BIRS, Meeting on Cluster Algebras
September 5-10, 2011, Banff, Canada

Continuous Cluster Categories

Gordana Todorov, Northeastern University, Boston, MA
Joint work with Kiyoshi Igusa, Brandeis Iniversity

Motivation: Continuous cluster categories of type **A** are uncountably infinite categories with cluster structures, where the clusters correspond to ideal geodesic triangulations of the hyperbolic plane in one case, or clusters of the cluster categories of type $\mathbf{A_n}$ in the other case. (The hyperbolic plane is the universal cover of once punctured surfaces.)

We define $\mathcal{A}_\mathbb{R}$ to be the category of k-representations of the real line $\mathbb{R}$, where k is a field. For each $a < b \in \mathbb{R}$ we denote by $V_{(a,b]}$ the special representation: $V_{(a,b]}(x) = k, \forall x \in (a,b]$ and $V_{(a,b]}(y) \to V_{(a,b]}(x)$ is 1_k for all $a < x < y \leq b$. We also define the full subcategory $\mathcal{B} \subset \mathcal{A}_\mathbb{R}$ to be additively generated by the indecomposable objects $\{V_{(a,b]} \mid a < 0 < b\}$.

A particularly nice correspondence between indecomposable objects of $\mathcal{B}$ and points in $\mathbb{R}^2$ is obtained by:

$$V_{(a,b]} \leftrightarrow M(x,y) \text{ where } (x,y) = (-ln(-a), ln(b)),$$

and we will use this correspondence to identify the k-representations of $\mathbb{R}$ with points in the plane $\mathbb{R}^2$.

For a positive real number $c \in \mathbb{R}$, we define the full subcategory $\mathcal{B}_{\geq c} \subset \mathcal{B}$ by defining the indecomposable objects of $(\mathcal{B}_{\geq c})$ as $\{M(x,y) \in \mathcal{B} \mid |x-y| \geq c\}$. *The continuous derived category* $\mathcal{D}_c$ is defined using "two way $\mathcal{B}_{\geq c}$-approximations" in $\mathcal{B}$ and defining triangulated structure on $\mathcal{D}_c := \mathcal{B}/\mathcal{B}_{\geq c}$.

For each positive real number $d \in \mathbb{R}$ we define functor $F_d : \mathcal{D}_c \to \mathcal{D}_c$ by $F_d(M(x,y)) = M(y+d, x+d)$ which can be used to define a triangulated automorphism of the doubled derived category $\mathcal{D}_c^{(2)}$. Using the functor F_d the orbit category of $\mathcal{D}_c^{(2)}$ is defined and denoted by $\mathcal{C}_{(c,d)} := \mathcal{D}_c^{(2)}/F_d$. With these definitions we have the following results.

Theorem: The orbit category $\mathcal{C}_{(c,d)}$ is triangulated if $c \leq d$.

Theorem: The orbit category $\mathcal{C}_{(c,d)}$ has a cluster structure if and only if either $c = d$ or $c = \frac{n+1}{n+3}d$ for some positive integer n.

Relation between continuous cluster category and ideal triangulation of hyperbolic plane by geodesics is obtained in the case $c = d = \pi$. To each representation $M(x,y) \in \mathcal{C}_{(\pi,\pi)}$ we associate geodesic starting at angle x and ending at $y+\pi$. With this correspondence, each cluster in the continuous cluster category $\mathcal{C}_{(\pi,\pi)}$ corresponds to an ideal triangulation of the hyperbolic plane.

Objects of $\mathcal{C}_{(c,\pi)}$ can also be viewed as representations of the circle S^1.