A MORE GENERAL MAXIMAL BERNSTEIN – TYPE INEQUALITY

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BERNSTEIN INEQUALITY

Let $X_1, X_2, \ldots,$ be a sequence of independent random variables such that for all $i \geq 1$, $EX_i = 0$ and for some $\kappa > 0$ and $v > 0$ for integers $m \geq 2$, $E\left|X_i\right|^m \leq vm!\kappa^{m-2}/2$.

The classic Bernstein inequality (cf. p. 855 of Shorack and Wellner (1986) says that in this situation for all $n \geq 1$ and $t \geq 0$

$$\mathbf{P}\left\{\left|\sum_{i=1}^{n} X_i\right| > t\right\} \leq 2\exp\left\{-\frac{t^2}{2vn + 2\kappa t}\right\}.$$

MAXIMAL VERSION

Moreover, (cf. Theorémè B.2 in Rio (2000) its maximal form also holds, i.e. we have

$$\mathbf{P}\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > t\right\} \leq 2\exp\left\{-\frac{t^2}{2vn+2\kappa t}\right\}.$$

GENERAL BERNSTEIN INEQUALITY

It turns out that, under a variety of assumptions, a sequence of not necessarily independent random variables $X_1, X_2, \ldots$, will satisfy a generalized Bernstein-type inequality of the following form: for suitable constants $A > 0$, $a > 0$, $b \geq 0$ and $0 < \gamma < 2$ for all $i \geq 0$, $n \geq 1$ and $t \geq 0$,

$$\mathbf{P}\{|S(i+1, i+n)| > t\} \leq A \exp\left\{-\frac{at^2}{n + bt^\gamma}\right\}, \tag{GB}$$

where for any choice of $1 \leq i \leq j < \infty$ we denote the partial sum $S(i, j) = \sum_{k=i}^{j} X_k$. Here are some examples.

BERNSTEIN EXAMPLE 1

Let $X_1, X_2, \ldots,$ be a stationary sequence satisfying

$$EX_1 = 0 \text{ and } VarX_1 = 1.$$

For each integer $n \geq 1$ set

$$S_n = X_1 + \cdots + X_n$$

and $B_n^2 = Var\,(S_n)$.

Assume that for some $\sigma_0^2 > 0$

we have $B_n^2 \geq \sigma_0^2 n$ for all $n \geq 1$.

Statulevičius and Jakimavičius (1988) prove that the partial sums satisfy GB with constants depending on the particular mixing and bounding condition that the sequence may fulfill.

BENTKUS AND RUDZKIS

Their Bernstein-type inequalities are derived via the following result of Bentkus and Rudzkis (1980) relating cumulant behavior to tail behavior:

For an arbitrary random variable ξ with expectation 0, whenever there exist $\gamma \geq 0$, $H > 0$ and $\Delta > 0$ such that its cumulants $\Gamma_k(\xi)$ satisfy $|\Gamma_k(\xi)| \leq (k!/2)^{1+\gamma} H/\Delta^{k-2}$ for $k = 2, 3, \ldots$, then for all $x \geq 0$

$$\mathbf{P}\{\pm\xi > x\}$$

$$\leq \exp\left\{-\frac{x^2}{2\left(H + \left(x/\Delta^{1/(1+2\gamma)}\right)^{(1+2\gamma)/(1+\gamma)}\right)}\right\}.$$

BERNSTEIN EXAMPLE 2

Doukhan and Neumann (2007) have shown using the result in Bentkus and Rudzkis (1980) cited in the previous example that if a sequence of mean zero random variables $X_1, X_2, \ldots,$ satisfies a general covariance condition then the partial sums satisfy GB.

Refer to their Theorem 1 and Remark 2, and also see Kallabis and Neumann (2006).

BERNSTEIN EXAMPLE 3

Assume that $X_1, X_2, \ldots,$ is a strong mixing sequence with mixing coefficients $\alpha(n)$, $n \geq 1$, satisfying for some $d > 0$, $\alpha(n) \leq \exp(-2dn)$. Also assume that $EX_i = 0$ for some $M > 0$ $|X_i| \leq M$, for all $i \geq 1$. Theorem 2 of Merlevéde, Peligrad and Rio (2009) implies that for some constant $D > 0$ for all $t \geq 0$ and $n \geq 1$,

$$\mathbf{P}\{|S_n| \geq t\} \leq \exp\left(-\frac{Dt^2}{nv^2 + M^2 + tM(\log n)^2}\right),$$

where $S_n = \sum_{i=1}^{n} X_i$ and

$$v^2 = \sup_{i>0}\left(Var(X_i) + 2\sum_{j>i}\left|cov(X_i, X_j)\right|\right).$$

EXPLANATION

To see how this last example satisfies GB, notice that for any $0 < \eta < 1$ there exists a $D_1 > 0$ such that for all $t \geq 0$ and $n \geq 1$,

$$nv^2+M^2+tM\left(\log n\right)^2 \leq n\left(v^2+M^2\right)+D_2t^{1+\eta}.$$

Thus GB holds with $\gamma = 1 + \eta$ for suitable $A > 0$, $a > 0$ and $b \geq 0$.

GENERAL MAXIMAL BERNSTEIN INEQUALITY

For any choice of $1 \leq i \leq j < \infty$ define

$$M(i,j) = \max\{|S(i,i)|, \ldots, |S(i,j)|\}.$$

Somewhat unexpectedly, if a sequence of random variables $X_1, X_2, \ldots,$ satisfies a Bernstein-type inequality of the form GB, then without any additional assumptions a modified version of it also holds for

$$M(m+1, m+n) = \max_{1 \leq i \leq n} |S(1+m, i+m)|.$$

GMB Inequality *Assume that for constants* $A > 0$, $a > 0$, $b \geq 0$ *and* $\gamma \in (0, 2)$, *inequality* GB *holds for all* $i \geq 0, n \geq 1$ *and* $t \geq 0$. *Then for every* $0 < c < a$ *there exists a* $C > 0$ *depending only on* A, a, b *and* γ *such that for all* $m \geq 0$, $n \geq 1$ *and* $t \geq 0$,

$$\mathbf{P}\{M(m+1, m+n) > t\} \leq C \exp\left\{-\frac{ct^2}{n + bt^\gamma}\right\}.$$

REMARK

Clearly $c < a$ can be chosen arbitrarily close to a.

The case $b = 0$ is a special case of Thereom 1 of Moricz (1979).

This result has appeared in Kevei and M (2011).

MOTIVATION

The GMB inequality was partially motivated by Theorem 2.2 of Móricz, Serfling and Stout (1982), who showed that whenever for a suitable positive function $g(i,j)$ of $(i,j) \in \{1,2,\dots\} \times \{1,2,\dots\}$, positive function $\phi(t)$ defined on $(0,\infty)$ and constant $K > 0$, for all $1 \leq i \leq j < \infty$ and $t > 0$,

$$\mathbf{P}\{|S(i,j)| > t\} \leq K \exp\{-\phi(t)/g(i,j)\},$$

then there exist constants $c > 0$ and $C > 0$ such that for all $m \geq 0$, $n \geq 1$ and $t > 0$,

$$\mathbf{P}\{M(m+1, m+n) > t\} \\ \leq C \exp\{-c\phi(t)/g(1,n)\}.$$

This inequality is clearly not applicable to obtain a maximal form of the generalized Bernstein inequality.

APPLICATIONS OF GMB INEQUALITY

An obvious application of the GMB inequality is the following bounded law of the iterated logarithm.

Bounded LIL *Under the assumptions of the previous theorem, with probability* 1,

$$\limsup_{n\to\infty} \frac{|S(1,n)|}{\sqrt{n \log\log n}} \leq \frac{1}{\sqrt{a}}.$$

OBSERVATION

In general one cannot replace "$\leq$ " by "$=$" our bounded LIL. To see this, let $Y, Z_1, Z_2, \ldots$ be a sequence of independent random variables such that Y takes on the value 0 or 1 with probability 1/2 and $Z_1, Z_2, \ldots$ are independent standard normals. Now define $X_i = YZ_i$, $i = 1, 2, \ldots$ It is easily checked that assumptions of the GMB inequality are satisfied with $A = 2$, $a = 1/2$, $b = 0$ and $\gamma = 1$.

When $Y = 1$ the usual law of the iterated logarithm gives with probability 1,

$$\limsup_{n\to\infty} |S(1,n)|/\sqrt{n \log\log n} = \sqrt{2} = 1/\sqrt{a}$$

whereas, when $Y = 0$ the above limsup is 0. This agrees with the bounded LIL, which says that with probability 1 the limsup is $\leq \sqrt{2}$.

However, we see that with probability 1/2 it equals $\sqrt{2}$ and with probability 1/2 it equals 0.

A MORE GENERAL MAXIMAL BERNSTEIN INEQUALITY

THEOREM Assume that there exist constants $A > 0$ and $a > 0$ and a sequence of non-decreasing non-negative functions $\{g_n\}_{n\geq 1}$ on $(0,\infty)$, such that for all $t > 0$ and $n \geq 1$, $g_n(t) \leq g_{n+1}(t)$ and for all $0 < \gamma < 1$

$$\lim_{n\to\infty} \inf\left\{\frac{t^2}{g_n(t)\log t} : g_n(t) > \gamma n\right\} = \infty,$$

where the infinum of the empty set is defined to be infinity, such that for all $m \geq 0$, $n \geq 1$ and $t \geq 0$,

$$P\{|S(m+1, m+n)| > t\} \leq A \exp\left\{-\frac{at^2}{n + g_n(t)}\right\}.$$

Then for every $0 < c < a$ there exists a $C > 0$ depending only on A, a and $\{g_n\}_{n\geq 1}$ such that for all $n \geq 1$, $m \geq 0$ and $t \geq 0$,

$$P\{M(m+1, m+n) > t\} \leq C \exp\left\{-\frac{ct^2}{n + g_n(t)}\right\}.$$

Note that the more general maximal Bernstein inequality implies the previous one by choosing

$$g_n(t) = bt^\gamma.$$

EXAMPLE 1 Assume that $X_1, X_2, \ldots,$ is a stationary Markov chain satisfying the conditions of Theorem 6 of Adamczak (2008) and let f be any bounded function f such that $Ef(X_1) = 0$.

This theorem implies that for suitable positive constants D, d_1, d_2 for all $t \geq 0$ and $n \geq 1$,

$$P(\{|S_n(f)| \geq t\} \leq D^{-1} \exp\left(-\frac{Dt^2}{nd_1 + td_2 \log n}\right),$$

where $S_n(f) = \sum_{i=1}^{n} f(X_i)$.

In this example one can verify that the assumptions of the theorem hold with

$A = D^{-1}$, $a = D/d_1$ and

$$g_n(t) = \left(\frac{td_2}{d_1}\right) \log n.$$

EXAMPLE 2 Assume that $X_1, X_2, \ldots,$ is a strong mixing sequence with mixing coefficients $\alpha(n)$, $n \geq 1$, satisfying for some $d > 0$, $\alpha(n) \leq \exp(-2dn)$. Also assume that $EX_i = 0$ for some $M > 0$ $|X_i| \leq M$, for all $i \geq 1$. Theorem 2 of Merlevéde, Peligrad and Rio (2009) implies that for some constant $D > 0$ for all $t \geq 0$ and $n \geq 1$,

$$\begin{aligned} &P\{|S_n| \geq t\} \\ &\leq \exp\left(-\frac{Dt^2}{nv^2 + M^2 + tM(\log n)^2}\right), \end{aligned}$$

where $S_n = \sum_{i=1}^n X_i$ and

$$v^2 = \sup_{i>0}\left(Var(X_i) + 2\sum_{j>i}\left|cov\left(X_i, X_j\right)\right|\right).$$

In this example the assumptions of the theorem hold with $A = 1$, $a = D/v^2$ and

$$g_n(t) = \frac{M^2}{v^2} + \left(\frac{tM}{v^2}\right)(\log n)^2,$$

which leads to the inequality valid for all $n \geq 1$ and $t > 0$

$$P\left\{\max_{1 \leq m \leq n} |S_m| \geq t\right\}$$

$$\leq C \exp\left(-\frac{cDt^2}{nv^2 + M^2 + tM(\log n)^2}\right)$$

for some constants $C \geq 1$ and $0 < c < 1$.

MOTIVATION OF MORE GMB

See Corollary 24 of Merlevéde and Peligrad (in press) for a closely related inequality that holds for all $n \geq 2$ and $t > K \log n$ for some $K > 0$.

They remark that their maximal inequality cannot be derived the Kevei and M (2011) GMB inequality. We formulated and proved our more GMB inequality to include results like theirs.