Theoretical and Applied Aspects of Nonnegative Matrices

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The workshop brought together young and experienced researchers who study nonnegative matrix theory and its applications. The speakers at the workshop presented recent progress, open problems and challenges involving nonnegative matrices and their generalizations. Specifically, discussed were eventually nonnegative matrices; combinatorial aspects of nonnegative matrix theory and its interplay with graph theory; numerical issues and applications in optimization and the solution of matrix equations. As desired and expected, new ideas, suggestions and alternative points of view were raised by the participants.

1 Presentation Highlights

The workshop opened with three overview talks, covering theory, numerical aspects and applications of nonnegative matrices. The subjects of interest were matrices possessing various types of nonnegativity properties: entrywise nonnegative matrices, eventually nonnegative matrices, completely positive matrices, and totally nonnegative matrices. Also discussed were co-positive matrices, P-matrices and positive semi-definite matrices.

Bryan Shader opened the workshop with an Olympics inspired talk. He presented a forward-looking survey of some of the ways combinatorics is using nonnegative matrix theory, including the chip-firing and rotor-routing models on directed graphs [5, 13]. A summary of relations between analytic and algebraic properties of a nonnnegative matrix with concepts of the digraph associated with the matrix was presented. He also linked various generalizations of the notion of primitivity in nonnegative matrix theory, which provide an avenue for the combinatorial analysis of products of nonnegative matrices [10, 11].

In **Leslie Hogben's**[1] and in **Craig Erickson's** talks, past and recent results on eventual nonnegativity and positivity were surveyed and discussed; see [6, 8, 9, 12]. The biggest challenge in this area remains to be the lack of a characterization of eventually nonnegative matrices that would lend itself to an applicable test. Indeed, unlike eventual positivity, the fulfilment of the so called weak Perron-Frobenius conditions is necessary but not sufficient.

Chun-Hua Guo introduced us to Algebraic Riccati Equations under the condition that the coefficients form a block matrix that is an M-matrix. These types of equations are inspired by transport theory and Markov models. There are challenges in this area pertaining to the design and analysis of solution algorithms, which will surely require the use of nonnegative matrix theory.

There were also participants who brought a different perspective to the workshop through their particular interests. **Pierre Maréchal** presented some of his recent work on minimizing the condition number among

[1]Leslie Hogben dedicated her talk to the late Uri Rothblum for his many and important contributions to nonnegative matrix theory.

all matrices in a compact subset of the positive semi-definite matrices. He showed that many classical applied problems admit such a formulation. The results can be applied under additional nonnegativity constraints, which made this presentation intriguing to those working in the theory and numerics of nonnegative matrices. **Shawn Wang** [1] applied nonnegative matrix theory to provide a rigorous proof for the convergence of a Gauss-Seidel type fixed point method that relies on nonnegative matrix products.

Colin Garnett told us all about ISBN as an error correcting code. Treating the ISBN numbers as a quotient lattice and using the Smith Normal Form, he described (and calculated the probability of solving accurately) the problem of generating a basis for the lattice from a given number of lattice points.

2 More open problems

The following open problems and ideas for future research were also identified and discussed.

Wayne Barrett presented and proved some fundamental and very elegant upper and lower bounds for the spectral radius of an adjacency matrix. These bounds linked the spectral radius directly to graphical constants such as the number of edges, the clique number, etc. Interestingly, the comparison of such lower and upper bounds provide new and non-trivial (proofs of) inequalities among these graphical quantities. The speaker then suggested the use of this technique for the discovery of more such relations, in the general case or in special cases of graphs. This idea exemplifies the interplay between matrix and graph theory which was prevalent throughout the workshop.

Naomi Shaked-Monderer talked about completely positive matrices. She explained why the maximum cp-rank is obtained not only in the interior of the cone of such matrices but also on the boundary. It is hoped that this new discovery will assist in obtaining a sharp upper bound for the cp-rank of $n \times n$ completely positive matrices and possibly resolve the conjecture that $\lfloor \frac{n^2}{4} \rfloor$ is the desired bound [2]. It was noted that the new result was recently used in the proof of this conjecture for $n = 5$.

Daniel Szyld [3, 4] connected positive and eventually positive matrices to properties of their spectral projections, and posed the challenge of characterizing an eventually nonnegative matrix A in terms of its spectral projection P and a decomposition of the type $A = \rho(A)P + Q$. During his talk on eventual nonnegativity and on matrices satisfying the Perron-Frobenius property, Daniel Szyld mentioned two auxiliary results regarding the mapping of one or two nonnegative vectors into the interior of the nonnegative orthant by an orthogonal matrix that is arbitrarily close to the identity. He asked whether these results can be generalized e.g., to three vectors in $\mathbb{R}^3$. That is, given nonnegative vectors u, v, w in $\mathbb{R}^3$, is there always an orthogonal matrix Q so that, for any given $\epsilon > 0$,

1. $Qu, Qv, Qw \in \operatorname{int} \mathbb{R}^3$
2. $\|Q - I\| < \epsilon$?

The following day Bryan Shader described a viable idea for producing a counterexample, taking into account that such an orthogonal matrix cannot be a reflection and thus can only be a rotation in $\mathbb{R}^3$. Finally, in Daniel Szyld's talk, questions regarding the shape of the cone of eventually nonnegative matrices came up and were discussed by the participants.

Shahla Nasserasr presented Garloff's conjecture on the checker-board interval of invertible totally nonnegative matrices (see [7]), and suggested a particular approach to resolving the conjecture. It was noted that results in interval regularity and interval P-matrices may be relevant.

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