Strong approximation for the empirical process in the dependent setting

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(joint work with J. Dedecker and E. Rio)

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Introduction

- Let $X = (X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables with common distribution function F. Define the empirical process of X by

$$R_X(s,t) = \sum_{1\le k\le t} \left(\mathbf{1}_{X_k\le s} - F(s)\right), \; s\in\mathbb{R}, \; t\in\mathbb{R}^+ .$$

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- For iid r.v's X_i with uniform distribution over $[0,1]$, Komlós, Major and Tusnády (1975) constructed a continuous centered Gaussian process K_X with covariance function

$$\mathbb{E}\big(K_X(s,t)K_X(s',t')\big) = (t\wedge t')(s\wedge s' - ss')$$

in such a way that

$$\sup_{s\in\mathbb{R},t\in[0,1]} |R_X(s,[nt]) - K_X(s,[nt])| = O(\log^2 n) \quad \text{almost surely.}$$

Previous results in the dependent case

- Berkes and Philipp (1977)- Yoshihara (1979): If $\alpha(n) = O(n^{-a})$ for some $a > 3$, and if F is continuous, there exists a Gaussian process, K_X, continuous such that

$$(*) \qquad \sup_{s\in\mathbb{R}, t\in[0,1]} |R_X(s,[nt]) - K_X(s,[nt])| = O(\sqrt{n}(\ln(n))^{-\lambda}) \quad \text{a.s,}$$

for some $\lambda > 0$. The covariance function Γ_X of K_X is given by $\Gamma_X(s,s',t,t') = \min(t,t')\Lambda_X(s,s')$ where

$$\Lambda_X(s,s') = \sum_{k\geq 0} \mathrm{Cov}(\mathbf{1}_{X_0\leq s}, \mathbf{1}_{X_k\leq s'}) + \sum_{k>0} \mathrm{Cov}(\mathbf{1}_{X_0\leq s'}, \mathbf{1}_{X_k\leq s}) \,.$$

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- Berkes, Hörmann and Shauer (2009): They obtained $(*)$ under a $S-$ mixing condition well adapted to function of iid sequences.

Dependence coefficients.

- We define (here $\mathcal{F}_0 = \sigma(X_i, i \le 0)$)

$$b(X_0, k) = \sup_{t \in \mathbf{R}} |\mathbb{P}(X_k \le t | X_0) - \mathbb{P}(X_k \le t)|$$

$$b(\mathcal{F}_0, i, j) = \sup_{(s,t) \in \mathbf{R}^2} |\mathbb{P}(X_i \le t, X_j \le s | \mathcal{F}_0) - \mathbb{P}(X_i \le t, X_j \le s)|$$

$\beta(\sigma(X_0), X_k) = \mathbb{E}(b(X_0, k))$ et $\beta_{2,Y}(k) = \sup_{i \ge j \ge k} \mathbb{E}(b(\mathcal{F}_0, i, j))$.

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- Dedecker (2010): If $\beta_{2,Y}(k) = O(n^{-a})$ for some $a > 1$, $\{n^{-1/2} R_X(s, n), s \in \mathbf{R}\} \Rightarrow G$ in $D(\mathbb{R})$.

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- Is it possible to obtain a strong approximation result under the condition: $\beta_{2,Y}(k) = O(n^{-a})$ for some $a > 1$?

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  2. Let $\Gamma_X(s, s', t, t') = \min(t, t')\Lambda_X(s, s')$. There exists a centered Gaussian process K_X with covariance function Γ_X, whose sample paths are almost surely uniformly continuous with respect to the pseudo metric

  $$d((s,t),(s',t')) = |F(s) - F(s')| + |t - t'| \, ,$$

  and such that for $\varepsilon = \delta^2/(22(\delta+2)^2)$,

  $$\sup_{s \in \mathbb{R}, t \in [0,1]} |R_X(s, [nt]) - K_X(s, [nt])| = O(n^{1/2-\varepsilon}) \quad \text{almost surely,}$$

Sketch of proof (1)

- Let P^* the probability on $\mathbb{R}$ whose density wrt P (law of X_0) is

$$\frac{1 + 4\sum_{k=1}^{\infty} b(x,k)}{C(\beta)} \text{ with } C(\beta) = 1 + 4\sum_{k=1}^{\infty} \beta(\sigma(X_0), X_k).$$

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- $Y_i = F_{P^*}(X_i-0)+\eta_i(F_{P^*}(X_i)-F_{P^*}(X_i-0))$ so that $F_{P^*}^{-1}(Y_i)=X_i$ almost surely

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- Then we have $R_X(\cdot,\cdot) = R_Y(F_{P^*}(\cdot),\cdot)$ and it suffices to study

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- $\mathrm{Var}\big(K_Y(u,n) - K_Y(v,n)\big) \le C(\beta) n |u - v|$

Sketch of proof (2)

- Notice that

$$\sup_{1\le k\le 2^{N+1}}\ \sup_{s\in[0,1]}\left|R(s,k)-K(s,k)\right|$$

$$\le \sup_{s\in[0,1]}\left|R(s,1)-K(s,1)\right| + \sum_{L=0}^{N} D_L\,.$$

where

$$D_L := \sup_{2^L<\ell\le 2^{L+1}}\ \sup_{s\in[0,1]}\left|(R(s,\ell)-R(s,2^L))-(K(s,\ell)-K(s,2^L))\right|.$$

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- It suffices to prove that for any $L\in\{0,\ldots,N\}$,

$$D_L = O(2^{L(\frac{1}{2}-\varepsilon)}) \text{ a.s.}$$

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- We have to take care of the quantities

$$D_{L,1} := \sup_{2^L < \ell \le 2^{L+1}} \sup_{s \in [0,1]} \left| (R(s,\ell) - R(\Pi_{r(L)}(s),\ell)) - (R(s,2^L) - R(\Pi_{r(L)}(s),2^L)) \right|$$

$$D_{L,2} := \sup_{2^L < \ell \le 2^{L+1}} \sup_{s \in [0,1]} \left| (K(s,\ell) - K(\Pi_{r(L)}(s),\ell)) - (K(s,2^L) - K(\Pi_{r(L)}(s),2^L)) \right|$$

and

$$D_{L,3} := \sup_{2^L < \ell \le 2^{L+1}} \sup_{s \in [0,1]} \Big| (R(\Pi_{r(L)}(s),\ell) - R(\Pi_{r(L)}(s),2^L)) - (K(\Pi_{r(L)}(s),\ell) - K(\Pi_{r(L)}(s),2^L)) \Big|$$

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- Let $\Lambda_{Y,L} = (\Lambda_Y(s_j, s_{j'}))_{j,j'=1,\cdots,2^{r(L)}}$ where

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- Let

$$d_{r(L)}(x, y) = \sup_{j=1,\cdots,2^{r(L)}} \left| x^{(j)} - y^{(j)} \right| .$$

Proof: construction of the Kiefer process (2).

- According to Rüschendorf (1985), there exists a random variable $V_{L,\ell} = \big(V_{L,\ell}^{(j)}\big)_{j=1,\cdots,2^{r(L)}}$ with law $\mathcal{N}(0, 2^{m(L)}\Lambda_{Y,L})$- that is measurable wrt $\sigma(\delta_{2^L+\ell 2^{m(L)}}) \vee \sigma(U_{L,\ell}) \vee \mathcal{F}_{2^L+(\ell-1)2^{m(L)}}$, independent of $\mathcal{F}_{2^L+(\ell-1)2^{m(L)}}$ and such that

$$\mathbb{E}\Big(d_{r(L)}(U_{L,\ell}, V_{L,\ell})\Big)$$

$$= \mathbb{E} \sup_{f \in \mathrm{Lip}(d_{r(L)})} \Big(\mathbb{E}\big(f(U_{L,\ell}) | \mathcal{F}_{2^L+(\ell-1)2^{m(L)}}\big) - \mathbb{E}\big(f(V_{L,\ell})\big)\Big)$$

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- We have then constructed Gaussian random variables $(V_{L,\ell})_{L\in\mathbb{N},\ell=1,\cdots,2^{L-m(L)}}$ in $2^{r(L)}$ that are mutually independent

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- According to Dudley and Philipp (1983), there exists a Kiefer process K_Y with covariance function Γ_Y such that

$$V_{L,\ell} = \Big(K_Y(s_j, 2^L+\ell 2^{m(L)}) - K_Y(s_j, 2^L+(\ell-1)2^{m(L)})\Big)_{j=1,\cdots,2^{r(L)-1}}.$$

Sketch of proof (4)

- Recall that

$$D_{L,3} := \sup_{2^L < \ell \leq 2^{L+1}} \sup_{s \in [0,1]} \Big| \left(R(\Pi_{r(L)}(s), \ell) - R(\Pi_{r(L)}(s), 2^L) \right)$$
$$- \left(K(\Pi_{r(L)}(s), \ell) - K(\Pi_{r(L)}(s), 2^L) \right) \Big|$$

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- Write that

$$A_{L,3} = \sup_{j \in \{1, \cdots, 2^{r(L)}\}} \sup_{k \le 2^{L-m(L)}} \Big| \sum_{\ell=1}^{k} (U_{L,\ell}^{(j)} - V_{L,\ell}^{(j)}) \Big| ,$$

$$B_{L,3} = \sup_{j \in \{1, \cdots, 2^{r(L)}\}} \sup_{k \le 2^{L-m(L)}} \sup_{\ell \in I_{L,k}} \Big| R(s_j, \ell) - R(s_j, 2^L + (k-1)2^{m(L)}) \Big| ,$$

$$C_{L,3} = \sup_{j \in \{1, \cdots, 2^{r(L)}\}} \sup_{k \le 2^{L-m(L)}} \sup_{\ell \in I_{L,k}} \Big| K(s_j, \ell) - K(s_j, 2^L + (k-1)2^{m(L)}) \Big| ,$$

Sketch of proof (5)

- We have that

$$\mathbb{P}\left(A_{L,3} \geq 2^{L(\frac{1}{2}-\varepsilon)}\right) \leq 2^{L-m(L)} 2^{L(\varepsilon-\frac{1}{2})} \mathbb{E}\Big(d_{r(L)}(U_{L,1}, V_{L,1})\Big)$$

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- *Proposition:* If $\beta_{2,X}(n) = O(n^{-1-\delta})$ for some $\delta > 0$ and if $4r(L) \leq m(L) \leq L$, for any $\ell \in \{1 \ldots, 2^{L-m(L)}\}$,

$$\mathbb{E}\Big(d_{r(L)}(U_{L,\ell}, V_{L,\ell})\Big) \ll 2^{\frac{m(L)+2r(L)}{(2+\delta)\wedge 3}} L^2 \,.$$

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- We will choose (for L large enough)

$$2^{2\varepsilon L-1} L^5 \leq 2^{r(L)} \leq 2^{2\varepsilon L} L^5$$

and

$$2^{L(1-2\varepsilon)} L^{-5} \leq 2^{m(L)} \leq 2^{1+L(1-2\varepsilon)} L^{-5}$$

On the proof of the Gaussian approximation

- Let $(N_{i,L})_{i\in\mathbb{Z}}$ be a sequence of iid $\sim \mathcal{N}(0, \Lambda_L)$. Assume that $(N_{i,L})_{i\in\mathbb{Z}}$ is independent of $\mathcal{F}_\infty \vee \sigma(\eta_i, i \in \mathbb{Z})$.

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- Then we have

$$\mathbb{E}\big(d_{r(L)}(U_{L,1}, V_{L,1})\big) = \sup_{g\in\mathrm{Lip}(d_{r(L)},\mathcal{F}_{2^L})} \mathbb{E}(g(U_{L,1}, \omega)) - \mathbb{E}(g(\widetilde{N}_L, \omega))\,.$$

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- There exists a Markov chain such that $\beta_{2,X}(k) > ck^{-1}$ for some positive constant c such that the finite dimensional marginals of the process $\{(n \ln n)^{-1/2} R_T(\cdot, n)\}$ converge in distribution to those of the degenerated Gaussian process G defined by

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- This shows that an approximation by a Kiefer process as in our main result cannot hold for this chain.

With a stronger coefficient!

- Let $\mathbf{X}_k = (X_j, j \geq k)$ and

$$\beta(k) = \| \sup_{\|f\|_\infty \leq 1} P_{\mathbf{X}_k|\mathcal{F}_0}(f) - P_{\mathbf{X}_k}(f) \|_1$$

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- If $\beta(n) = O(n^{-1-\delta})$ for some $\delta > 0$, the rate in the strong approximation result should be

$$n^{\frac{1}{(2+\delta)\wedge 3}} (\log n)^{7(d+1)/3}$$

if the variables are in $\mathbb{R}^d$.

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