Moderate deviations for the eigenvalue counting function of Wigner matrices

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joint work with Hanna Döring

WIGNER matrices

- random Hermitian matrices M_n of size n
- for $i < j$: the real and imaginary parts of $(M_n)_{ij}$ are iid, with mean 0 and variance $1/2$
- $(M_n)_{ii}$ are iid with mean 0 and variance 1

example: entries are Gaussian:
Gaussian Unitary Ensemble (GUE)

Condition (C) on M_n

M_n satisfies condition (C) if :

- the real part ξ and the imaginary part $\widetilde{\xi}$ of $(M_n)_{ij}$ are independent
- and have an exponential decay:
  there are two constants C and C' such that

$$P(\xi \geq t^C) \leq e^{-t} \quad \text{and} \quad P(\widetilde{\xi} \geq t^C) \leq e^{-t}$$

  for all $t \geq C'$

can possibly be relaxed (not necessarily identically distributed; finite moment condition: $\mathbb{E}|\xi|^C, \mathbb{E}|\widetilde{\xi}|^C < \infty$ for C suff. large)

known results

GUE: the joint law of the eigenvalues is known

allowing for a lot of descriptions of their limiting behavior both in the global and local regimes

$$W_n := \frac{1}{\sqrt{n}} M_n, \quad A_n := \sqrt{n} M_n \quad \text{coarse/fine-scale}$$

W_n: placing all eigenvalues in a bounded interval ($[-2, 2]$)

A_n: keeping the spacing between adjacent eigenvalues to be roughly of unit size

known results

$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$: eigenvalues of W_n

(global) WIGNER theorem:
(under substantially more general hypotheses true)

$\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$ converges weakly almost surely as $n \to \infty$ to law

$$\varrho(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\, 1_{[-2,2]}$$

$I \subset \mathbb{R}$:

$$\frac{1}{n}N_I(W_n) := \frac{1}{n}\sum_{j=1}^{n} 1_{\{\lambda_j \in I\}} \to \varrho(I) \quad \text{a.s.}$$

fluctuation level: global

GUE: M_n', $W_n' := \frac{1}{\sqrt{n}} M_n'$

Theorem (Costin-Lebowitz; 1995)

Let I_n be an intervall and $\mathbb{V}(N_{I_n}(W_n')) \to \infty$ as $n \to \infty$, then

$$\frac{N_{I_n}(W_n') - \mathbb{E}(N_{I_n}(W_n'))}{\sqrt{\mathbb{V}(N_{I_n}(W_n'))}} \to N(0,1)$$

in distribution.

applying orthogonal polynomial techniques and/or the particular determinantal structure of GUE

wave of new results

Recent wave of progress:

It has been conjectured, since the 1960s, by Wigner, Dyson, Mehta and many others, that the local statistics (the convergence of distribution functions) are universal, in the sense that they hold not only for the GUE, but for any other Wigner random matrix also.

Erdös, Schlein, Yau / Tao, Vu, 2009/2010/2011...

statistics

local: distribution of the gaps between consecutive eigenvalues: who many $1 \leq i \leq n$ are there such that $\lambda_{i+1} - \lambda_i \leq s$?

k-point correlation functions

distribution of individual λ_i

Gaudin, sin-kernel due to Dyson, Tracy-Widom

What about large and moderate deviations (global/local)?

(joint projects with L. Erdös, Th. Kriecherbauer)

large deviations (LDP): iid summands

$(X_i)_i$ i.i.d.

$$S_n := \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$P\big(S_n \sim x\big) \approx \exp\big(-n\ I(x)\big)$$

with rate function

$$I(x) = \sup_{z\in\mathbb{R}}\{z\,x - \log Ee^{zX_1}\}$$

large deviations: CRAMÉR...

moderate deviations (MDP): iid summands

$(X_i)_i$ i.i.d.

$$S_n^{a_n} := \frac{1}{a_n\sqrt{n}} \sum_{i=1}^{n} X_i \qquad \text{with} \qquad 1 \ll a_n \ll \sqrt{n}$$

$$P\big(S_n^{a_n} \sim x\big) \approx \exp\big(-a_n^2 \; I(x)\big)$$

with rate function

$$I(x) = \frac{x^2}{2\mathbb{V}(X_1)}$$

moderate deviations

some universality

(global) deviation results for spectral statistics

- LDP and MDP for empirical measures of eigenvalues in the Gaussian case
- LDP and MDP for empirical measures of eigenvalues for special Gaussian divisible ensembles:

$$(1-t)^{1/2} M_n' + t^{1/2} V_n$$

  V_n deterministic selfadjoint matrix with convergent spectral measure
  Ben Arous, Guionnet, Dembo, 1997, 2002
- LDP for empirical measures for other symmetries
  E., Stolz, 2007,2011
- traces of powers of Bernoulli random matrices
  Döring, E., 2009

no universal version for Wigner matrices!

First result

Let $\mathbb{V}(N_{I_n}(W'_n)) \to \infty$:

Theorem (GUE)

For any $(a_n)_n$ *with* $1 \ll a_n \ll \sqrt{\mathbb{V}(N_{I_n}(W'_n))}$

$$\frac{N_{I_n}(W'_n) - \mathbb{E}(N_{I_n}(W'_n))}{a_n \sqrt{\mathbb{V}(N_{I_n}(W'_n))}}$$

satisfies a MDP with speed a_n^2 *and rate function* $x^2/2$.

consider cumulants or log-LAPLACE transform; use determinantal structure of GUE

also: moderate deviation estimates of CRAMÉR type

version with numerics

GUSTAVSSON, 2005, showed:
for $I = [y, \infty)$ with $y \in (-2, 2)$

$$\mathbb{E}[N_I(W_n')] = n\varrho(I) + O(\frac{\log n}{n}) \quad \text{and} \quad \mathbb{V}(N_I(W_n')) = \big(\frac{1}{2\pi^2} + o(1)\big) \log n$$

applying strong asymptotics for orthogonal polynomials with respect to exponential weights due to DEIFT et. al.

hence

$$\frac{N_I(W_n') - n\varrho(I)}{a_n\sqrt{\frac{1}{2\pi^2}\log n}}$$

satisfies the same MDP

universal MDP (global)

main result:

Theorem

Let M_n be a WIGNER *matrix whose entries satisfy condition (C) and match the corresponding entries of GUE up to order 4. Let $I = I(y) = [y, \infty)$ for $y \in (-2, 2)$.*
For any $(a_n)_n$ with $1 \ll a_n \ll \sqrt{\mathbb{V}(N_I(W_n))}$

$$\frac{N_I(W_n) - \mathbb{E}(N_I(W_n))}{a_n \sqrt{\mathbb{V}(N_I(W_n))}}$$

satisfies a MDP with speed a_n^2 and rate function $x^2/2$.

the same is true with numerics (under finite moment condition!)

matching 4 moments

complex random variables X and Y match to order k if

$$\mathbb{E}[\mathrm{Re}(X)^m \, \mathrm{Im}(X)^l] = \mathbb{E}[\mathrm{Re}(Y)^m \, \mathrm{Im}(Y)^m]$$

for all $m, l \geq 0$ such that $m + l \leq k$.

matching the corresponding entries of GUE up to order 4:
fix third and fourth moment

due to the famous Four Moment Theorem of TAO and VU

Moderate deviations for a local statistic

on the way proving our result:

let $t(x) \in [-2, 2]$ defined for $x \in [0, 1]$ by

$$x = \int_{-2}^{t(x)} d\varrho(t)$$

consider $i = i(n)$ such that $i/n \to a \in (0, 1)$: λ_i is in the bulk

$t(i/n)$: expected location of the i-th eigenvalue

$$\frac{\sqrt{\log n}}{\pi\sqrt{2}} \frac{1}{n\varrho(t(i/n))}$$

standard deviation (mean eigenvalue spacing)

GUE and Wigner

Theorem

Let $i/n \to a \in (0,1)$, $1 \ll a_n \ll \sqrt{\log n}$. Let W_n be a Wigner matrix whose entries satisfy a finite moment condition and match the corresponding entries of GUE up to order 4. Then

$$\sqrt{\frac{4-t(i/n)^2}{2}}\frac{\lambda_i(W_n)-t(i/n)}{a_n\frac{\sqrt{\log n}}{n}}$$

satisfies a MDP with speed a_n^2 and rate function $x^2/2$.

Gustavsson: CLT

Proof

(1): GUE: transfer the $N_{I_n}(W_n')$–MDP to $\lambda_i(W_n')$:

use the tight relation: for $I(y) = [y, \infty)$

$$N_{I(y)}(W_n') \leq n - i \quad \Leftrightarrow \quad \lambda_i(W_n') \leq y$$

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$$I_n := \left[t(i/n) + \xi\, a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}, \infty \right)$$

$$\begin{aligned} P_n\left(\frac{\lambda_i(W_n') - t(i/n)}{a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}} \le \xi \right) &= P_n\big(N_{I_n}(W_n') \le n - i\big) \\ &= P_n\left(\frac{N_{I_n}(W_n') - \mathbb{E}[N_{I_n}(W_n')]}{a_n\, (\mathbb{V}(N_{I_n}(W_n')))^{1/2}} \le \frac{n - i - \mathbb{E}[N_{I_n}(W_n')]}{a_n\, (\mathbb{V}(N_{I_n}(W_n')))^{1/2}} \right) \end{aligned}$$

Asymptotics

remember:

$$\mathbb{E}[N_{I_n}(W_n')] = n\,\varrho(I_n) + O\big(\frac{\log n}{n}\big)$$

here: I_n depends on a_n and with strong asymptotics for orthogonal polynomials

$$n\,\varrho(I_n) = n - i - \xi a_n (\log n)^{1/2} \frac{1}{\sqrt{2}\pi} + O\big(\frac{a_n^2 \log n}{n}\big)$$

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moreover we apply

$$\mathbb{V}(N_{I_n}(W_n')) = \big(\frac{1}{2\pi^2} + o(1)\big)\log n$$

Proof

(2): next transfer the local MDP universally:
apply the Four Moment Theorem:

Theorem (Tao, Vu)

Let M_n be Wigner whose entries satisfy a moment condition and match the corresponding entries of GUE up to order 4. Then there is a small constant c_0 such that

$$P\big(\lambda_i(A_n') \in I_-\big) - n^{-c_0} \le P\big(\lambda_i(A_n) \in I\big) \le P\big(\lambda_i(A_n') \in I_+\big) + n^{-c_0}$$

and

$$\frac{1}{a_n^2} \log n^{-c_0} \to -\infty$$

by assumption

Proof

(3): reverse strategy to go back to $N_I(W_n)$

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(3): reverse strategy to go back to $N_I(W_n)$

- deep fact: $\mathbb{E}(N_I(W_n))$ and $\mathbb{V}(N_I(W_n))$ have identical asymptotic behaviour to the ones for GUE matrices!
- Unfortunately, the Four Moment Theorem does not give this!
- Indeed, the Four Moment Theorem deals with a finite number of eigenvalues, whereas the computation of $\mathbb{E}(N_I(W_n))$ and $\mathbb{V}(N_I(W_n))$ involves all the eigenvalues of the matrix
- To achieve the result one can apply recent results by Erdös, Yau and Yin providing suitable localization properties of the eigenvalues in the bulk: therefore need assumption (C)