Slepian's inequality

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Let $X = (X_1, \ldots, X_k)$ be a random vector such that $X_1, \ldots, X_k \in L^2(P)$. We let $\bar{X}_i := X_i - EX_i$ denote the centered random variables and we let

$$\sigma_{ij}^X := E(\bar{X}_i\, \bar{X}_j) \text{ and } \pi_{ij}^X := E(\bar{X}_i - \bar{X}_j)^2$$

denote *the covariances* and *the squared intrinsic metrics* of X . Note that $\pi_{ii}^X = 0$ and $\pi_{ij}^X = \sigma_{ii}^X + \sigma_{jj}^X - 2\,\sigma_{ij}^X$.

Let $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$ be Gaussian vectors with zero means and set $\theta_{ij} := \sigma_{ij}^Y - \sigma_{ij}^X$ and $\gamma_{ij} = \pi_{ij}^X - \pi_{ij}^Y$. Let $f : \mathbf{R}^k \to \mathbf{R}$ be a given function satisfying a certain set of "regularity conditions". Then Slepian's inequality states:

$$(1) \qquad \sum_{i=1}^k \sum_{j=1}^k \theta_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \;\Rightarrow\; Ef(X) \leq Ef(Y)$$

and we have an important variant of (1); due to X. Fernique (1974), stating:

$$(2) \qquad \sum_{i=1}^k \sum_{j=1}^k \gamma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \;\Rightarrow\; Ef(X) \leq Ef(Y)$$

again under a certain set of "regularity conditions" which are a bit different from the ones implying (1).

Since $\gamma_{ij} = 2\theta_{ij} - \theta_{ii} - \theta_{jj}$, we have

$$\sum_{i=1}^k \sum_{j=1}^k \gamma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = 2\sum_{i=1}^k \sum_{j=1}^k \theta_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - 2\sum_{i=1}^k \theta_{ii} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^k \frac{\partial f}{\partial x_j}\right)(x)$$

Hence, if $\sum_{j=1}^k \frac{\partial f(x)}{\partial x_j}$ is constant, then (1) implies (2); for instance, if $f(x + te) = at + f(x)$ for some $a \in \mathbf{R}$ where $e = (1, 1, \ldots, 1)$

X. Fernique proved (2) when $\gamma_{ij} \geq 0$ and $f(x) = \phi(Q(x))$ where $Q(x) = \max_{1 \leq i,j \leq n} |x_i - x_j|$ and $\phi : \mathbf{R}_+ \to \mathbf{R}$ is convex and increasing. Note that $Q(x + te) = Q(x)$.

Slepian's inequality (the smooth case)

Let $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$ be Gaussian vectors with zero means and set $\theta_{ij} := \sigma_{ij}^Y - \sigma_{ij}^X$ and $\gamma_{ij} = \pi_{ij}^X - \pi_{ij}^Y$. Let $f : \mathbf{R}^k \to \mathbf{R}$ be a differentiable function such that $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}$ are Fréchet differentiable and let $\kappa : \mathbf{R}^k \to [0, \infty]$ be a Borel function and set

$$\kappa^\diamond(x, y) := \sup_{(s,t)\in S_+^2} (1 + \|sx - ty\|) \cdot \kappa(sx + ty) \ \ \forall\, x, y \in \mathbf{R}^k$$

$$\|h\|_\kappa := \inf\{\, c \geq 0 \mid \ |f(x)| \leq c\, \kappa(x) \ \ \forall\, x \in \mathbf{R}^k\} \ \ \forall\, h : \mathbf{R}^k \to \mathbf{R}$$

where $S_+^2 = \{(s, t) \in \mathbf{R}^2 \mid s \geq 0 \ , \ t \geq 0 \ , \ s^2 + t^2 = 1\}$. Suppose that

(a) $\displaystyle\int_{\mathbf{R}^k} P_X(dx) \int_{\mathbf{R}^k} \kappa^\diamond(x, y)\, P_Y(dy) \, < \, \infty$

(b) $\| \frac{\partial f}{\partial x_i} \|_\kappa < \infty$ and $\| \sum_{j=1}^k \theta_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \|_\kappa < \infty \ \ \forall\, i = 1, \ldots, k$

Then we have

(1) $\displaystyle\sum_{i=1}^k \sum_{j=1}^k \theta_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \ \Rightarrow \ Ef(X) \leq Ef(Y)$

and if there exists $a \in \mathbf{R}$ such that

(c) $f(x + t\, e) = at + f(x) \ \ \forall\, t \in \mathbf{R} \ \forall\, x \in \mathbf{R}^k$

where $e = (1, 1, \ldots, 1)$, then we have

(2) $\displaystyle\sum_{i=1}^k \sum_{j=1}^k \gamma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \ \Rightarrow \ Ef(X) \leq Ef(Y)$

Remark: Let $Q : \mathbf{R}^k \to [0, \infty)$ be any given seminorm on $\mathbf{R}^k$ and set $\kappa(x) := e^{Q(x)^2}$. Then we have

$$\kappa^\diamond(x, y) \leq \left(1 + \sqrt{\|x\|^2 + \|y\|^2}\right) \cdot e^{Q(x)^2 + Q(y)^2}$$

$$\leq (1 + \|x\|)\, e^{Q(x)^2} \cdot (1 + \|y\|)\, e^{Q(y)^2}$$

Let λ_X and λ_Y be the largest eigen value of Σ_X and Σ_Y , respectively, and set $\lambda := \max(\lambda_X, \lambda_Y)$. Then we have

$\kappa(x) := e^{\alpha\, ||x||^2}$ satisfies (a) for all $0 \leq \alpha < \frac{1}{2\lambda}$

Schwartz distributions

Slepian's inequality is often used to prove inequalities of the form

$$P(X_1 \le t_1 \,, \ldots, \, X_k \le t_k) \le P(Y_1 \le t_1 \,, \ldots, \, Y_k \le t_k)$$
$$\text{or } \; P(X_1 \ge t_1 \,, \ldots, \, X_k \ge t_k) \le P(Y_1 \ge t_1 \,, \ldots, \, Y_k \ge t_k)$$

which means that the function f is an indicator function of some set $A \subseteq \mathbf{R}^k$; for instance $A = \{(x_1, \ldots, x_k) \mid x_i \le t_i \; \forall \, i = 1, \ldots, k\}$.

Let $D(\mathbf{R}^k)$ denote the set of all infinitely often continuously differentiable functions $f : \mathbf{R}^k \to \mathbf{R}$ with compact support with its usual inductive limit topology and let $D^*(\mathbf{R}^k)$ denote *the Schwartz distributions*; i.e. the set of continuous linear functionals $\xi : D(\mathbf{R}^k) \to \mathbf{R}$. If $\xi \in D^*(\mathbf{R}^k)$, we write $\xi \ge 0$ if $\xi(\phi) \ge 0$ for all non-negative functions $\phi \in D(\mathbf{R}^k)$.

If $f : \mathbf{R}^k \to \mathbf{R}$ be locally λ_k-integrable, then $f(\phi) := \int_{\mathbf{R}^k} f(y)\phi(y)\, dy$ and

$$\partial_{i_1,\ldots,i_n} f(\phi) = (-1)^n \int_{\mathbf{R}^k} f(x) \, \frac{\partial^n \phi}{\partial x_{i_1} \cdots \partial x_{i_n}}(x) \, dx \;\; \forall \, \phi \in D(\mathbf{R}^k)$$

are Schwartz' distributions associated to f and its "partial derivative" $\frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}}$.

Following Kahane, Ledoux and Talagrand, we shall interpret the condition: $\sum_i \sum_j a_{ij} \frac{\partial^2 f}{\partial x_i \, \partial x_j} \ge 0$ in distribution sense; i.e. as $\sum_i \sum_j a_{ij} \, \partial_{ij} f \ge 0$.

If f is twice differentiable and f , $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \, \partial x_j}$ are locally Lebesgue integrable for all $1 \le i, j \le k$, we have

$$\sum_{i=1}^k \sum_{j=1}^k a_{ij} \, \partial_{ij} f \ge 0 \;\Leftrightarrow\; \sum_{i=1}^k \sum_{j=1}^k a_{ij} \, \frac{\partial^2 f}{\partial x_i \, \partial x_j}(x) \ge 0 \;\; \lambda_k\text{-a.e.}$$

Let $\Delta_i^u f(x) = f(x + u e_i) - f(x)$ denote the difference operator for $x \in \mathbf{R}^k$, $u \in \mathbf{R}$ and $i = 1, \ldots, k$ where $e_1, \ldots, e_k$ are the standard unit vectors. If $\epsilon_1, \epsilon_2, \ldots > 0$ and $\delta_1, \delta_2, \ldots > 0$ are positive sequences satisfying $\epsilon_n \to 0$ and $\delta_n \to 0$ and $f : \mathbf{R}^k \to \mathbf{R}$ is locally λ_k-integrable, we have

$$\sum_{i=1}^k \sum_{j=1}^k a_{ij} \, \partial_{ij} f \ge 0 \;\Leftrightarrow\; \sum_{i=1}^k \sum_{j=1}^k a_{ij} \, \Delta_i^{\epsilon_n} \, \Delta_j^{\delta_n} f(x) \ge 0 \;\; \lambda_k\text{-a.e.} \;\; \forall \, n \ge 1$$

An example

Let $f(x) = -1_\Delta(x)$ where $\Delta = \{(x_1, \dots, x_k) \mid x_1 = \cdots = x_k\}$ and $k \geq 2$. Then we have $\partial_{ij} f = 0$ for all $1 \leq i, j \leq k$. Let $X_1, \dots, X_k$ be independent $N(0,1)$-variables and set $X = (X_1, \dots, X_k)$ and $Y = (X_1, \dots X_1)$. Then we have

$$\theta_{ij} = \sigma_{ij}^Y - \sigma_{ij}^X = 1 - \delta_{ij} \geq 0 \text{ and } \theta_{ii} = 0$$

$$Ef(X) = 0 \text{ and } Ef(Y) = -1$$

Showing that Theorem 3.11 p.74 in Ledoux and Talagrand, *Probability in Banach Spaces*, is false. However, their corollaries 3.12–3.14 are correct but with a different proof.

Approximate directional continuity

Recall that $K \subseteq \mathbf{R}^k$ is *starshaped* if $\alpha x \in K$ for all $x \in K$ and all $0 \le \alpha \le 1$. Let $K \subseteq \mathbf{R}^k$ be a bounded, starshaped Borel set with non-empty interior and let $f : \mathbf{R}^k \to \mathbf{R}$ be a function. If $x \in \mathbf{R}^k$, we say that f is *continuous at* x *along* K if

(*)
$$\lim_{n\to\infty} \left\{ \sup_{y\in K} |f(x + \tfrac{y}{n}) - f(x)| \right\} = 0$$

We let $C^K(f)$ denote the set of all $x \in \mathbf{R}^k$ satisfying (*). We say that f is *approximately continuous at* x *along* K if f is locally Lebesgue integrable and

(**)
$$\lim_{n\to\infty} \int_K |f(x + \tfrac{y}{n}) - f(x)|\, dy = 0$$

We let $C^K_{ap}(f)$ denote the set of all $x \in \mathbf{R}^k$ satisfying (**).

We say that f is *right continuous at* x if f is continuous at x along the unit cube $[0,1]^k$, and we say that f is *left continuous at* x if f is continuous at x along the negative unit cube $[-1,0]^k$.

Fact: If f is locally Lebesgue integrable, we have $C^K(f) \subseteq C^K_{ap}(f)$ and $\lambda_k(\mathbf{R}^k \setminus C^K_{ap}(f)) = 0$.

A lemma

Let $(U_0, U_1, \ldots, U_k)$ be a $(k+1)$-dimensional Gaussian vector with mean zero. Set $U = (U_1, \ldots, U_k)$ and $\theta = (\theta_1, \ldots, \theta_k)$ where $\theta_i = \mathrm{cov}(U_0, U_i) = E(U_0 U_i)$. Let $h : \mathbf{R}^k \to \mathbf{R}$ be a Borel function such that directional derivative $\frac{\partial h}{\partial \theta}(x) = \lim_{u \to 0} \frac{1}{u}(h(x + u\theta) - h(x))$ exists for all $x \in \mathbf{R}^k$ and

$$E|U_0\, h(U)| < \infty \text{ and } E|\tfrac{\partial h}{\partial \theta}(U)| < \infty$$

Then we have

(1) $\quad E\{U_0\, h(U)\} = E\{\frac{\partial h}{\partial \theta}(U)\}$

Proof: If $\theta = 0$, then $\frac{\partial h}{\partial \theta}(x) = 0$ and U_0 and U are independent and since $EU_0 = 0$, we see that (1) holds trivially. In general, we set $V_0 = \sigma^{-2}\, U_0$ and $V_i = U_i - \theta_i\, V_0$. Then $(V_0, V_1, \ldots, V_k)$ is Gaussian with mean zero and V_0 and $W := (V_1, \ldots, V_k)$ are independent. By integration by parts, we have

$$\int_{-\infty}^{\infty} h(z + t\theta)\, \sigma^2\, t\, e^{-(\sigma t)^2/2}\, dt = \int_{-\infty}^{\infty} \frac{\partial h}{\partial \theta}(z + t\theta)\, e^{-(\sigma t)^2/2}\, dt$$

for all $z \in \mathbf{R}^k$ for which the integrals exist and since V_0 has density $\frac{\sigma}{\sqrt{2\pi}}\, e^{-(\sigma t)^2/2}$, we obtain (1) by integrating with respect to P_W

Slepian's inequality (the general case)

Let $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$ be Gaussian vectors with zero means and set $\theta_{ij} := \sigma_{ij}^Y - \sigma_{ij}^X$ and $\gamma_{ij} = \pi_{ij}^X - \pi_{ij}^Y$. Let $f : \mathbf{R}^k \to \mathbf{R}$ and $\kappa : \mathbf{R}^k \to [0, \infty]$ be a Borel functions and let $K \subseteq \mathbf{R}^k$ be a bounded, starshaped Borel set with non-empty interior satisfying

(a) $\displaystyle\int_{\mathbf{R}^k} P_X(dx) \int_{\mathbf{R}^k} \kappa^\diamond(x,y)\, P_Y(dy) < \infty$

(b) $\|F\|_\kappa < \infty$ where $F(x) = \sup_{y \in K} |f(x+y)|$

(c) $P(X \in C_{ap}^K(f)) = 1 = P(Y \in C_{ap}^K(f))$

Then we have

(1) $\displaystyle\sum_{i=1}^k \sum_{j=1}^k \theta_{ij}\, \partial_{ij} f(x) \geq 0 \;\Rightarrow\; Ef(X) \leq Ef(Y)$

and if there exists $a \in \mathbf{R}$ such that

(d) $f(x + t\,e) = at + f(x) \;\; \forall\, t \in \mathbf{R} \;\forall\, x \in \mathbf{R}^k$

where $e = (1, 1, \ldots, 1)$, then we have

(2) $\displaystyle\sum_{i=1}^k \sum_{j=1}^k \gamma_{ij}\, \partial_{ij} f(x) \geq 0 \;\Rightarrow\; Ef(X) \leq Ef(Y)$

Remark: Let $\mathcal{R}_X$ and $\mathcal{R}_Y$ denote the ranges of Σ_X and Σ_Y . Then $\mathcal{R}_X$ and $\mathcal{R}_Y$ are linear subspaces of $\mathbf{R}^k$ and we let $\lambda_{\mathcal{R}_X}$ and $\lambda_{\mathcal{R}_Y}$ denote the Lebesgue measures on $\mathcal{R}_X$ and $\mathcal{R}_Y$, respectively. Then (c) is equivalent to

(c*) $\lambda_{\mathcal{R}_X}(\mathcal{R}_X \setminus C_{ap}^K(f)) = 0 = \lambda_{\mathcal{R}_Y}(\mathcal{R}_Y \setminus C_{ap}^K(f))$

Since $\mathbf{R}^k \setminus C_{ap}^K(f)$ is a λ_k-null set, we see that (c) holds if Σ_X and Σ_Y are non-singular.

Integral orderings

Let $(S,\mathcal{B})$ be a measurable space and let $\Phi \subseteq \mathbf{R}^S$. If X and Y are S-valued random functions, it is custom to define *the* Φ*-integral ordering* as follows:

$$X \leq_{\Phi} Y \;\Leftrightarrow\; E\phi(X) \leq E\phi(Y) \;\; \forall \phi \in \Phi \text{ so that the expectations exists}$$

There is deficiency with this ordering: It is NOT a preordering.

Example: Let $k = 1$ and let Φ be the set of all increasing convex functions on $\mathbf{R}$. If X is a random variable with $EX^+ = \infty$, we have $X \leq_\Phi Y$ and $Y \leq_\Phi X$ for every random variable Y .

The deficiency can be repaired by the usual modification:

$$X \preceq_{\Phi} Y \;\Leftrightarrow\; E^*\phi(X) \leq E^*\phi(Y) \;\; \forall \phi \in \Phi$$

Then $X \preceq_\Phi Y$ implies $X \leq_\Phi Y$ and the converse implication holds if $\phi(X) \in L^1(P)$ and $\phi(Y) \in L^1(P)$ for all $\phi \in \Phi$. Passing to the distributions measures $P_X(B) = P(X \in B)$, leads us to the following:

Let $\mathrm{Pr}(S,\mathcal{B})$ denote the set of all probability measures on $(S,\mathcal{B})$. Let Φ be a set of real-valued functions on S . Then we introduce *the* Φ*-integral ordering* on $\mathrm{Pr}(S,\mathcal{B})$ as above:

$$\mu \preceq_\Phi \nu \text{ if and only if } \int^* \phi \, d\mu \leq \int^* \phi \, d\nu \text{ for all } \phi \in \Phi$$

and if $\mathcal{Q} \subseteq \mathrm{Pr}(S,\mathcal{B})$, we define *the maximal generator* as follows

$$D^\Phi(\mathcal{Q}) := \{ f \in \mathbf{R}^S \mid \int^* f \, d\mu \leq \int^* f \, d\nu \;\; \forall \mu,\nu \in \mathcal{Q} \text{ so that } \mu \preceq_{\Phi} \nu \}$$

Supermodularity

$f : \mathbf{R}^k \to \mathbf{R}$ is *supermodular* if $\Delta_i^s \Delta_j^t f(x) \geq 0$ for all $x \in \mathbf{R}^k$, all $s, t > 0$ and all $1 \leq i \neq j \leq k$, or equivalently if

$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y) \;\; \forall\, x, y \in \mathbf{R}^k$$

where $\wedge$ and $\vee$ are the usual lattice operation on $\mathbf{R}^k$.

We say that f is *submodular* if $-f$ is supermodular, and f is *modular* if f is supermodular and submodular.

Let $\xi_1, \ldots, \xi_k : \mathbf{R} \to \mathbf{R}$ be either all increasing or all decreasing. Let $J \subseteq \mathbf{R}$ be an interval and let $\varphi : J \to \mathbf{R}$ be an increasing convex function. Then we have

(1) f is modular if and only if there exist functions $f_1, \ldots, f_k : \mathbf{R} \to \mathbf{R}$ such that $f(x_1, \ldots, x_k) = f_1(x_1) + \cdots + f_k(x_k)$

(2) If f is increasing and supermodular, then f is Borel measurable and if $f(\mathbf{R}^k) \subseteq J$, then $\varphi(f(x))$ is supermodular

(3) If $\mathbf{R}_+ \subseteq J$, then $\varphi(\max_{1 \leq i,j \leq k} |x_i - x_j|)$ is submodular

(4) If f is supermodular, then $f(\xi_1(x_1), \ldots, \xi_k(x_k))$ is supermodular

(5) $\max(\xi_1(x_1), \ldots, \xi_k(x_k))$ is submodular and if $\psi : \mathbf{R} \to \mathbf{R}$ is decreasing, then $\psi(\max(x_1, \ldots, x_k))$ is supermodular

(6) $\min(\xi_1(x_1), \ldots, \xi_k(x_k))$ is supermodular

(7) If $\xi_1, \ldots, \xi_k$ are non-negative, then $\prod_{i=1}^k \xi_i(x_i)$ is supermodular

(8) If f is bounded and supermodular, then there exists a bounded modular function f_0 and a bounded, increasing, supermodular function f_1 such that $f(x) = f_0(x) + f_1(x)$ for all $x \in \mathbf{R}^k$

Let $I_1, \ldots, I_k \subseteq \mathbf{R}$ be intervals with left endpoint $-\infty$ and let $J_1, \ldots, J_k \subseteq \mathbf{R}$ be intervals with right endpoint $+\infty$. Set $A = I_1 \times \cdots \times I_k$ and $B = J_1 \times \cdots \times J_k$. Then we have

(8) 1_A, 1_B and $1_{A \cup B}$ are supermodular

The supermodular ordering

We let $\preceq_{\rm sm}$ denote the integral ordering induced by the set of all supermodular Borel functions.

We let $\preceq_{\rm bsm}$ denote the integral ordering induced by the set of all bounded supermodular Borel functions.

We let $\preceq_{\rm ism}$ denote the integral ordering induced by the set of all increasing supermodular functions.

We let $\preceq_{\rm m}$ denote the integral ordering induced by the set of all modular Borel functions.

We let $\preceq_{\rm bm}$ denote the integral ordering induced by the set of all bounded modular Borel functions.

Let $\mathrm{Sm}(\mathbf{R}^k)$ denote the set of all supermodular functions on $\mathbf{R}^k$ and let $C_b^\infty(\mathbf{R}^k)$ denote the set of all bounded, infinitely often continuously differentiable function on $\mathbf{R}^k$ with bounded partial of all orders. Then we have

(1) $X \preceq_{\rm bm} Y \;\Leftrightarrow\; X_i \sim Y_i \;\; \forall\, i = 1, \ldots, k$

(2) $X \preceq_{\rm bsm} Y \;\Leftrightarrow\; X \preceq_{\rm ism} Y$ and $X_i \sim Y_i \;\; \forall\, i = 1, \ldots, k$

(3) $Ef(X) \leq Ef(Y) \;\; \forall\, f \in \mathrm{Sm}(\mathbf{R}^k) \cap C_b^\infty(\mathbf{R}^k) \;\Rightarrow\; X \preceq_{\rm bsm} Y$

and if $k = 1, 2$, we have

(4) $X \preceq_{\rm m} Y \;\Leftrightarrow\; X \preceq_{\rm bm} Y$

But (4) fails if $k \geq 3$.

Consider the setting of Slepian's inequality and suppose that $\theta_{ii} = 0$ for all $i = 1, \ldots, k$ and $\theta_{ij} \geq 0$ for all $1 \leq i, j \leq k$. By (3) and Slepian's inequality, we see that $X \preceq_{\rm bsm} Y$

In the modern literature is often claimed that have

$$U \preceq_{\rm bsm} V \;\Leftrightarrow\; U \preceq_{\rm sm} V$$

and as a consequence, that Slepian's inequality implies $X \preceq_{sm} Y$. The first claim fails for $k \geq 3$ and I don't know if the second claim is true but in view of the next example, conjecture that it fails for $k \geq 3$ ..

A strange example

Let $U, X_1, \ldots, X_k$ be random variables such that $X_i \sim U$ for all $i = 1, \ldots, k$. A.H. Chen (1980) showed that

(1) $Ef(X_1, \ldots, X_k) \leq Ef(U, \ldots, U)$

for all supermodular functions $f : \mathbf{R}^k \to \mathbf{R}$ satisfying a certain set of regularity conditions. In the modern literature it is often claimed that these regularity conditions are not needed. The following example (due to G. Simons (1977) who used it another context) shows that we DO need some regularity conditions:

Let U be a strictly positive random variable having with density*:*

(*) $f(x) = \frac{2}{\pi(1+x^2)}$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$

(*The one-sided Cauchy distribution)*. Since U is strictly positive, we may define

$$V = (U - \tfrac{1}{U}) \cdot (1_{\{U>1\}} - 1_{\{U\leq 1\}})$$

A straight forward computation shows that U , $\frac{1}{U}$ and $\frac{1}{2}V$ have the same density given by (*). Set $f(x,y,z) = x + y - 2z$ for $(x,y,z) \in \mathbf{R}^3$. Then we have

$$f(U,U,U) = 0 \quad , \quad f(U, \tfrac{1}{U}, \tfrac{1}{2}V) = 2U\,1_{\{U\leq 1\}} + \tfrac{2}{U}\,1_{\{U>1\}}$$

$$0 < f(U, \tfrac{1}{U}, \tfrac{1}{2}V) \leq 2 \quad , \quad Ef(U,U,U) = 0 < Ef(U, \tfrac{1}{U}, \tfrac{1}{2}V) = \tfrac{2\,\log 2}{\pi}$$

which means that (1) fails.