Kinetic limits and imaging models for waves in random media

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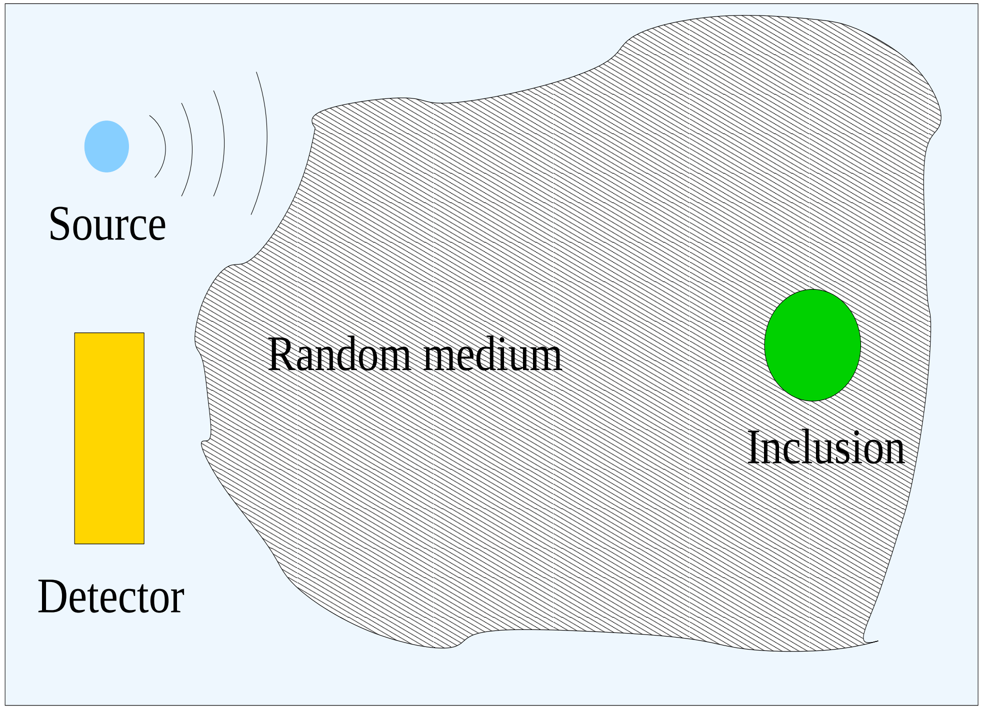
Outline

- Introduction
  - Modeling of our inverse problem of interest
  - Identification of the relevant physical scales
- Examples of theoretical results
  - Quantification of the statistical instabilities for some reduced models
- Numerical simulations

Our generic problem : from detector measurements, image a buried inclusion in a random medium.

- The random medium models an unknown heterogeneous medium (atmosphere, forest, ocean, etc)
- We are interested in a regime in which the interaction between the wave and the medium is strong
- To make this precise, we need to identify the main scales.



4 important parameters :

- the overall distance of propagation L
- the wavelength λ
- the correlation length l
- the strength of the fluctuations σ

We use the standard wave equation to model the propagation :

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0, \qquad \kappa(\mathbf{x}) \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = 0, \qquad \text{on } \mathbb{R}^d,$$

$$p(0, \mathbf{x}) = p_0 \left(\mathbf{x}, \frac{\mathbf{x}}{\lambda} \right), \quad \mathbf{v}(0, \mathbf{x}) = 0,$$

where we assume κ takes the form (κ_0 is the background compressibility)

$$\kappa(\mathbf{x}) = \kappa_0 + \sigma V \left(\frac{\mathbf{x}}{l} \right), \qquad \mathbb{E}(V) = 0, \qquad \mathbb{E}(V(\mathbf{x}+\mathbf{y})V(\mathbf{y})) = R(\mathbf{x})$$

V being a mean-zero stationary process with correlation function R.

Scalings

- We make first the high frequency assumption that

$$\frac{\lambda}{L} = \varepsilon \ll 1.$$

- Several choices are possible for l : we assume here that it is of order of the wavelength so that the interaction between the wave and the medium is maximal. Thus $l \approx \lambda$. The alternatives $l \ll \lambda$ or $\lambda \ll l$ lead to different asymptotical regimes.
- Weak coupling regime : the strength of the fluctuations is supposed to be $\sqrt{\varepsilon}$. This is the intensity that provides an effect of order one of the medium over large propagation distances.

Therefore κ reads

$$\kappa(\mathbf{x}) = \kappa_0 + \sqrt{\varepsilon} V \left(\frac{\mathbf{x}}{\varepsilon} \right).$$

There is another important parameter : the mean free path $c_0\Sigma^{-1}$ (depends on λ and R), that can be interpreted as the average distance between 2 interactions of the wave and the medium.

- If $L \leq c_0\Sigma^{-1}$, coherent regime : the wave has weakly interacted with the medium, the wave front can me measured. In this regime, interferometry methods perform well (L. Borcea, G. Papanicolaou, K. Sølna, C. Tsogka)
- If $L \geq c_0\Sigma^{-1}$, incoherent regime : the wave strongly interacts with the medium, the wave front is not available. We need a model to describe the multiple interactions.

In our configuration : $L \approx 320\lambda$, $c_0\Sigma^{-1} \approx 40\lambda$, so that $L \approx 8c_0\Sigma^{-1}$. This is not the diffusive regime yet.

Asymptotic models

We already have a model : the wave equation. Not appropriate here, its solutions strongly depend on the randomness.

We need a model that weakly depends on the random medium : transport equations. They generally describe quantities quadratic in the wavefield as the wave energy

$$\mathcal{E}^\varepsilon(t,\mathbf{x}) := \frac{1}{2}\left(\kappa(\mathbf{x})|p|^2(t,\mathbf{x}) + \rho_0|\mathbf{v}|^2(t,\mathbf{x})\right).$$

It is well-known, at least formally [Ryzhik-Papanicolaou-Keller 96], that

$$\lim_{\varepsilon\to 0}\mathbb{E}\{\mathcal{E}^\varepsilon\}(t,\mathbf{x}) = \int_{\mathbb{R}^d} a(t,\mathbf{x},\mathbf{k})d\mathbf{k},$$

where a is an amplitude solution to a radiative transfert equation

$$\frac{\partial a}{\partial t} + c_0\,\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}a + \Sigma\, a = Q(a), \qquad \hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}, \qquad \Sigma = Q(1)$$

$$Q(a) = \int_{\mathbb{R}^d} a(t,\mathbf{x},\mathbf{p})\sigma(\mathbf{k},\mathbf{p})\delta(c_0|\mathbf{p}| - c_0|\mathbf{k}|)d\mathbf{p}, \quad \sigma(\mathbf{k},\mathbf{p}) = \frac{\pi c_0^2|\mathbf{k}|^2}{2(2\pi)^d}\hat{R}(\mathbf{k}-\mathbf{p})$$

Asymptotic models

Our approach then consists in solving an inverse transport problem using measurements of $\mathcal{E}^\varepsilon$ and the model for a (that only depend on $\hat{R}$) rather than using the wave description.

Main Problem : statistical stability. a describes the limit of $\mathbb{E}\{\mathcal{E}^\varepsilon\}$ and not that of $\mathcal{E}^\varepsilon$. In practical experiments, one cannot compute averages since one has access to one realization of the random medium only, the physical medium.

One therefore expects the following self-averaging property to hold when $\varepsilon \to 0$

$$\mathcal{E}^\varepsilon \sim \mathbb{E}\{\mathcal{E}^\varepsilon\}. \tag{1}$$

Quantifying precisely this relation is essential for our imaging problem since

$$\delta\mathcal{E}^\varepsilon = \mathcal{E}^\varepsilon - \mathbb{E}\{\mathcal{E}^\varepsilon\}$$

is the main source of noise.

Quantifying the instability $\delta\mathcal{E}^\varepsilon$ is a very difficult task, essentially possible in simplified settings, as the paraxial approximation in which the wave propagation is described by a Schrödinger equation.

There are many results in the literature adressing the convergence of $\mathbb{E}\{\mathcal{E}^\varepsilon\}$ (or of $\mathcal{E}^\varepsilon$ in probability) for the Schrödinger equation, see e.g. Erdös-Yau, Bal-Papanicolaou-Ryzhik, Fannjianng, Poupaud-Vasseur, without analyzing the convergence rate.

For some reduced models, we were able to make (1) precise so as to

- obtain optimal error estimates
- quantify the dependence on some parameters of the problem as the concentration of initial conditions or the size of detector
- characterize the first-order corrector to transport (at least its covariance)
- obtain convergence when the random medium has some long-range interactions.

Some typical results

We suppose the propagation is described by the following Schrödinger equation :

$$\left(i\eta \frac{\partial}{\partial z} + \frac{\eta^2}{2}\Delta_{\mathbf{x}} + \sqrt{\eta} V^\eta \left(z, \frac{\mathbf{x}}{\eta} \right) \right) \psi_\eta(z, \mathbf{x}) = 0, \quad z > 0, \quad \mathbf{x} \in \mathbb{R}^{d-1} \equiv \mathbb{R}^D,$$

$\psi_\eta(0, \mathbf{x}) = \psi_\eta^0(\mathbf{x})$ bounded in L^2.

The dependence -or not- on z of V is crucial for the mathematical analysis

- Simplest case : V is a white noise in z, regular in $\mathbf{x}$
- Most difficult case : V is independent of z

We choose an initial condition of the form (a pure state)

$$\psi_\eta^0(\mathbf{x}) = \frac{1}{\eta^{\frac{D\alpha}{2}}} \chi\left(\frac{\mathbf{x} - \mathbf{x}_0}{\eta^\alpha} \right) e^{i\frac{(\mathbf{x}-\mathbf{x}_0)\cdot \mathbf{k}_0}{\eta}}.$$

The main tool in the analysis is the Wigner transform

$$W^\eta(z, \mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} e^{i\,\mathbf{k}\cdot\mathbf{y}}\, \psi_\eta(z, \mathbf{x} - \frac{\eta \mathbf{y}}{2}) \psi_\eta^*(z, \mathbf{x} + \frac{\eta \mathbf{y}}{2})\, d\mathbf{y}.$$

Some typical results

The Wigner transform provides a phase space description of the propagation and

$$\int_{\mathbb{R}^D} W^\eta(z,\mathbf{x},\mathbf{k})d\mathbf{k} = |\psi_\eta(z,\mathbf{x})|^2 \equiv \text{wave energy}$$

The related initial condition reads

$$W_{\eta 0}(\mathbf{x},\mathbf{k}) = \frac{1}{\eta^D} W_0\left(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha}, \frac{\mathbf{k}-\mathbf{k}_0}{\eta^{1-\alpha}}\right).$$

Another important tool is the scintillation function J_η , defined as

$$J_\eta(z,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) = \mathbb{E}\{W_\eta(z,\mathbf{x},\mathbf{k})W_\eta(z,\mathbf{y},\mathbf{p})\} - \mathbb{E}\{W_\eta(z,\mathbf{x},\mathbf{k})\}\mathbb{E}\{W_\eta(z,\mathbf{y},\mathbf{p})\},$$

whose weak convergence to zero implies the convergence in probability :

$$\mathbb{P}\Big(|\langle W_\eta(z),\varphi\rangle - \langle \mathbb{E}\{W_\eta\}(z),\varphi\rangle| \geq \delta\Big) \leq \frac{1}{\delta^2}\langle J_\eta(z), \varphi\otimes\varphi\rangle.$$

When V is white-noise in time, J_η satisfies a closed-form equation (it is a consequence of the fast decorrelation in time) and a complete analysis of the limit $\eta \to 0$ is possible.

The Itô-Schrödinger equation reads :

$$d\psi_\eta(z,\mathbf{x}) = \frac{1}{2} i\eta\Delta_{\mathbf{x}}\psi_\eta(z,\mathbf{x})dz + i\psi_\eta(z,\mathbf{x}) \circ dB\left(\frac{\mathbf{x}}{\eta}, z\right), \quad (z,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^D$$

where B is a Wiener process with correlation function

$$\mathbb{E}\{B(\mathbf{x},z)B(\mathbf{y},z')\} = R(\mathbf{x}-\mathbf{y}) z \wedge z', \qquad R \in L^1 \cap L^\infty.$$

The Wigner transform satisfies the random Wigner equation

$$(dW_\eta + \mathbf{k}\cdot\nabla_{\mathbf{x}} W_\eta\, dz)(z,\mathbf{x},\mathbf{k}) =$$
$$\frac{i}{(2\pi)^D}\int_{\mathbb{R}^d} d\mathbf{p}\, e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\eta}} \left(W_\eta(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}) - W_\eta(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\right) \circ d\hat{B}(\mathbf{p},z),$$

with $\hat{B} = \mathcal{F}_{\mathbf{x}\to\mathbf{p}} B$.

The key point the is that J_η satisfies a closed-form equation :

$$\Big(\frac{\partial}{\partial z} + \mathcal{T}_2 + 2R(0) - \mathcal{Q}_2 - \mathcal{K}_\eta\Big) J_\eta = \mathcal{K}_\eta a_\eta \otimes a_\eta \quad \text{on } \mathbb{R}^{2D} \times \mathbb{R}^{2D},$$

$$J^\eta(z=0) = 0,$$

$$\mathcal{T}_2 = \mathbf{k}\cdot\nabla_{\mathbf{x}} + \mathbf{p}\cdot\nabla_{\mathbf{y}}, \qquad a_\eta = \mathbb{E}\{W_\eta\},$$

$$\mathcal{Q}_2 h = \int_{\mathbb{R}^{2D}} \Big(\hat{R}(\mathbf{k}-\mathbf{k}')\delta(\mathbf{p}-\mathbf{p}') + \hat{R}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{k}-\mathbf{k}')\Big) h(\mathbf{x},\mathbf{k}',\mathbf{y},\mathbf{p}') d\mathbf{k}' d\mathbf{p}',$$

$$\mathcal{K}_\eta h = \sum_{\epsilon_i,\epsilon_j=\pm 1} \epsilon_i \epsilon_j \int_{\mathbb{R}^D} \hat{R}(\mathbf{u}) e^{i\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{u}}{\eta}} h\left(\mathbf{x}, \mathbf{k}+\epsilon_i\frac{\mathbf{u}}{2}, \mathbf{y}, \mathbf{p}+\epsilon_j\frac{\mathbf{u}}{2}\right) d\mathbf{u}.$$

Theorem [Bal - P., CPDE 2010] For all $\alpha \in [0,1]$, $J_\eta \to 0$ weakly in $\mathcal{S}'$ and more precisely

$$J_\eta = g_1(\eta,\alpha,D)J_1 + g_2(\eta,\alpha,D)J_2 + r_\eta,$$

where r_η is negligible in $\mathcal{S}'$ compared to the first 2 terms. J_1 and J_2 satisfy 4-transport equations of the form

$$\Big(\frac{\partial}{\partial z} + \mathcal{T}_2 + 2R(0) - \mathcal{Q}_2\Big) J_i = S_i(\alpha,D) \quad \text{on } \mathbb{R}^{2d}\times\mathbb{R}^{2d},$$

$$J_i(z=0) = J_i^0(\alpha,D),$$

The data for J_1 are linear with respect to $\hat{R}$ (single scattering) while that of J_2 are quadratic (double scattering). Higher-order scattering events therefore produce negligible instabilities compared to that of the simple and double scattering.

The single scattering contribution dominates when $\alpha > \frac{2}{3}$.

The Itô-Schrödinger regime

When $\alpha > \frac{1}{2}$, $S_i = 0 \Rightarrow$ instabilities generated by an initial condition

(This is consistant with a recent result of Komorowski-Ryzhik)

When $\alpha \leq \frac{1}{2}$, $J_i^0 = 0 \Rightarrow$ instabilities generated by a source term.

Most stable configuration : $\alpha = 0 \Rightarrow g_2 = \eta^D$.

Least stable configuration : $\alpha = 1 \Rightarrow g_1 = \eta$.

We also have the following result : consider a test function of the form

$$\varphi_{s_1}(\mathbf{x}, \mathbf{k}) = \frac{1}{\eta^{s_1 D}} \varphi\Big(\frac{\mathbf{x}}{\eta^{s_1}}, \mathbf{k}\Big).$$

Then, for $\alpha = 0$

$$\langle J_\eta, \varphi_{s_1} \otimes \varphi_{s_1} \rangle = \mathcal{O}\big(\eta^{d(1-s_1)}\big).$$

Hence, when the initial condition has a support of order one compared to η, statistical stability holds when the detector is of size $\varepsilon^{-s_1 D}$, $s_1 < 1$.

The Schrödinger equation with time-independent potential reads in the weak coupling regime :

$$\left(i\eta \frac{\partial}{\partial z} + \frac{\eta^2}{2}\Delta_{\mathbf{x}} + \sqrt{\eta} V\left(\frac{\mathbf{x}}{\eta}\right)\right)\psi_\eta(z,\mathbf{x}) = 0, \quad z > 0, \quad \mathbf{x} \in \mathbb{R}^D,$$

$\psi_\eta(0,\mathbf{x}) = \psi_\eta^0(\mathbf{x})$ bounded in L^2.

We know from Erdös-Yau (and Spöhn) that when $D \geq 2$, $\mathbb{E}\{W_\eta\}$ converges to a solution to a radiative transfer equation. Up to our knowledge, the convergence of the whole process W_η is an open problem.

Nevertheless, it is possible to analyze the scintillation of the single and double scattering contributions (that are expected to be dominant in some regimes according the Itô-Schrödinger case) and also obtain optimal estimates.

The random Wigner equation reads :

$$\left(\frac{\partial}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\right) W_\eta(z,\mathbf{x},\mathbf{k}) =$$
$$\frac{i}{\sqrt{\eta}(2\pi)^D}\int_{\mathbb{R}^d} d\mathbf{p}\hat{V}(\mathbf{p})\, e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\eta}} \left(W_\eta(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}) - W_\eta(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\right)$$

We then formally expand W_η in terms of multiple scattering and only retain the terms at most quadratic in $\hat{R}$. The corresponding scintillation is

$$J_\eta = J_\eta^S + J_\eta^D.$$

We consider again an initial condition of the form

$$W_{\eta 0}(\mathbf{x},\mathbf{k}) = \frac{1}{\eta^d} W_0\left(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha}, \frac{\mathbf{k}-\mathbf{k}_0}{\eta^{1-\alpha}}\right).$$

Examples of results

- Again, the single scattering contribution dominates when $\alpha > \frac{2}{3}$.
- Single scattering
  - Most stable configuration : $\alpha = 0 \Rightarrow \eta^{d+1}$ (to be compared with η^{d+2} for Itô-S.).
  - Least stable configuration : $\alpha = 1$ ($\alpha \geq \frac{1}{2}$ in 1D) $\Rightarrow \eta$ (same as Itô-S.)
  - The single scattering is always stable, even in 1D
  - Localization is generated in 1D by higher order scattering events
- Double scattering
  - Least stable configuration for $D \geq 2$: $\eta^{\frac{d}{2}}$.
  - Least stable configuration for $D = 1$: scintillation of order 1 when $\alpha = 0$
  - Instability in 1D when $\alpha = 0$
  - Stability in 1D when $\alpha > 0$

Statistical stability

The conclusion of the theoretical analysis is that the Wigner function (i.e. the energy) is self-averaging in many configurations for reduced models of propagation.

Besides, the instabilities critically depend on some parameters of the problem as the initial conditions or the detector.

Owing these results, we (formally) assume that they can be extrapolated to the wave equation and then perform reconstructions.

Imaging results

Imaging procedure (in 2D) :

- We measure the energy $\mathcal{E}^\varepsilon$ for one realization of the random medium.
- We assume we can form differential measurements (ie both in absence and presence of the inclusion)
  - this removes a substantial amount of noise
  - imaging with direct measurements only is possible if the inclusion is large enough or if the mean free path is important
- We find the deterministic transport prediction that best fits the (weakly) random measurements.
- The procedure is carried out over 20 realizations to quantify the variance of the reconstructed parameters.

Imaging results

- **First step : estimation of the transport parameters**

For simplicity, we use a random medium associated with an isotropic cross-section, and an initial condition with only one frequency content.

- one parameter to reconstruct, the mean free time Σ^{-1}.
- we minimize over Σ^{-1}

$$\int_0^T |\mathcal{E}^\varepsilon(t) - \mathcal{A}(t)|^2 dt$$

$$\mathcal{E}^\varepsilon(t) = \int_{\mathcal{D}} \mathcal{E}^\varepsilon(t,\mathbf{x})d\mathbf{x} \qquad \mathcal{A}(t) = \int_{\mathcal{D}}\int_{S^1} a(t,\mathbf{x},|\mathbf{k}_0|\hat{\mathbf{k}})d\mathbf{x}d\hat{\mathbf{k}}$$

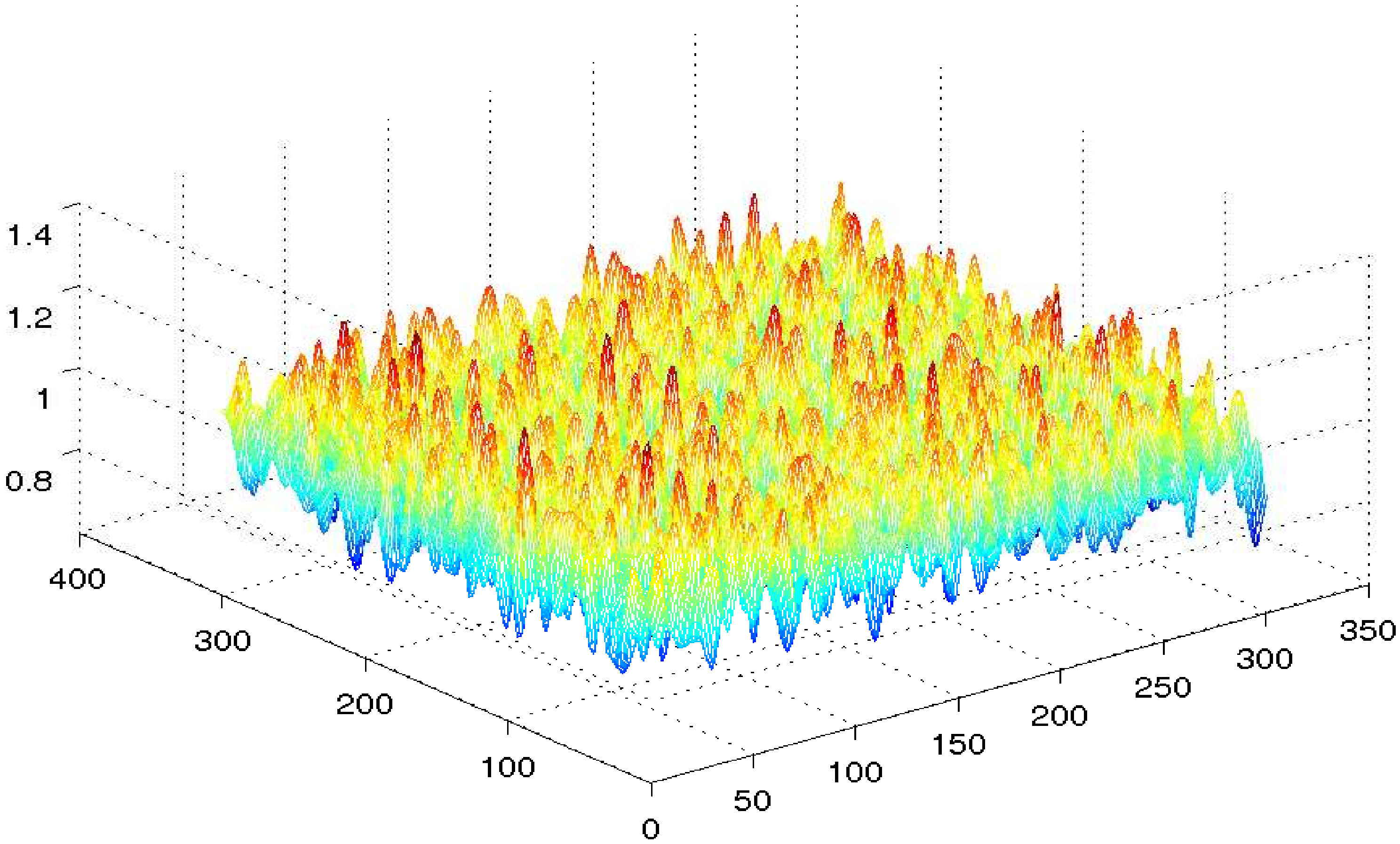
- **Second step : reconstruction of the inclusion**

We suppose the inclusion is spherical and perfectly reflecting.

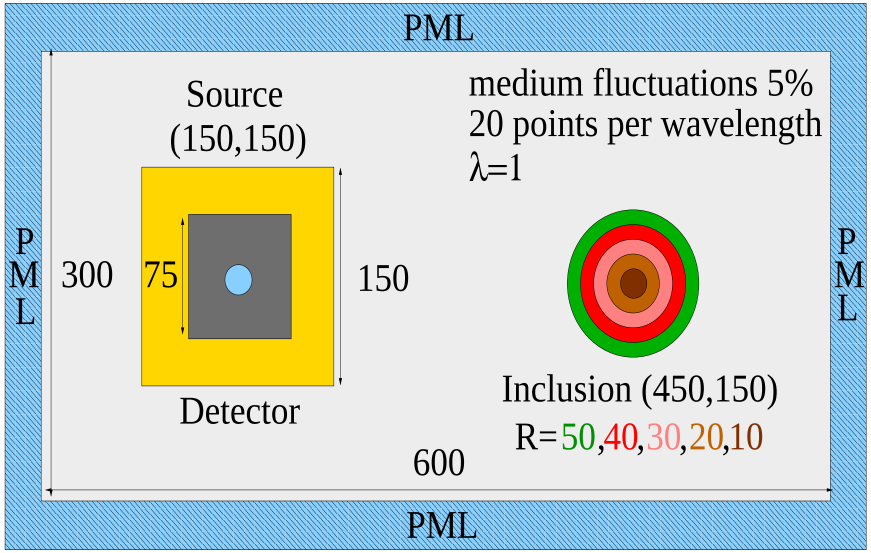
- we minimize over (X, Y, R)

$$\int_0^T |\delta\mathcal{E}^\varepsilon(t) - \delta\mathcal{A}[X,Y,R](t)|^2 dt$$

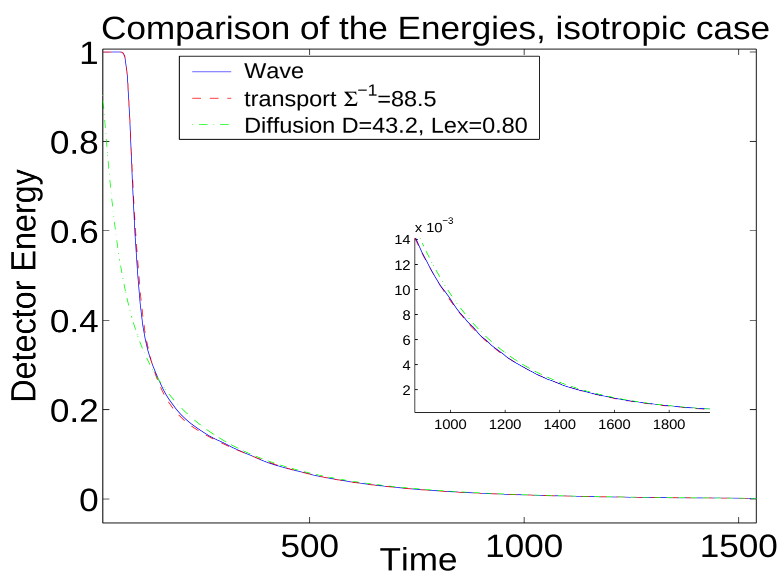
Example of random media

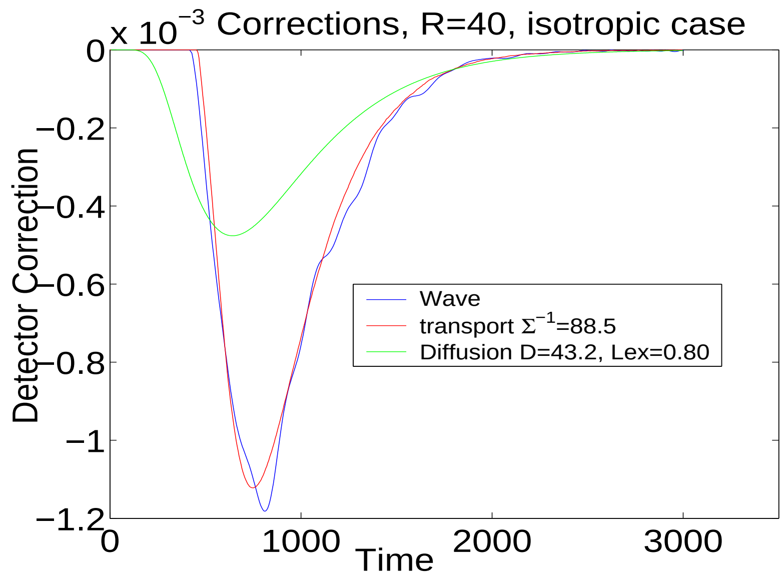


Validity of transport



- $\lambda = 1$
- Isotropic cross section, mean free path ≈ 90
- Isotropic initial condition with only one frequency content

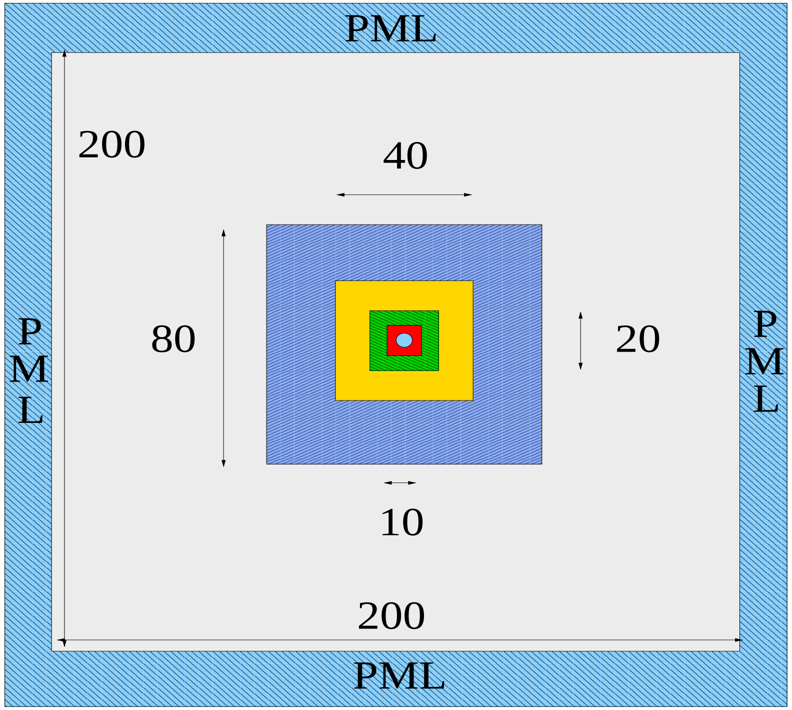


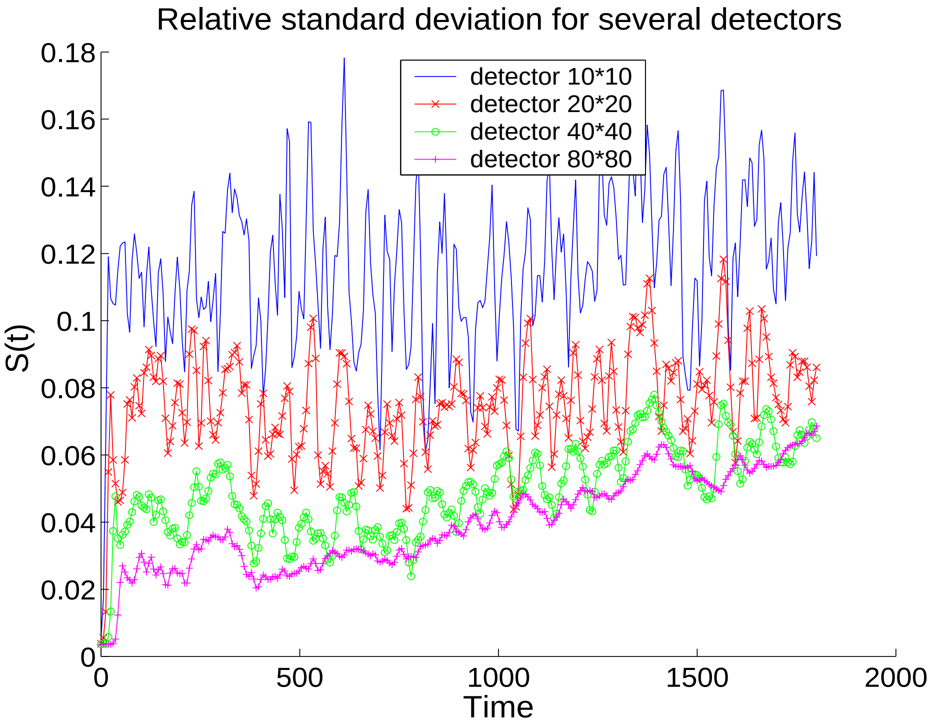


Statistical Stability

Wavelength $\lambda = 1$, mean free path ≈ 40, isotropic cross section.

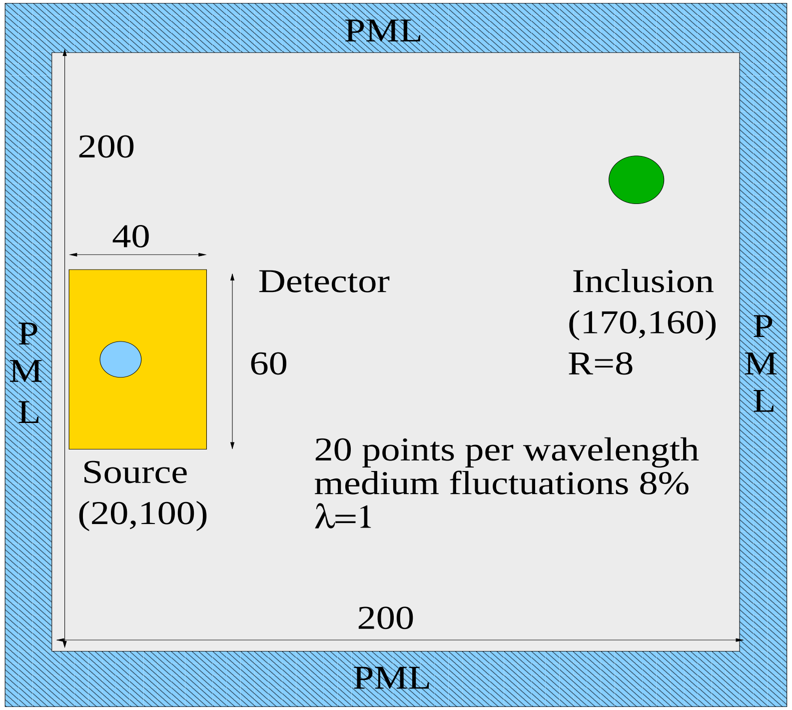
Display of $S(t) = \dfrac{\sigma\{\mathcal{E}^\varepsilon\}(t)}{\mathbb{E}\{\mathcal{E}^\varepsilon\}(t)}$ for 20 realizations.

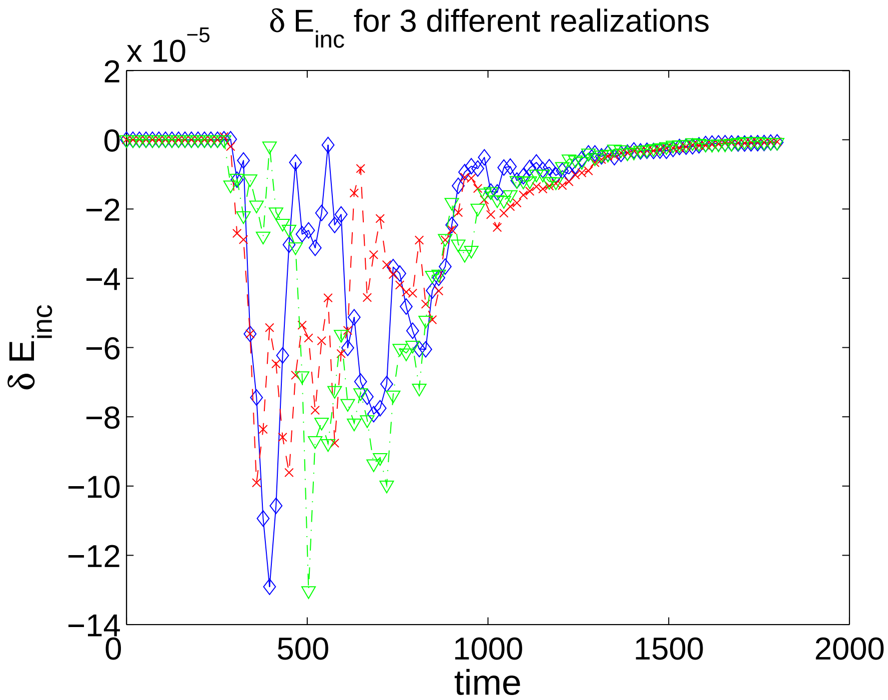




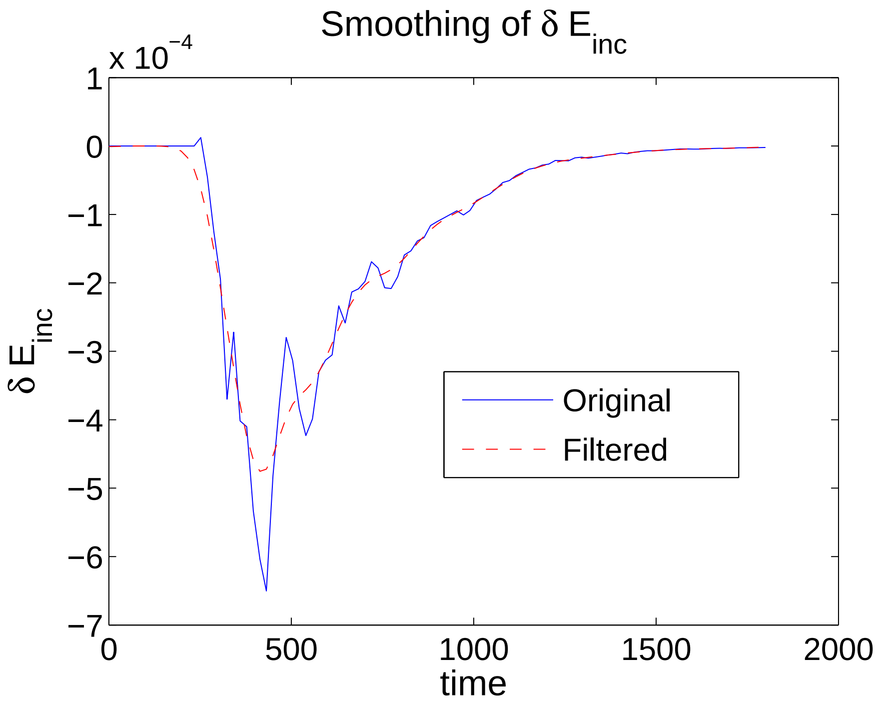
Measurements (mean free path ≈ 40)

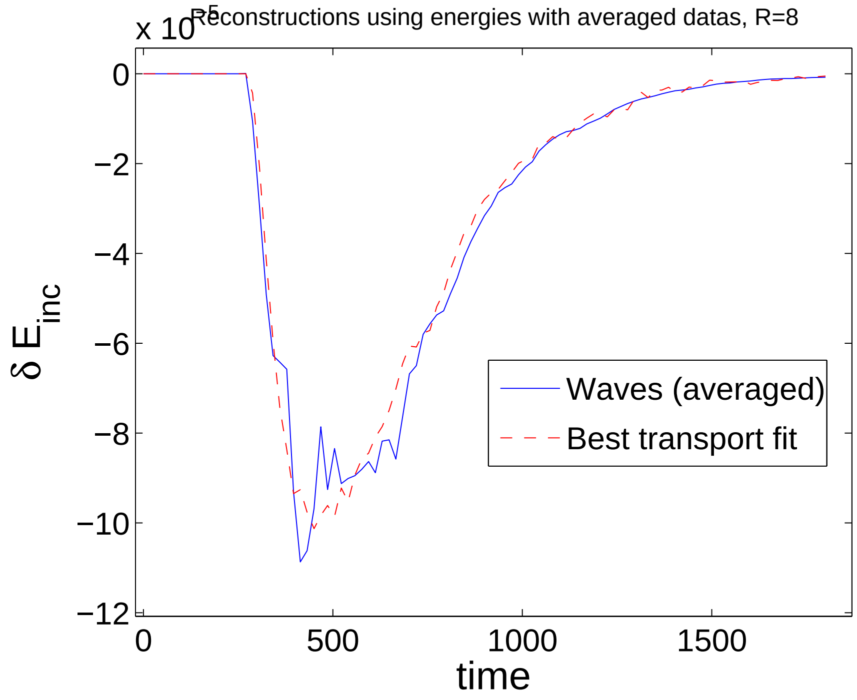
$\delta\mathcal{E}_{\text{inc}} =$ differential measurements of the wave energy





Measurements (mean free path ≈ 40)

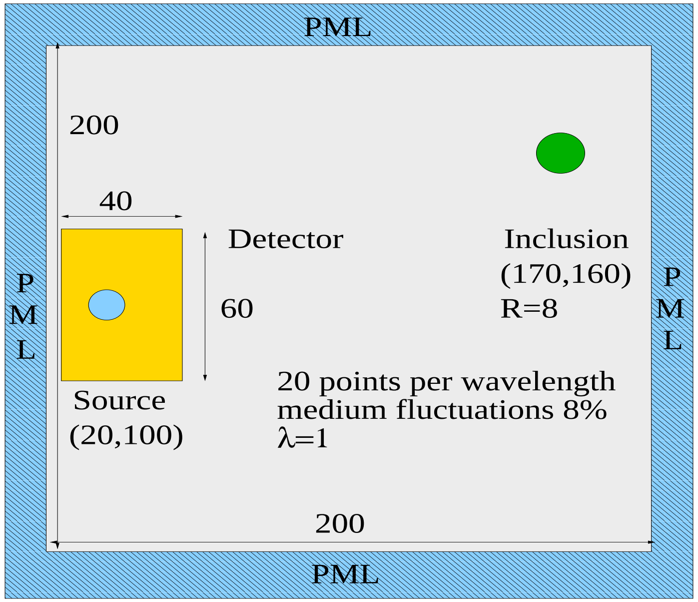


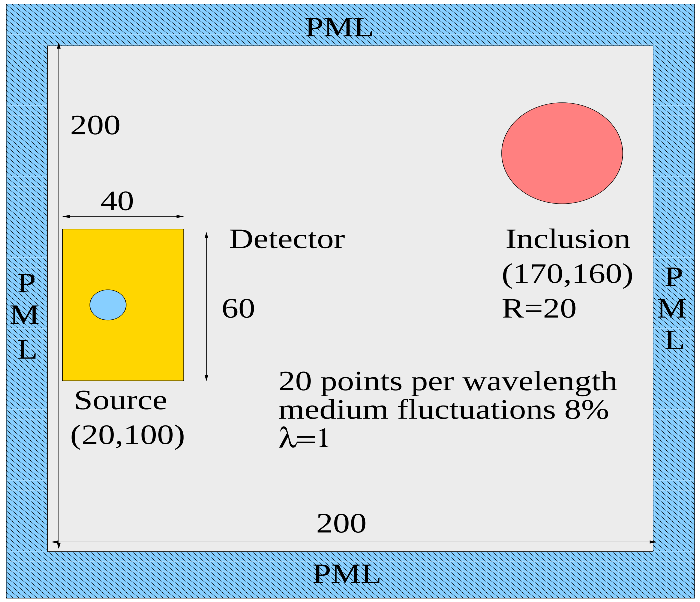


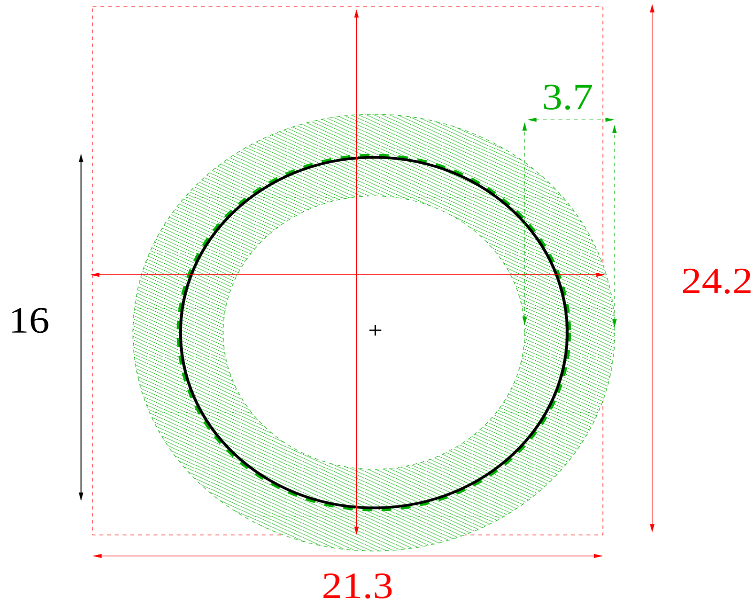
Exact parameters are $(X, Y, R) = (170, 160, 8)$.

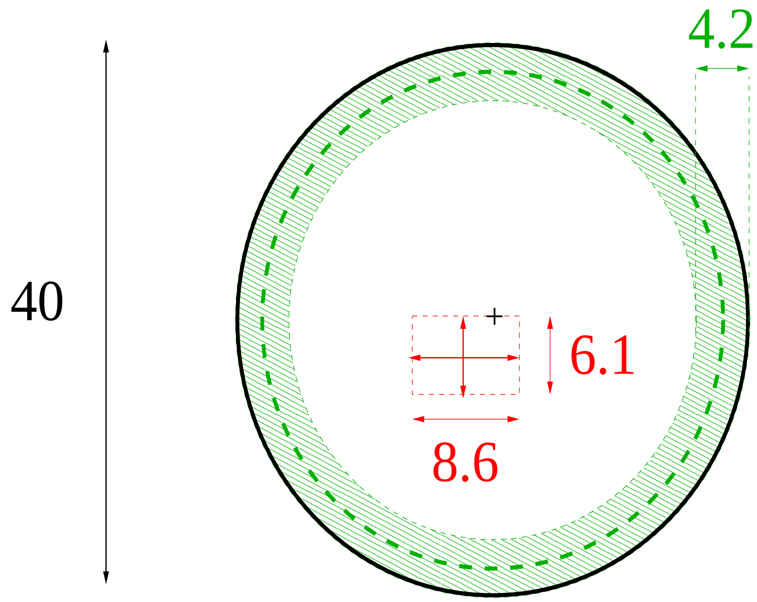
Reconstructed parameters with averaged datas $(168.6, 157.9, 7.6)$.

Reconstructions (mean free path ≈ 40)









Improving the stability

The stability can be improved if one is able to form the correlation of the wavefield in presence and in absence of the inclusion :

$$\mathcal{C}^{\varepsilon}(t,\mathbf{x}) = \frac{1}{2}\Big(\kappa_1^{\frac{1}{2}}\kappa_2^{\frac{1}{2}}(\mathbf{x})p_{\varepsilon}^{1}(t,\mathbf{x})\,p_{\varepsilon}^{2}(t,\mathbf{x}) + \rho_0\mathbf{v}_{\varepsilon}^{1}(t,\mathbf{x})\cdot\mathbf{v}_{\varepsilon}^{2}(t,\mathbf{x})\Big)\,.$$

In this case, the corresponding transport solution satisfies Dirichlet boundary conditions at the inclusion boundary rather than specular conditions for the energies.

Formal calculations then show that, in the transport regime, the correlation should be more stable than the energy.

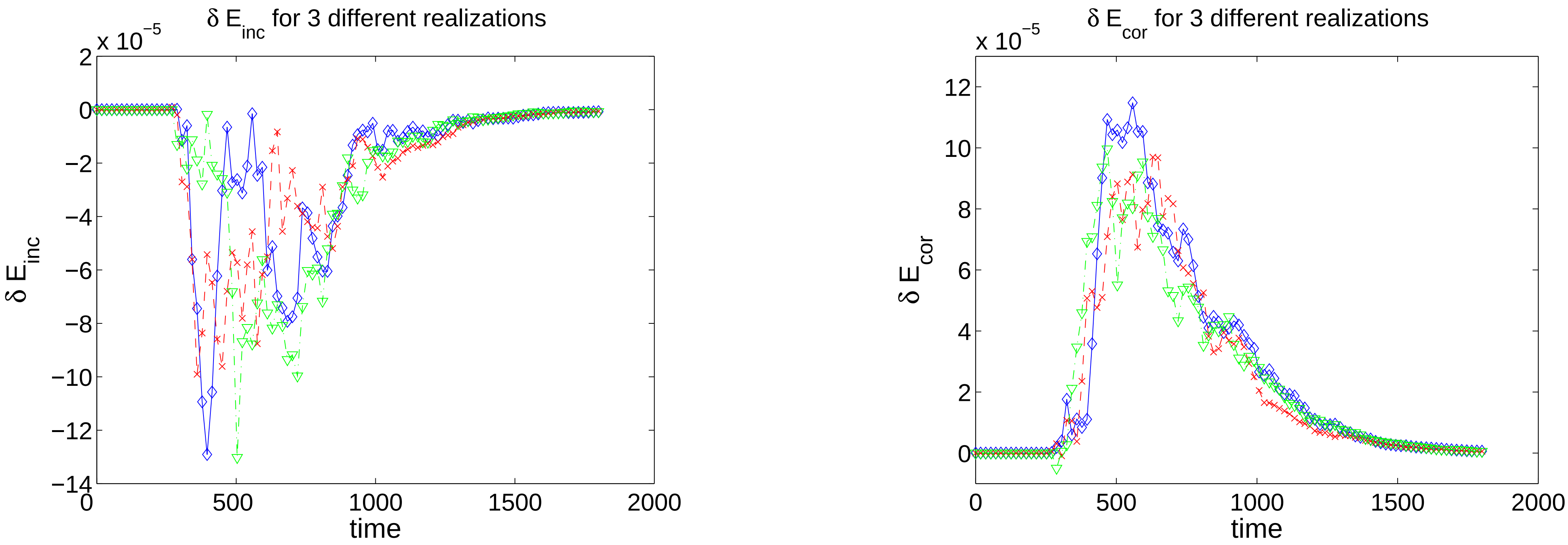
In the diffusion regime, we have, for $d \geq 3$:

$$\delta E^{\varepsilon} = \mathcal{O}(R^{d}), \qquad \delta C^{\varepsilon} = \mathcal{O}(R^{d-2})$$

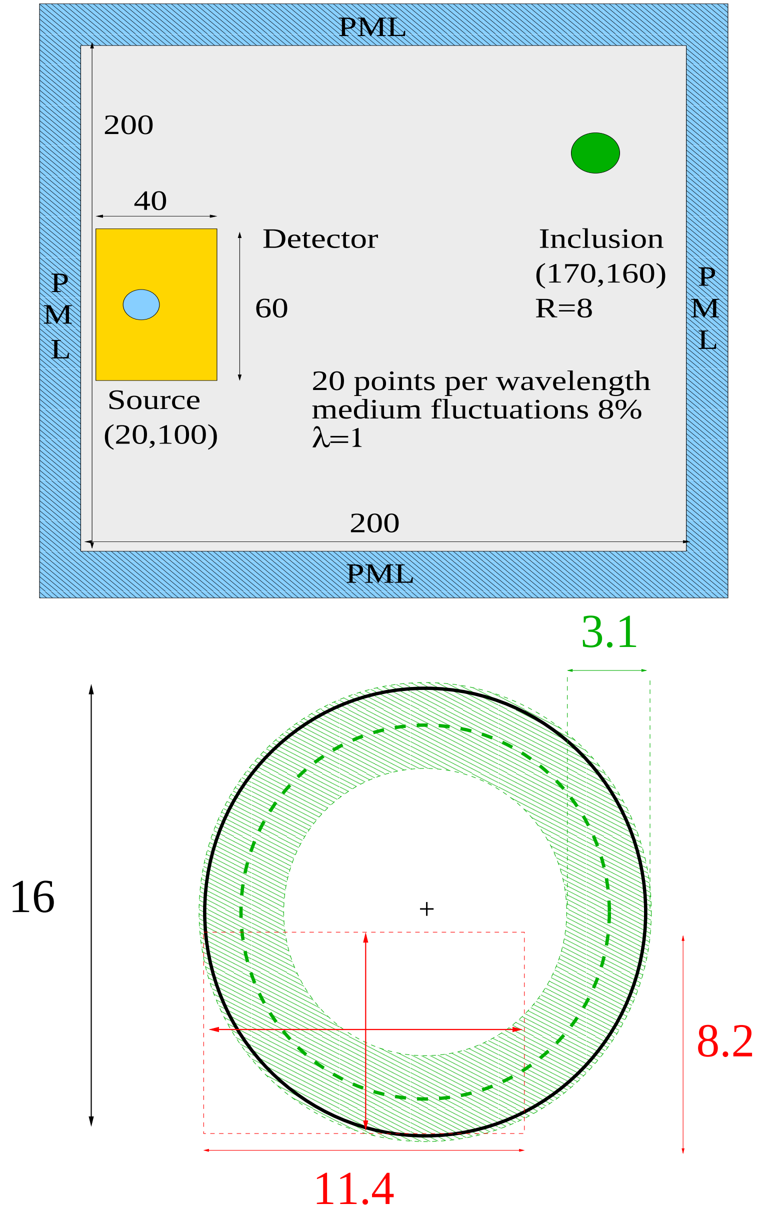
so that

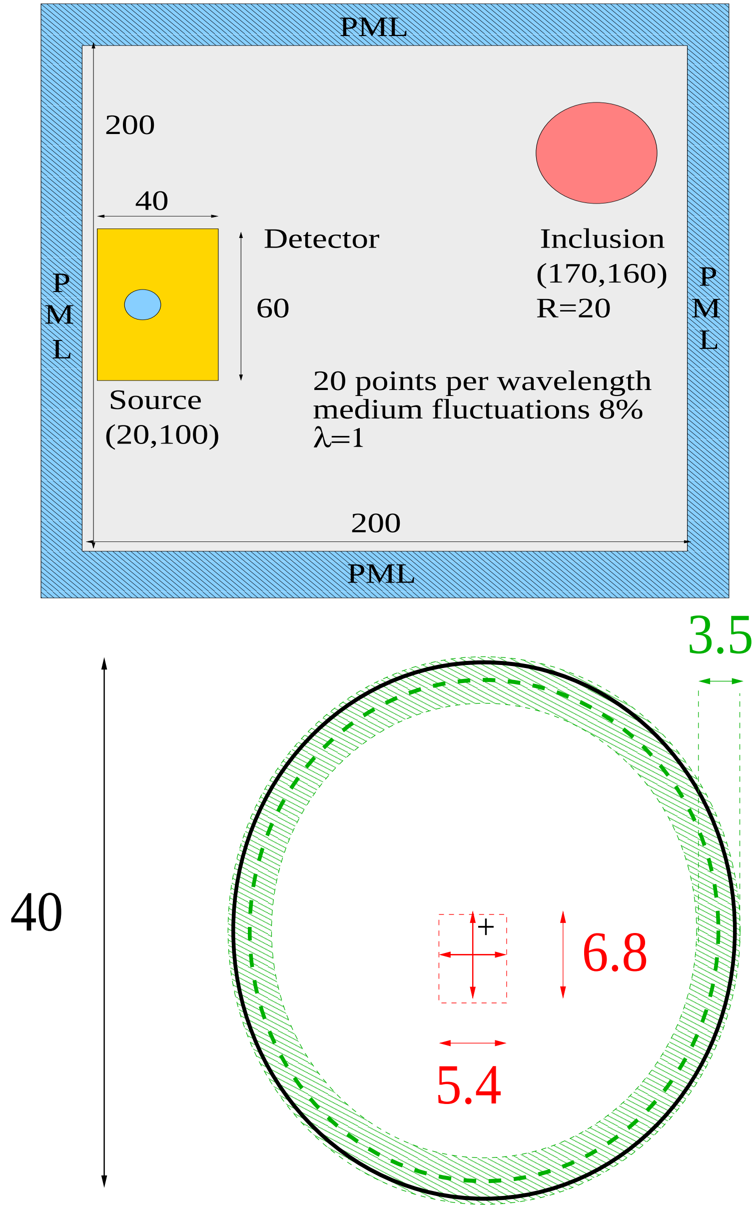
$$\delta E^{\varepsilon} \ll \delta C^{\varepsilon}$$

Correlations (mean free path ≈ 40)

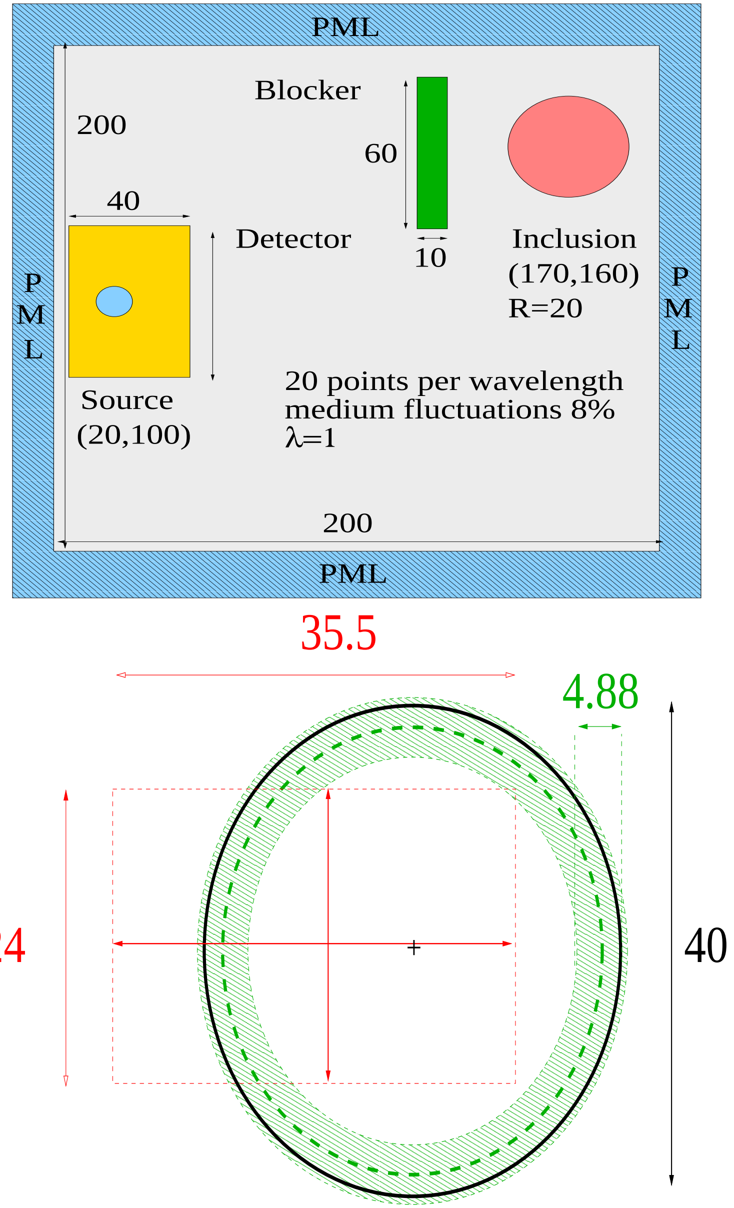


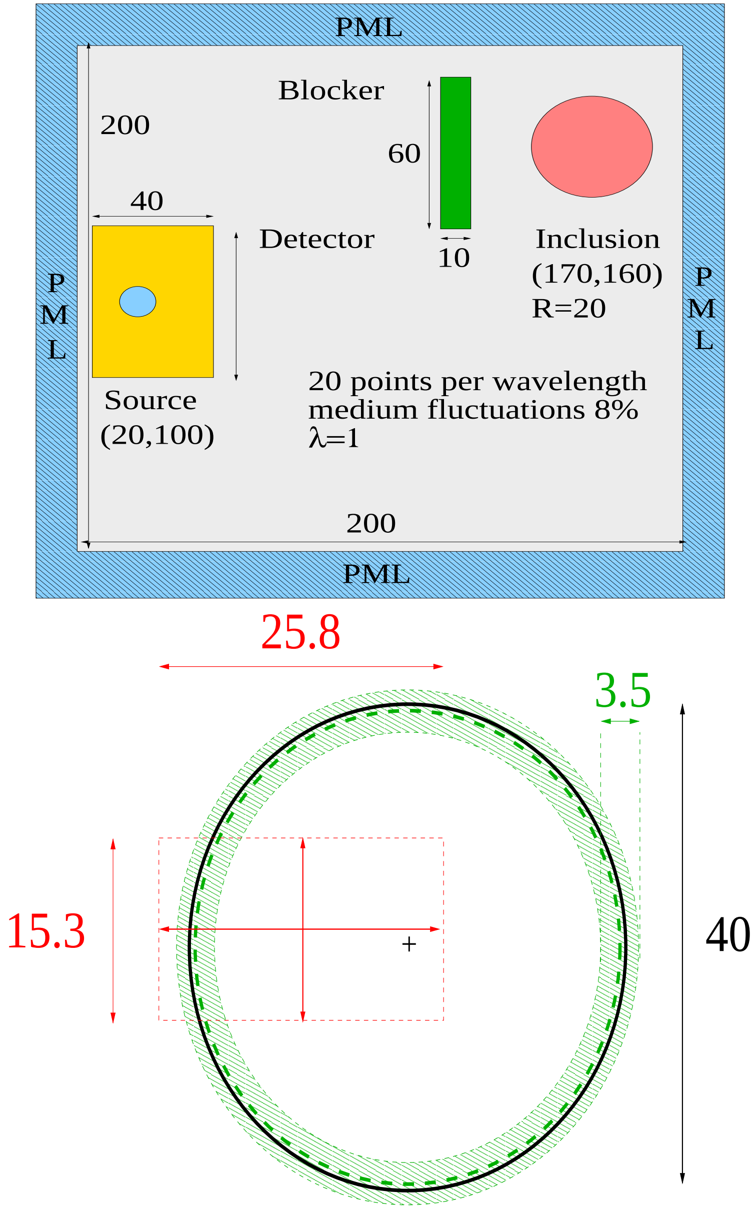
Reconstructions (mean free path ≈ 40)





Reconstructions (mean free path ≈ 40)





Conclusion

- Models for imaging in strongly heterogeneous environments were proposed and validated
- They consist in solving an inverse transport problem rather than imaging using a wave description
- Quantifying their precision requires a careful analysis of the limit random wave equation $\longrightarrow$ transport equation
- In simplified settings some optimal rates of convergence were obtained and the corrector was characterized

Some open questions

- Kinetic limit for the wave equation
- Comparison transport-based and interferometry methods