Almost sure invariance principles for unbounded functions of expanding maps of the interval

Jérôme Dedecker

Université Paris-Descartes
Joint work with S. Gouëzel and F. Merlevède

HDP, BIRS, October 2011

1. Uniformly Expanding maps

Consider a map T from $[0,1]$ to $[0,1]$, piecewise monotonic and piecewise $\mathcal{C}^2$, mapping each interval of the partition to $[0,1]$, and such that $|T'| > 1$ on each interval of the partition.

Example :

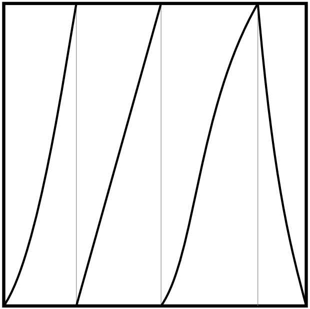

FIG.1 Graph of a uniformly expanding map, with $d = 4$

Examples

The most studied example is the "doubling map" : $T_0(x) = \{2x\}$. Clearly, the Lebesgue measure is invariant by T_0.

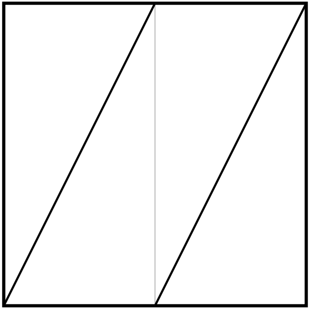

FIG.2 Graph of T_0

Numerable partition of $[0,1]$ can also be considered, as for the Gauss map $T(x) = \{x^{-1}\}$.

2. Intermittent maps

Let T be uniformly expanding, except at 0, where the right derivative is equal to 1. More precisely, the behavior around zero is : $T'(0) = 1$ and $T''(x) \sim cx^{\gamma-1}$ as $x \to 0$, with $c > 0$ and $\gamma \in]0, 1[$.

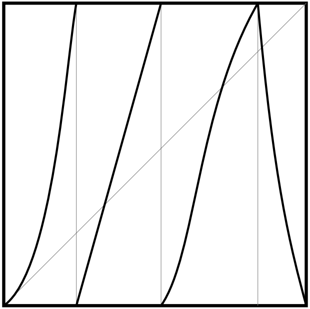

FIG.3 Graph of an intermittent map, with $d = 4$

Examples

Among the most studied examples, let us cite the LSV maps (Liverani, Saussol and Vaienti, 1999) :

$$\text{for } 0 < \gamma < 1, \quad T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$

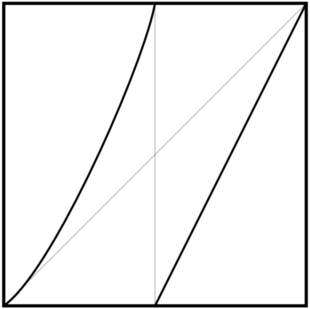

FIG.4 Graph of T_γ

3. The associated Markov Chain

- If T is either uniformly expanding or intermittent with $\gamma < 1$, there exists an unique T-invariant absolutely continuous probability measure ν.
- Define then the Perron-Frobenius operator K with respect to ν : for any f, g in $L^2(\nu)$

$$\nu(f \circ T \cdot g) = \nu(f \cdot K(g)),$$

which means exactly that

$$\mathbb{E}(g|T) = K(g)(T),$$

so K is a transition Kernel.

- Define then the Markov chain (X_i) with invariant measure ν and transition kernel K : on $([0,1], \nu)$ the n-tuple $(T, T^2, \ldots, T^n)$ is distributed as $(X_n, X_{n-1}, \ldots, X_1)$. See Hennion and Hervé (2001).

The example of T_0

- In the case of T_0, ν is the Lebesgue measure over $[0,1]$. By definition of K, one has that

$$K(f)(x) = \frac{1}{2}\Big(f\Big(\frac{x}{2}\Big) + f\Big(\frac{x+1}{2}\Big)\Big)$$

- The chain with transition K is the AR(1) process

$$X_{n+1} = \frac{1}{2}(X_n + \xi_{n+1}),$$

where (ξ_i) is iid with law $\mathcal{B}(1/2)$ and independent of X_0. Hence

$$X_n = \sum_{i=0}^{\infty} \frac{\xi_{n-i}}{2^{i+1}}.$$

Dependence properties of the Markov chain

- Recall that the chain (X_i) (or equivalently the sequence (T^i) on $([0,1],\nu)$) is not α-mixing in the sense of Rosenblatt (1956).
- For any Borel sets A, B of $[0,1]$, let

$$D_n(A,B,T) = |\nu(A \cap T^{-n}(B)) - \nu(A)\nu(B)|.$$

The sequence (T^i) would be α-mixing if

$$\lim_{n\to\infty} \sup_{A,B\in\mathcal{B}([0,1])} D_n(A,B,T) = 0\,.$$

- Taking $A_n = T^{-n}(B)$, then $D_n(A_n,B,T) = |\nu(B) - (\nu(B))^2|$ and consequently

$$\sup_{A,B\in\mathcal{B}([0,1])} D_n(A,B,T) \geq \sup_{B\in\mathcal{B}([0,1])} |\nu(B) - (\nu(B))^2| > 0$$

since ν is not a Dirac mass.

Dependence properties of the Markov chain

Let F be the distribution function of X_0, and let $G_k(t) = \mathbf{1}_{X_k \leq t} - F(t)$. With C. Prieur (2007), we have introduced the coefficients

$$
\begin{aligned}
\phi_{1,X}(k) &= \sup_{t \in \mathbb{R}} \|\mathbb{E}(G_k(t)|X_0)\|_\infty , \\
\phi_{2,X}(k) &= \phi_{1,X}(k) \vee \sup_{i > j \geq k} \sup_{t,s \in \mathbb{R}} \Big\| \mathbb{E}(G_i(t)G_j(s)|X_0) - \mathbb{E}(G_i(t)G_j(s)) \Big\|_\infty . \\
\alpha_{1,X}(k) &= \sup_{t \in \mathbb{R}} \|\mathbb{E}(G_k(t)|X_0)\|_1 , \\
\alpha_{2,X}(k) &= \alpha_{1,X}(k) \vee \sup_{i > j \geq k} \sup_{t,s \in \mathbb{R}} \Big\| \mathbb{E}(G_i(t)G_j(s)|X_0) - \mathbb{E}(G_i(t)G_j(s)) \Big\|_1 .
\end{aligned}
$$

Dependence properties of the Markov chain

- Assume that T is uniformly expanding. Then, using the contraction properties of K in the space of BV functions (see Lasota and Yorke (1974)), we have proved with C. Prieur (2007) that

$$\phi_{2,X}(n) \leq C\rho^n, \quad \text{with } \rho < 1\,.$$

- If T is intermittent, starting from a construction by S. Gouëzel (2004, 2007), we have proved with S. Gouëzel and F. Merlevède (2010) that

$$An^{(\gamma-1)/\gamma} \leq \alpha_{2,X}(n) \leq Bn^{(\gamma-1)/\gamma}\,.$$

The lower bound is taken from Sarig (2002).

4. Invariance principles for uniformly expanding maps

- Let H be a tail function. Let $\mathcal{F}(H,\nu)$ be the closed convex envelop in $\mathbb{L}^1(\nu)$ of the set of functions f which are monotonic on some open interval of $]0,1[$ and 0 elsewhere, and such that $\nu(|f|>t) \leq H(t)$.
- Assume that T is uniformly expanding, and that f belongs to $\mathcal{F}(H,\nu)$, where

$$\int_0^\infty xH(x)dx < \infty \, .$$

- Then the series

$$\sigma^2(f) = \mathrm{Var}_\nu(f) + 2\sum_{k>0} \mathrm{Cov}_\nu(f, f\circ T^k)$$

converges absolutely.

Invariance principles for uniformly expanding maps

Moreover

- The process

$$\Big\{\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}(f\circ T^k-\nu(f)), t\in[0,1]\Big\}$$

 converges in distribution in the Skorokhod topology to $\sigma(f)W$, where W is a standard Brownian motion.

- Enlarging $([0,1],\nu)$ if necessary, there exists a sequence $(Z_i)_{i\geq 1}$ of i.i.d. Gaussian random variables with mean zero and variance $\sigma^2(f)$ such that

$$\Big|\sum_{k=1}^{n}(f\circ T^k-\nu(f)-Z_k)\Big| = o(\sqrt{n\ln(\ln(n))})\,,\ \text{almost surely.}$$

Invariance principles for uniformly expanding maps

- In particular, if f is monotonic from $]0,1[$ to $\mathbb{R}$, these invariance principles hold as soon as $\nu(f^2) < \infty$, without any regularity condition on f.
- Hofbauer and Keller (1982) proved that T can be written as a function of a β-mixing process. They obtained rates of convergence in the strong invariance principle for BV functions f, by using a general result by Philipp and Stout (1975) for function of mixing sequences.
- For the map T_0 : the weak invariance principle holds if

$$\int_0^1 \frac{\omega_2(f,t)}{t\sqrt{|\ln(t)|}} dt < \infty ,$$

where ω_2 is the modulus of continuity of f in $\mathbb{L}^2(dx)$. See Maxwell and Woodroofe (2000) for the CLT and Peligrad and Utev (2005) for the weak invariance principle.

5. Invariance principles for intermittent maps

- Assume that T is an intermittent map with $\gamma < 1/2$, and that f belongs to $\mathcal{F}(H, \nu)$, where

$$\int_0^\infty x(H(x))^{\frac{1-2\gamma}{1-\gamma}} dx < \infty .$$

- The process

$$\Big\{ \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (f \circ T^k - \nu(f)), t \in [0,1] \Big\}$$

converges in distribution in the Skorokhod topology to $\sigma(f)W$, where W is a standard Brownian motion.

- Enlarging $([0,1], \nu)$ if necessary, there exists a sequence $(Z_i)_{i\geq 1}$ of i.i.d. Gaussian random variables with mean zero and variance $\sigma^2(f)$ such that

$$\Big| \sum_{k=1}^{n} (f \circ T^k - \nu(f) - Z_k) \Big| = o(\sqrt{n \ln(\ln(n))}) \, , \text{ almost surely.}$$

Invariance principles for intermittent maps

Since the density of ν is such that $0 < c \leq h(x)/x^{-\gamma} \leq C < \infty$, these invariance principles hold if

- the function f is positive, non-increasing over $]0, 1[$, and

$$f(x) \leq \frac{C}{x^{(1-2\gamma)/2} |\ln(x)|^a} \quad \text{for } a > 1/2$$

- f is positive, non-decreasing over $]0, 1[$, and

$$f(x) \leq \frac{C}{(1-x)^{(1-2\gamma)/(2-2\gamma)} |\ln(1-x)|^a} \quad \text{for } a > 1/2$$

Hence, the function f can "blow up" more quickly in the neighborhood of 1 than in the neighborhood of 0.

Invariance principles for intermittent maps

- Gouëzel (2004) proved that, if $\gamma < 1/2$ and $f(x) = x^{-(1-2\gamma)/2}$, then

$$\frac{1}{\sqrt{n\ln(n)}}\sum_{k=1}^{n}(f\circ T^k - \nu(f)) \text{ converges in distribution to a normal law,}$$

  which shows that our previous result is close to the optimal.
- In fact, if $(H(x))^{\frac{1-2\gamma}{1-\gamma}} \leq Cx^{-2}$, which is the case studied by Gouëzel, one can prove that, for any $b > 1/2$,

$$\frac{1}{\sqrt{n}(\ln(n))^b}\sum_{k=1}^{n}(f\circ T^k - \nu(f)) \text{ converges to 0 almost everywhere.}$$

  This is the same phenomenon as in the iid case, when the CLT holds with the normalization $\sqrt{n\ln(n)}$ (see Feller (1968)).

Invariance principles for intermittent maps

- For Hölder continuous functions and $\gamma < 1/2$, Melbourne and Nicol (2005) obtained rates of convergence in the strong invariance principle. They use the construction of Young (1999) to go back to the uniformly expanding case, and then the arguments of Hofbauer and Keller (1982) and Philipp and Stout (1975).
- F. Merlevède and E. Rio (2011) obtained the following rates of convergence for unbounded functions. Let $p \in]2,3[$ and $\gamma \in]0,1/p[$. Let $f \in \mathcal{F}(H,\nu)$ where

$$\int_0^\infty x^{p-1}(H(x))^{\frac{1-p\gamma}{1-\gamma}}\,dx < \infty\,.$$

Then, enlarging $([0,1],\nu)$ if necessary, one can build a sequence $(Z_i)_{i\geq 1}$ of iid Gaussian with mean zero and variance $\sigma^2(f)$ such that

$$\Big|\sum_{k=1}^n (f\circ T^k - \nu(f) - Z_k)\Big| = O(n^{1/p}(\ln n)^{1/2-1/p}) \text{ almost surely.}$$

6. Sketch of proof of the strong invariance principle

- For the Markov chain, the WIP follows from its dependence properties and appropriate covariance inequalities. Because of the equality in distribution, the WIP in true also for the iterates (T^i).
- For the Markov chain, the ASIP may be proved by using a martingale approximation, an appropriate maximal inequality, and a result by Samek and Volný (2000).
- This martingale approximation cannot be used to prove the result for (T^i), because of the time reversion. See the discussion in Melbourne and Nicol (2005).

Sketch of proof of the strong invariance principle

- First step : there exists g_M whose variation norm is less than M, and such that

$$\sum_{n=1}^{\infty} \frac{1}{n} \nu\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k}((f-g_M)\circ T^i - \nu(f-g_M))\Big| \ge C(M)\sqrt{n \ln\ln(n)}\Big) < \infty$$

with $C(M) \to 0$, in such a way that, almost surely

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{\sqrt{n \ln\ln(n)}} \Big|\sum_{i=1}^{k}((f-g_M)\circ T^i - \nu(f-g_M))\Big| = 0\,.$$

- Let $H_M = \min(H(M), H)$. For intermittent maps with $\gamma < 1/2$,

$$C(M) = C\sum_{n>0} \int_0^{\alpha_{2,X^{(n)}}} (H_M^{-1}(t))^2 dt \le D\int_0^{\infty} x(H_M(x))^{\frac{1-2\gamma}{1-\gamma}} dx\,.$$

Hence $C(M) \to 0$ as soon as

$$\int_0^{\infty} x(H(x))^{\frac{1-2\gamma}{1-\gamma}} dx < \infty\,.$$

Sketch of proof of the strong invariance principle

- Second step : we apply to the BV function g_M the strong approximation result of Merlevède and Rio (2011) :
  If T is either uniformly expanding, or intermittent with $\gamma < 1/2$, then enlarging $([0,1], \nu)$ if necessary, one can build a sequence $(Z_{i,M})_{i \geq 1}$ of iid Gaussian with mean zero and variance $\sigma^2(g_M)$ such that

$$\Big| \sum_{k=1}^{n} (g_M \circ T^k - \nu(g_M) - Z_{k,M}) \Big| = O(n^{1/2-\epsilon}) \text{ almost surely.}$$

  Moreover $\sigma^2(g_M) \to \sigma^2(f)$, and for any $L \in \mathbb{N}$, the σ-algebras $\sigma(Z_{i,M})_{i<2^L, M \in \mathbb{N}}$ and $\sigma(Z_{i,M})_{i \geq 2^L, M \in \mathbb{N}}$ are independent.
- Last step : one can find $M(i) \to \infty$ such that the ASIP holds with

$$Z_i = \frac{\sigma(f)}{\sigma(g_{M(i)})} Z_{i,M(i)}.$$