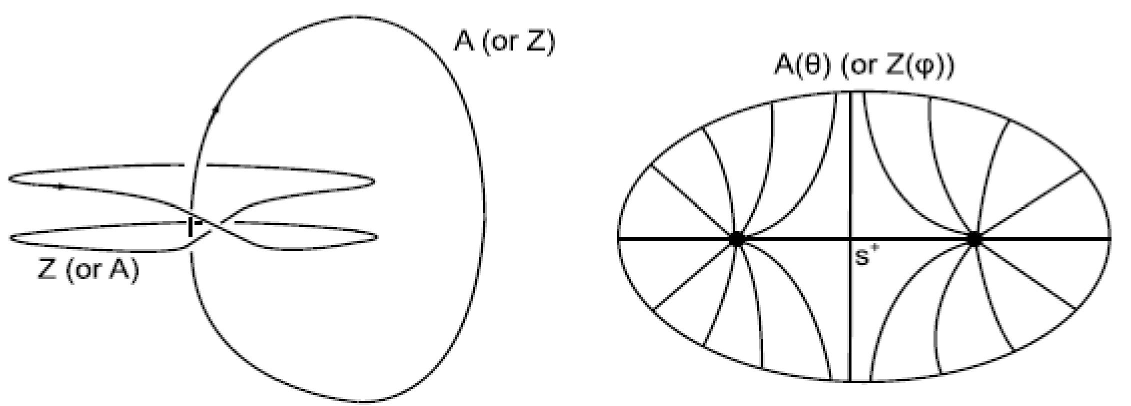

positive (1)-Morton double

Iterated doubling pairs
or Morton doubling pairs

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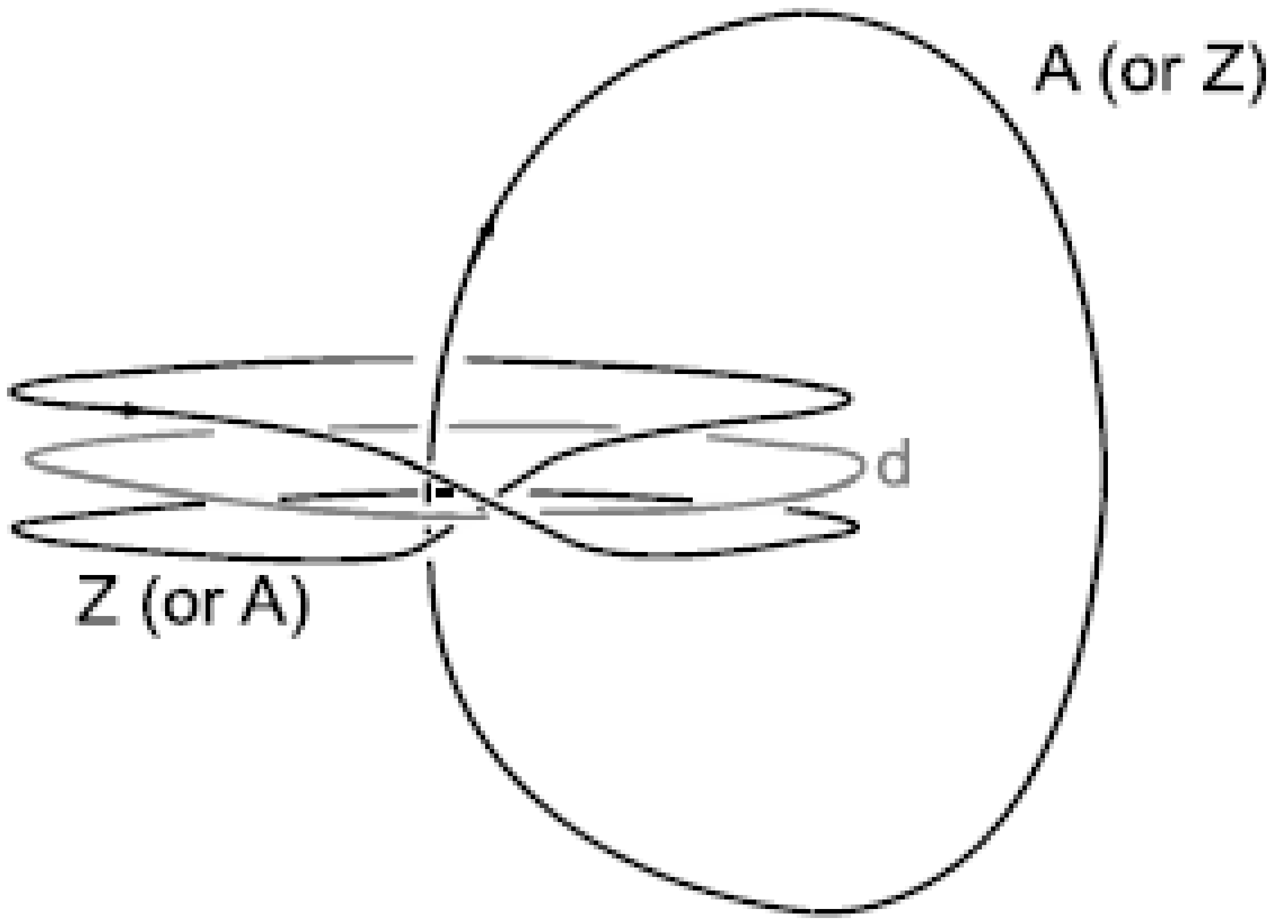
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d
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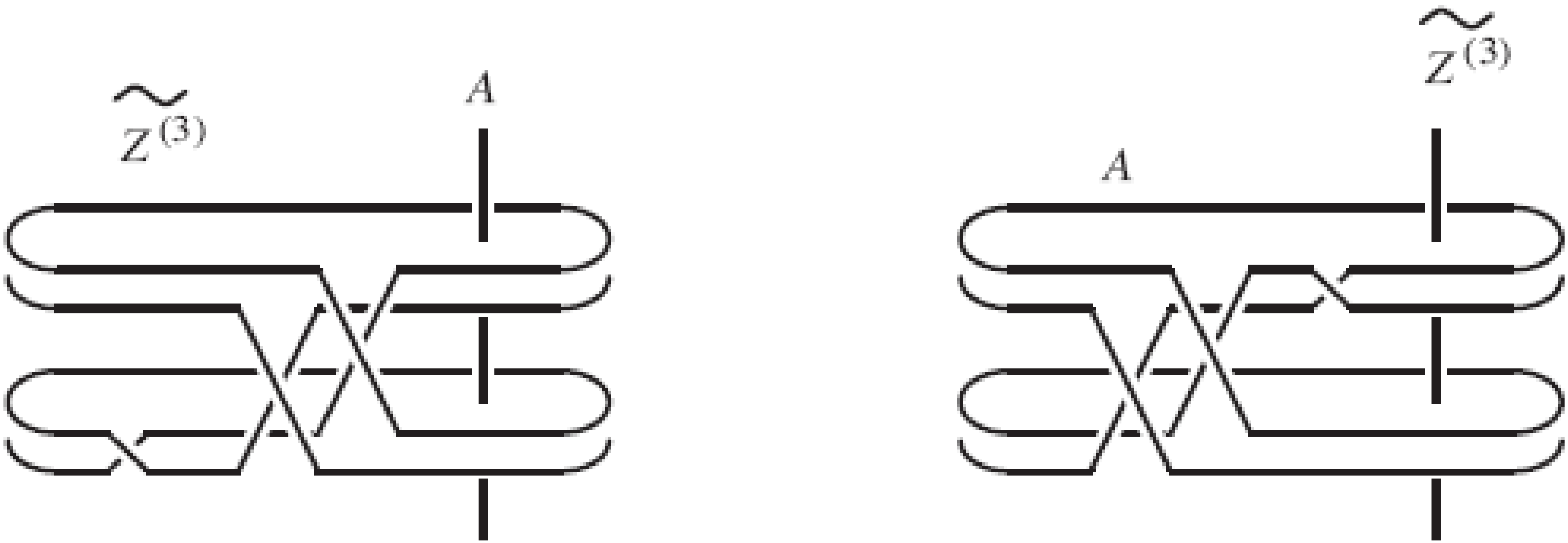

Two positive Morton doubling of Hopf link.

Dual-Foliated Branched Surfaces

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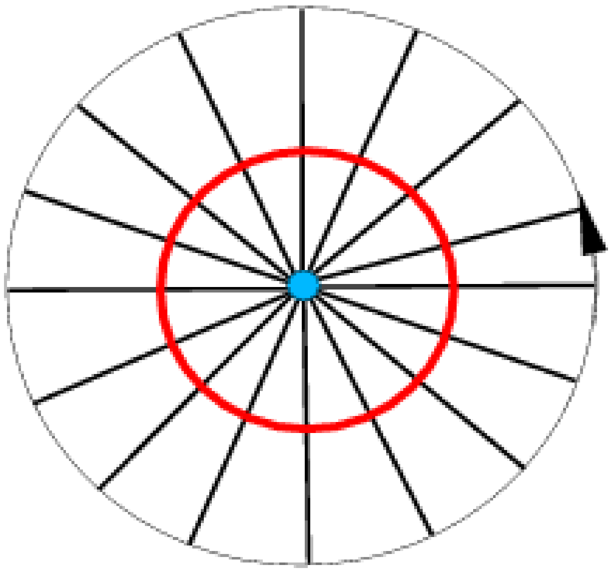
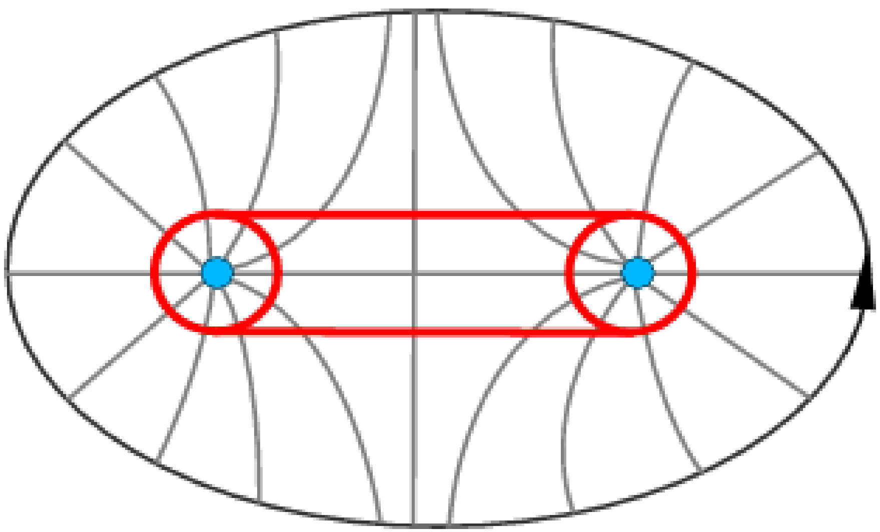
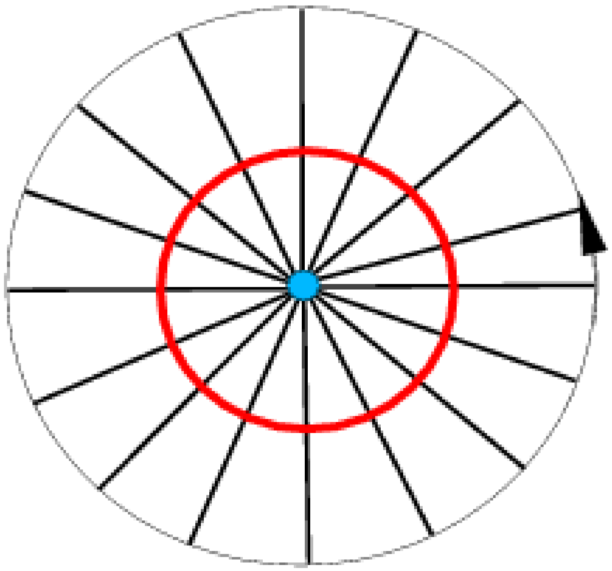
- The core of N_i is the derived component $d_i (\subset N_i \setminus N_{i+1})$.
- $\mathcal{B}_{A,Z}$ is *dual foliated* by $\{A(\theta)\}$ and $\{Z(\phi)\}$.

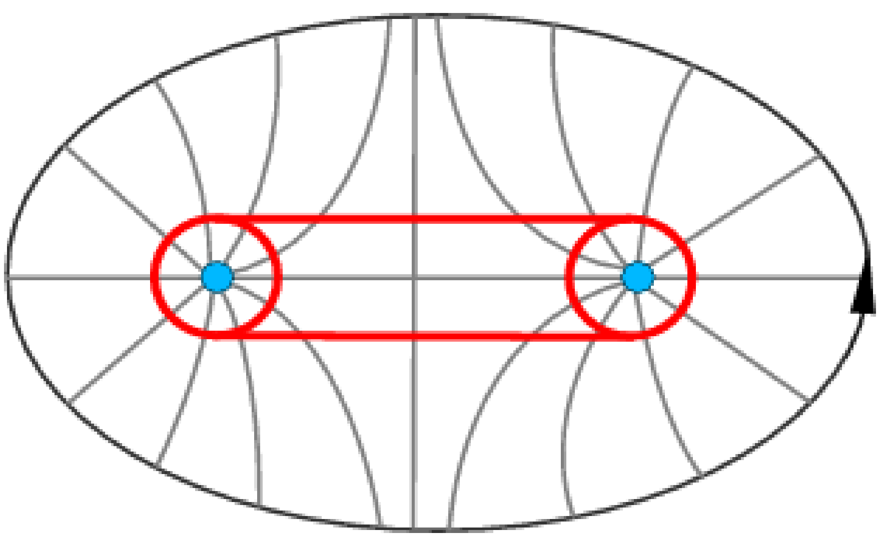


The suspension transverse train track (which is just a circle) for the Hopf link gives us the branched surface.
In this case the branched surface corresponds to the peripheral torus for *A* and *Z*.



The suspension transverse train track for the (1)-Morton double gives us the branched surface.
In this case the branched surface is the union of the peripheral tori for *A* and *Z*.





The dual foliation comes from the intersection of the branched surface with the two braid fibrations for the *(A,Z)* pair.

Now consider the following configuration:

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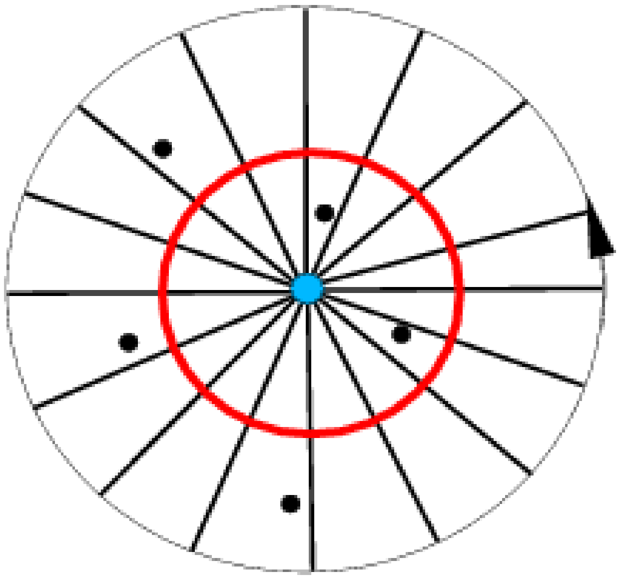
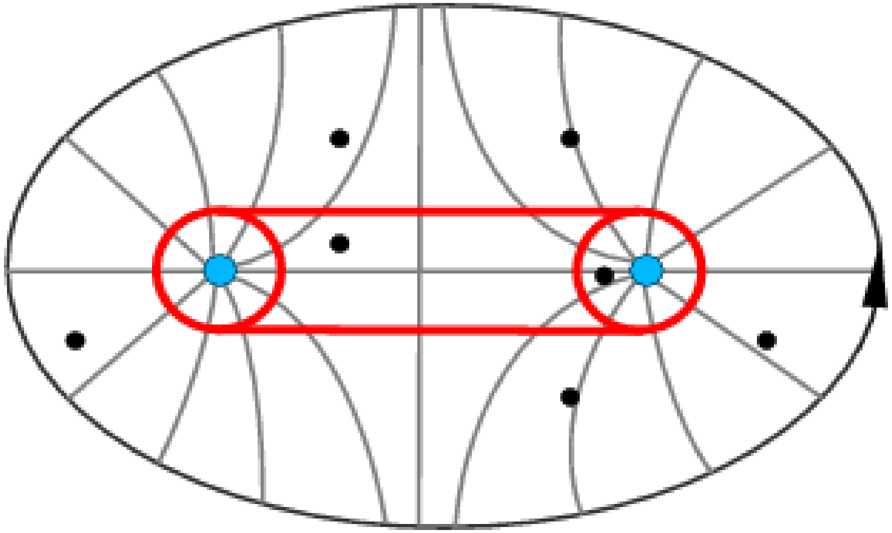
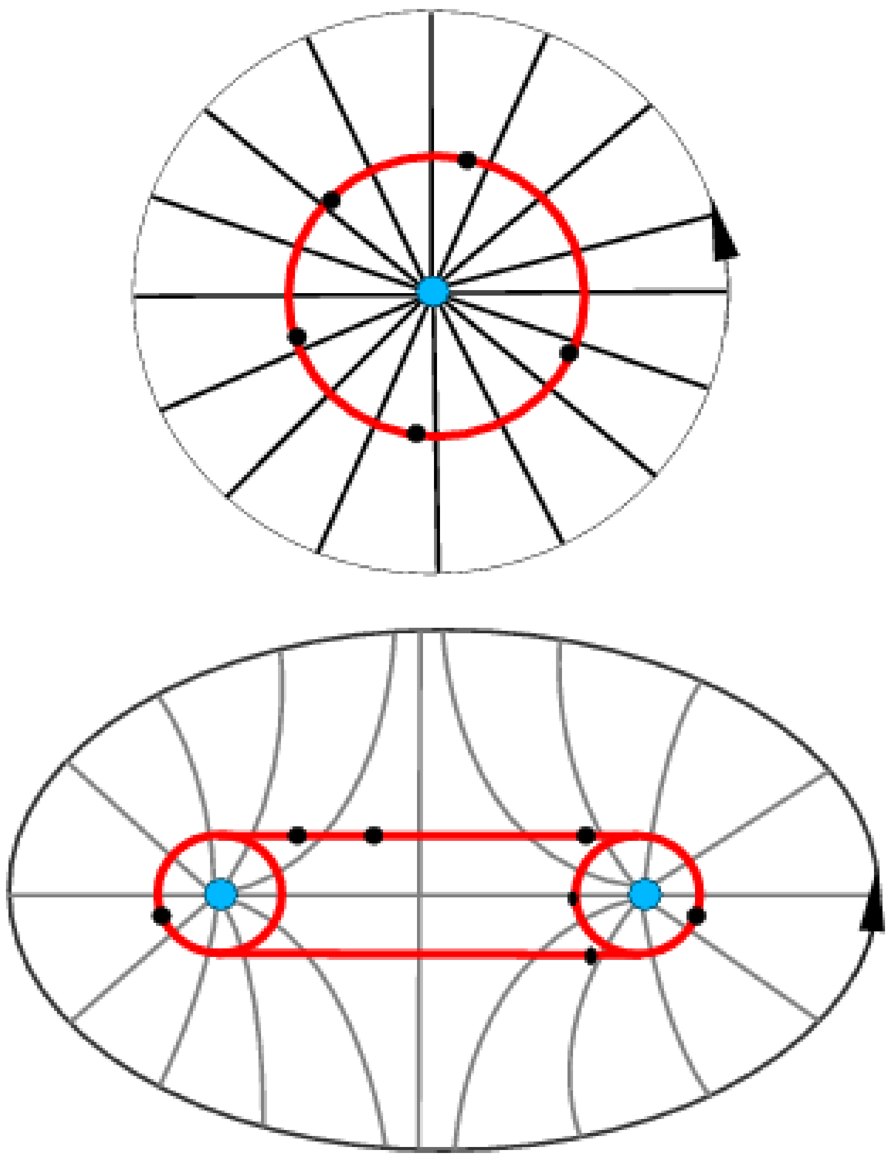
- *A is a braid axis for the oriented link $L \sqcup Z$ and $\{A(\theta)\}$ is a corresponding braid fibration. Thus, L is necessarily transverse to each $A(\theta)$.*
- *Similarly, Z is a braid axis for the oriented link $A \sqcup L$ and $\{Z(\phi)\}$ is a corresponding braid fibration. Thus, L is necessarily transverse to each $Z(\phi)$.*





The black dots where the link *L* is intersecting the foliated disc fibers. As we push forward in the braid fibration we will see these dots move around in the disc fiber always positively transverse to the foliation of the fiber.

We can use the leaves of the foliation to project the link *L* onto the train track and, thus, the branched surface.



In the cyclic movie that runs as we move forward in the fibrations associated with either *A* or *Z* will will see the black dots of *L* race around the oriented train track in the positive direction. A crossing of the projection onto the branched surface occurs whenever one dot passes other.

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- Then L is a closed braid L_A with respect to the axis A and fibration $\{A(\theta)\}$; and, a closed braid L_Z with respect to the axis Z and the fibration $\{Z(\phi)\}$.

Theorem 3 (Lafountain-Matsuda-M) *Let L be carried by a dual-foliated branched surface $\mathcal{B}_{A,Z}$ and suppose (A, Z) is a positive interated doubling pair. Then the self-linking numbers for L_A and L_Z are equal. In particular, if the braid indexes are equal then their algebraic lengths are the same.*

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Theorem 4 (Lafountain-Matsuda-M) *Let L be carried by a dual-foliated branched surface $\mathcal{B}_{A,Z}$ and suppose (A, Z) is a negative interated doubling pair. Assume that braid indexes are both minimal then their self-linking number and, thus, algebraic lengths are the same. L_A and L_Z are equal.*

Question: Given closed braid respresentatives β, β' of an oriented link L, is there an iterated doubling pair (A, Z) such that the assoicated $\mathcal{B}_{A,Z}$ carries L with L_A braid isotopic to β and L_Z braid isotopic to β'?

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Anwser: Almost.

Theorem 5 (Lafountain-Matsuda-M, preliminary version) *Let β, β' be two closed braid representatives of an oriented link L. Then there exists a β'' such that:*

1. *β'' is obtained from β' through a sequence of destabilizations, exchange moves and braid isotopies.*
2. *There is an iterated doubling pair (A, Z) such that the associated $\mathcal{B}_{A,Z}$ carries L.*
3. *L_A is braid isotopic to β and L_Z is braid isotopic to β''.*