On the uniform convergence of random series in Skorohod space and representations of càdlàg infinitely divisible processes

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Outline

1. Background and motivation
2. Convergence of random series in the Skorohod space $D[0,1]$
3. Series representation of càdlàg infinitely divisible processes
4. Some corollaries

Motivation

1. A.N. Shiryaev (1980): When is a Gaussian process a semimartingale?
2. F. Knight (1992) solved this problem, very elegantly, for Gaussian moving averages.
3. ABOC and J. Pedersen (2009) characterized when a class of infinitely divisible moving averages are semimartingales.

Recall that $X = \{X(t)\}_{t\in[0,1]}$ is said to be infinitely divisible if for any $0 \leq t_1 < \cdots < t_n \leq 1$, the random vector

$$(X(t_1), \ldots, X(t_n))$$

has an infinitely divisible distribution.

- ABOC and J. Rosiński (2011, forthcoming) give a complete characterization when an infinitely divisible process is a semimartingale and and provide an explicit canonical decomposition.

The key is a detailed analysis of jumps combined with stochastic analysis. Series representations of general infinitely divisible processes provide a way to study the jump process. For this method to work, we needed the validity of the Itô-Nisio Theorem for $D[0,1]$ equipped with the supremum norm $\|\cdot\|$.

Is the Itô-Nisio Theorem true for $(D[0,1], \|\cdot\|)$?

The solution may also be of an independent interest.

The Itô-Nisio Theorem (function space version)

- Let T be a set and $(F, \|\cdot\|)$ be a Banach space of functions form T into $\mathbb{R}$ such that $\delta_t : x \mapsto x(t)$ is continuous for all t, and $x \mapsto \|x\|$ is measurable with respect to the cylindrical σ-algebra $\sigma(\delta_t : t \in T)$.
- Let $X_j = \{X_j(t)\}_{t \in T}$, $j \in \mathbb{N}$, be independent and symmetric stochastic processes with paths in F, and set

$$S_n = \sum_{j=1}^{n} X_j, \qquad n \geq 1.$$

Theorem (Itô and Nisio (1968))

Assume that F is separable. Then the following are equivalent:

1. $\lim_{n \to \infty} S_n$ *exists a.s. in F*
2. S_n *converges in finite dimensional distributions to process S with paths in F.*

The Skorohod space $D[0,1]$

1. A function $f : [0,1] \to \mathbb{R}$ is said to be càdlàg if it is right-continuous with left-hand limits.
2. Let $D[0,1]$ be the Skorohod space of càdlàg functions, that is,

$$D[0,1] = \Big\{ f : [0,1] \to \mathbb{R} \mid f \text{ is càdlàg} \Big\}.$$

3. For all $f \in D[0,1]$ let $\|f\| = \sup_{t \in [0,1]} |f(t)|$ denote its sup norm.
4. Note that all Lévy processes, all martingales and most Markov processes have paths in $D[0,1]$.

The Itô-Nisio Theorem *does not* hold for general non-separable Banach spaces F. Indeed, one can reduce from [2] that the Itô-Nisio Theorem is not true in the following non-separable Banach spaces:

$$\ell^\infty(\mathbb{N}), \quad BV_p \text{ for } p > 1, \quad C^\alpha[0,1] \text{ for } \alpha \in (0,1),$$

where

BV_p is the space of functions of bounded p-variation,
$C^\alpha[0,1]$ is the space of α-Hölder continuous functions.

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[1] Jain, N. C. and D. Monrad (1983). Gaussian measures in B_p. *Ann. Probab. 11*(1), 46–57.

$D[0,1]$ in Skorohod's topology

(1) The usual proofs of the Itô-Nisio Theorem relies on the fact that all probability measures μ on a separable Banach space are convex tight, that is, for all $\epsilon > 0$ there exists a convex compact set K such that $\mu(K^c) \leq \epsilon$.

(2) $D[0,1]$ equipped with Skorohod's topology d_S is a separable complete metric space.

(3) However, the addition operation is not continuous in $(D[0,1], d_S)$, and as a consequence of this, probability measures on $(D[0,1], d_S)$ are *not* convex tight, cf. Daffer and Taylor [3]. Hence a straightforward extension of the Itô-Nisio Theorem to $(D[0,1], d_S)$ seems not possible.

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[2] Daffer, P. Z. and R. L. Taylor (1979). Laws of large numbers for $D[0,1]$. *Ann. Probab.* 7(1), 85–95.

A related result:

Kallenberg [3] has shown that in $(D[0,1], d_S)$ a series of independent random elements converges in law if and only if it converges a.s.

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[3] Kallenberg, O. (1974). Series of random processes without discontinuities of the second kind. *Ann. Probab. 2*, 729–737.

The set-up

1. Let
$$S_n(t) = \sum_{j=1}^{n} X_j(t), \qquad t \in [0,1],\ n \geq 1,$$
where X_j are independent càdlàg processes.
2. All of the following results are shown for processes $\{X_j(t)\}_{t\in[0,1]}$ taking values in a separable Banach space E, however, we will only focus on the real-valued case.

The Itô-Nisio Theorem for $D[0,1]$

Theorem (ABOC & Jan Rosiński)

Suppose that S_n converges in finite dimensional distributions to a càdlàg process.
Then there exists a càdlàg process S such that

(i) *If $\{X_n\}$ are symmetric, then $S_n \to S$ uniformly on $[0,1]$ a.s.*

(ii) *If $\{X_n\}$ are not symmetric, then there exist $c_n \in D[0,1]$ such that*

$$S_n + c_n \to S \quad \text{uniformly on } [0,1] \text{ a.s.} \tag{1}$$

(iii) *Moreover, if the family $\{S(t) : t \in T\}$ is uniformly integrable and the functions $t \mapsto \mathbb{E}[X_n(t)]$ are càdlàg, then one can take c_n in (1) given by $c_n(t) = \mathbb{E}[S(t) - S_n(t)]$.*

Next we will use the above theorem to prove uniform convergence of series representations of càdlàg infinitely divisible processes.

Let $X = \{X(t)\}_{t\in[0,1]}$ be an infinitely divisible process without Gaussian part with the following Lévy-Khintchine representation: For all $\theta_1, \ldots, \theta_n \in \mathbb{R}$,

$$\mathbb{E}\exp\left\{i\sum_{j=1}^{n}\theta_j X(t_j)\right\} = \exp\left\{i\sum_{j=1}^{n}\theta_j b(t_j) + \int_{\mathbb{R}^n}\left(e^{i\sum_{j=1}^{n}\theta_j x_j} - 1 - i\sum_{j=1}^{n}\theta_j [\![x_j]\!]\right)\nu_{t_1,\ldots,t_n}(dx_1\cdots dx_n)\right\},$$

where $\{b(t_j)\} \subseteq \mathbb{R}$, $\nu_{t_1,\ldots,t_n}$ are Lévy measures on $\mathbb{R}^n$ and $[\![x]\!] = x/(1 \vee |x|)$ is a continuous truncation function.

1. Let $\{\gamma_j\}$ be an i.i.d. sequence of standard exponential random variables and $\Gamma_j = \sum_{i=1}^{j} \gamma_i$ for $j \geq 1$. Let $\mathcal{V}$ be a measurable space and $\{V_j\}$ be an i.i.d. sequence in $\mathcal{V}$ and set $V = V_1$. Assume that $\{V_j\}$ are $\{\Gamma_j\}$ are independent.
2. Let $X = \{X(t)\}_{t\in[0,1]}$ be an infinitely divisible process without Gaussian part with shifts $\{b(t)\}$ and Lévy measures $\nu_{t_1,\dots,t_n}$. Let $H\colon [0,1] \times \mathbb{R}_+ \times \mathcal{V} \to \mathbb{R}$ be a measurable function such that for every $t_1, \dots, t_n \in [0,1]$ and $B \in \mathcal{B}(\mathbb{R}^n)$

$$\nu_{t_1,\dots,t_n}(B) = \int_0^\infty \mathbb{P}\Big(\big(H(t_1, r, V), \dots, H(t_n, r, V)\big) \in B\backslash\{0\}\Big)\, dr,$$

$H(\cdot, r, v) \in D[0,1]$ for every (r, v), and $r \mapsto \|H(\cdot, r, v)\|$ is nonincreasing for every $v \in \mathcal{V}$.

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[4] Rosiński, J. (1990). On series representations of infinitely divisible random vectors. *Ann. Probab.* *18*(1), 405–430.

Theorem (ABOC & Jan Rosiński)

Assume that $X = \{X(t)\}_{t\in[0,1]}$ *has càdlàg paths. Then, with probability 1,*

$$Y(t) := b(t) + \sum_{j=1}^{\infty}[H(t,\Gamma_j,V_j) - C_j(t)]$$

converges uniformly in $t \in [0,1]$*, where*

$$C_j(t) = \int_{\Gamma_{j-1}}^{\Gamma_j} \mathbb{E}[\![H(t,r,V)]\!]\,dr,$$

and $\{Y(t)\} \overset{d}{=} \{X(t)\}$*.*

Moreover, we may choose $\{\Gamma_j, V_j\}$ *such that*

$$\{X(t)\}_{t\in[0,1]} \overset{in}{=} \{Y(t)\}_{t\in[0,1]}$$

where $\overset{in}{=}$ *means that the two processes are indistinguishable.*

For each càdlàg function $f\colon [0,1] \to \mathbb{R}$ let $\Delta f(t) = f(t) - f(t-)$ denote its jump at t.

Corollary

Assume, in addition, that the shift function $t \mapsto b(t)$ *and the truncation functions* $t \mapsto C_j(t)$ *are continuous.*
Then

$$\{\Delta X(t)\}_{t\in[0,1]} \overset{in}{=} \Big\{ \sum_{j=1}^{\infty} \Delta H(t, \Gamma_j, V_j) \Big\}_{t\in[0,1]}.$$

The jump structure of general stable processes

Let $\alpha \in (0,2)$ and M be a symmetric α-stable random measure with σ-finite control measure m, i.e.,

$$\mathbb{E}e^{i\theta M(A)} = e^{-m(A)|\theta|^\alpha}.$$

Let $X = \{X(t)\}_{t\in[0,1]}$ be a symmetric α-stable process of the form

$$X(t) = \int_S f(t,s)\, M(ds), \tag{2}$$

where $f_t = f(t,\cdot)$ are deterministic M-integrable functions.

When does a stable process has paths in $D[0,1]$?

Theorem (ABOC and Jan Rosiński)

Let $X = \{X(t)\}_{t\in[0,1]}$ be a symmetric α-stable process of the form (2) with $\alpha \in (1,2)$ and finite control measure m.
Assume that there exist $\beta_1, \beta_2 > \frac{1}{2}, p_1 > \alpha, p_2 > \frac{\alpha}{2}$ and increasing continuous functions $F_1, F_2 \colon [0,1] \to \mathbb{R}$ such that for all $0 \le t_1 \le t \le t_2 \le 1$,

$$\int |f_{t_2} - f_{t_1}|^{p_1}\, dm \le [F_1(t_2) - F_1(t_1)]^{\beta_1},$$

$$\int |(f_t - f_{t_1})(f_{t_2} - f_t)|^{p_2}\, dm \le [F_2(t_2) - F_2(t_1)]^{2\beta_2}.$$

Then X has a càdlàg modification.

This extends a recent result by [5] from $p_1 = p_2 = 2$ and removes a technical condition. Our proof relies on different methods.

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[5] Davydov, Y. and C. Dombry (2011). On the convergence of Le Page series in Skohorod space. *arXiv:1107.2193v1*.

Corollary

Let $X = \{X(t)\}_{t\in[0,1]}$ *be a symmetric* α*-stable process of the form* (2). *Assume that* X *is càdlàg and continuous in probability.*

(i) *The largest jump in absolute value of* X, $\sup_{t\in[0,1]}|\Delta X(t)|$, *is Fréchet distributed with shape parameter* α *and scale parameter*

$$c_\alpha\Big(\int \sup_{t\in[0,1]} |\Delta f(t,s)|^\alpha \, m(ds)\Big)^{1/\alpha}, \quad c_\alpha \text{ is a constant.}$$

For all $p > 0$ and all càdlàg functions $g\colon [0,1] \to \mathbb{R}$ let

$$V_p(g) = \sum_{s \in [0,1]} |\Delta g(s)|^p.$$

Corollary (continued)

(ii) *We have that $V_p(X) < \infty$ a.s. if and only if either $t \mapsto f(t,s)$ is continuous for m-a.e. s, or that $p > \alpha$ and*

$$\sigma := \int V_p(f(\cdot, s))^{\alpha/p}\, m(ds) < \infty.$$

In this case, $V_p(X)$ is respectively zero a.s., or a positive (α/p)-stable random variable with scale parameter $\sigma^{p/\alpha}$ and shift parameter 0.

For an α-stable Lévy process we have $S = [0,1]$, $m = \mathrm{Leb}$ and $f(t,s) = \mathbf{1}_{\{s \le t\}}$. Since $V_p(f(\cdot, s)) \equiv 1$, we have $V_p(X) < \infty$ a.s. if and only if $p > \alpha$.

Thank you!