Linear Algebra and Association Schemes

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Outline

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Facing up to Association Schemes

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Schur Idempotents

An association scheme $\mathcal{A}$ consists of a set $A_0, \dots, A_d$ of 01-matrices such that:

1. $A_0 = I$ and $\sum_i A_i = J$.
2. $A_i^T \in \mathcal{A}$ for all i.
3. For all i and j the product $A_i A_j$ lies in the span $\mathbb{R}[\mathcal{A}]$ of the matrices in $\mathcal{A}$.
4. $A_i A_j = A_j A_i$ for all i and j.
5. $A_i \circ A_j = \delta_{i,j} A_i$.

Note that $\mathbb{R}[\mathcal{A}]$ is also closed under Schur multiplication, by (5). We call $\mathbb{R}[\mathcal{A}]$ the Bose-Mesner algebra of the scheme.

Symmetric Schemes

The scheme is symmetric if each matrix A_i is symmetric; this is the only case we will consider here. Hence we can view $A_1, \ldots, A_d$ as the adjacency matrices of graphs with common vertex set V. We set $v = |V|$.

Matrix Idempotents

The span $\mathbb{R}[\mathcal{A}]$ is a commutative semisimple algebra and therefore it has a basis of matrices $E_0, \ldots, E_d$ such that

1. $E_0 = \frac{1}{v} J$
2. $\overline{E_i} \in \{E_0, \ldots, E_d\}$ for all i.
3. For all i and j the product $E_i \circ E_j$ lies in $\mathbb{R}[\mathcal{A}]$.
4. $E_i E_j = \delta_{i,j} E_i$.

The Change of Basis Matrix

There are scalars $p_i(j)$ such that

$$A_i = \sum_{j=0}^{d} p_i(j) E_j.$$

The change of basis matrix is denoted by P and is called the matrix of eigenvalues of the scheme.

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The name is well chosen because, since $E_j E_r = \delta_{j,r} E_r$, we have

$$A_i E_r = p_i(r) E_r$$

and so $p_i(r)$ is an eigenvalue of A_i.

Outline

The Inner Product

If $\mathcal{M}$ is a complex vector space of matrices, we define an inner product by

$$\langle M, N\rangle = \operatorname{tr}(M^* N) = \operatorname{sum}(\overline{M} \circ N)$$

Orthogonal Bases

Since

$$\langle A_i, A_j \rangle = \operatorname{sum}(A_i \circ A_j) = \delta_{i,j} \operatorname{sum}(A_i) = \delta_{i,j} v v_i$$

and

$$\langle E_i, E_j \rangle = \operatorname{tr}(E_i E_j) = \delta_{i,j} \operatorname{tr}(E_i) = \delta_{i,j} m_i$$

we have two orthogonal bases for $\mathbb{R}[\mathcal{A}]$.

Projections

The Bose-Mesner algebra $\mathbb{R}[\mathcal{A}]$ is a subspace of the space of $v \times v$ matrices and so we may use any orthogonal basis of $\mathbb{R}[\mathcal{A}]$ to compute the orthogonal projection $\widehat{M}$ of a matrix M onto $\mathbb{R}[\mathcal{A}]$:

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$$\widehat{M} = \sum_{i=0}^{d} \frac{\langle M, A_i \rangle}{\langle A_i, A_i \rangle} A_i = \sum_{j=0}^{d} \frac{\langle M, E_j \rangle}{\langle E_j, E_j \rangle} E_j.$$

An Application

Suppose S is a subset of the vertices of $\mathcal{A}$ with characteristic vector x. Set $M = xx^T$. We compute $\widehat{M}$.

- $\langle xx^T, A_i \rangle = \operatorname{tr}(xx^T A_i) = \operatorname{tr}(x^T A_i x) = x^T A_i x$.

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Theorem

If $M = xx^T$ *then*

$$\widehat{M} = \sum_{i=0}^{d} \frac{x^T A_i x}{vv_i} A_i = \sum_{i=0}^{d} \frac{x^T E_j x}{m_j} E_j.$$

Outline

The Identity

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Theorem (Koppinen)

$$\sum_i \frac{1}{vv_i} A_i \otimes A_i = \sum_j \frac{1}{m_j} E_j \otimes E_j =: \mathcal{K}.$$

Using Koppinen

We have

$$(xx^T \otimes I)\mathcal{K} = \sum_i \frac{1}{vv_i}(xx^T A_i) \otimes A_i = \sum_j \frac{1}{m_j}(xx^T E_j) \otimes E_j$$

and if we apply $\mathrm{tr} \otimes I$ to each side, we get:

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The Clique-Coclique Bound

Suppose S and C are subsets of V with characteristic vectors x and y respectively and

$$(x^T A_i x)(y^T A_i y) = 0. \qquad (i = 1, \ldots, d)$$

Then on one hand $(x \otimes y)^T \mathcal{K}(x \otimes y)$ is equal to

$$\sum_{i=0}^{d} \frac{(x^T A_i x)(y^T A_i y)}{v v_i} = \frac{(x^T A_0 x)(y^T A_0 y)}{v} = \frac{|C|\,|S|}{v}$$

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and, on the other it is equal to

$$\sum_{i=0}^{d} \frac{(x^T E_j x)(y^T E_j y)}{m_j} \geq (x^T E_0 x)(y^T E_0 y) = \frac{|C|^2 |S|^2}{v^2}.$$

The Clique-Coclique Bound, ctd

Corollary

If C is a clique and S a coclique, then

$$|C|\,|S| \le v;$$

if equality holds then $(x^T E_j x)(y^T E_j y) = 0$ for $j = 1, \ldots, d$.

Orthogonality of Eigenvalues

If we multiply each version of $\mathcal{K}$ by $E_r \otimes E_s$ then recalling that $A_i E_r = p_i(r) E_r$, we get

$$\left(\sum_i \frac{p_i(r) p_i(s)}{v v_i} \right) E_r \otimes E_r = \delta_{r,s}\, E_r \otimes E_r$$

and hence

$$\sum_i \frac{p_i(r) p_i(s)}{v_i} = v \delta_{r,s}.$$

Outline

Projections Again

If $u_1, \ldots, u_m$ is an orthogonal basis for a subspace U, then orthogonal projection onto U is represented by

$$\sum_i \frac{1}{\langle u_i, u_i \rangle} u_i u_i^*.$$

Note that $u_i u_i^*$ is an element of $\mathrm{End}(U)$ and

$$\mathrm{End}(U) \cong U \otimes U^* \cong U \otimes U.$$

The Proof

Applying this to our pair of orthogonal bases, we get

$$\sum_i \frac{1}{vv_i} A_i \otimes A_i = \sum_j \frac{1}{m_j} E_j \otimes E_j.$$

An Interpretation

1. Orthogonal projection onto $\mathbb{R}[\mathcal{A}]$ is an endomorphism on the space $\mathcal{M}_{v\times v}$ of $v \times v$ real matrices.

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3. It is not unreasonable to denote this endomorphism by:
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4. $\mathcal{K} \in \mathrm{End}(\mathcal{M}_{v\times v})$.

A Stranger Interpretation

Matrix and Schur multiplication are linear maps

$$\mathcal{M}_{v\times v}\otimes\mathcal{M}_{v\times v}\to\mathcal{M}_{v\times v}$$

which we denote by μ and σ. As they are linear, they have adjoint maps (coproducts) μ^* and σ^* respectively from $\mathcal{M}^*$ to $\mathcal{M}^*\otimes\mathcal{M}^*$. Koppinen says that

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Pseudocyclic Schemes

Definition

An association scheme on d classes is pseudocyclic if

$$m_1 = \cdots = m_d$$

and

$$v_1 = \cdots = v_d.$$

Examples: Cyclotomic Schemes

Let $\mathbb{F}$ be a finite field and let R be a subgroup of the multiplicative group of $\mathbb{F}$ (such that $-1 \in R$).

Definition

The vertices of the cyclotomic scheme are the elements of $\mathbb{F}$, two distinct vertices u and v are adjacent in the i-th graph of the scheme if their difference is in the i-th coset of R.

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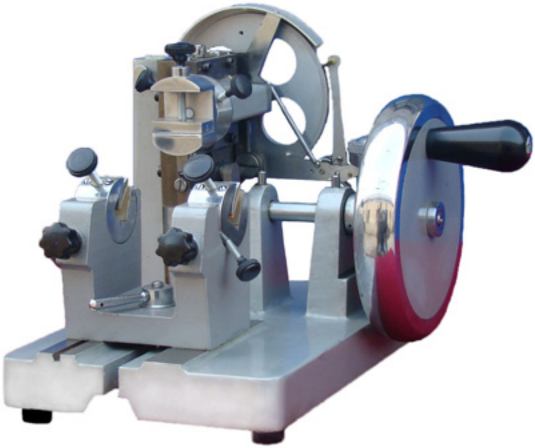
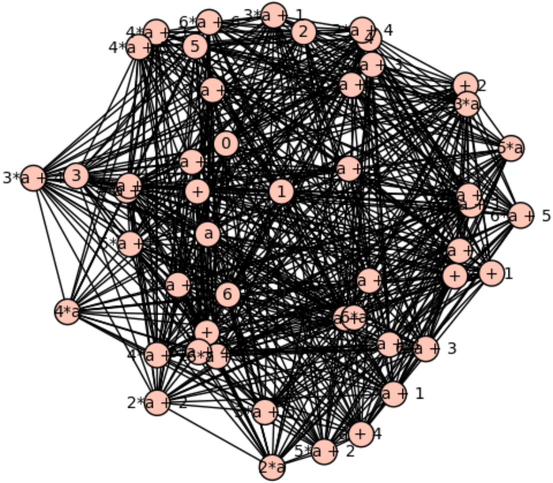
If $|\mathbb{F}| \equiv 1$ mod 4 and R is the set of non-zero squares in $\mathbb{F}$, then the graphs in the scheme are the Paley graph and its complement.

No Cyclotome Picture, but...



OK, An Actual Example: Paley(49)



Pseudocyclic to Strongly Regular

Theorem

If $\mathcal{A}$ is a pseudocyclic scheme with d classes, then

$$\sum_{i=1}^{d} A_i \otimes A_i$$

is the adjacency matrix of a strongly regular graph.

Proof.

Set $m = (v-1)/d$. Then

$$\mathcal{K} = \frac{1}{v} I + \frac{1}{vm} \sum_{i=1}^{d} A_i^{\otimes 2} = \frac{1}{v} J + \frac{1}{m} \sum_{j=1}^{d} E_j^{\otimes 2}.$$

□

Outline

A Quantum of Knowledge

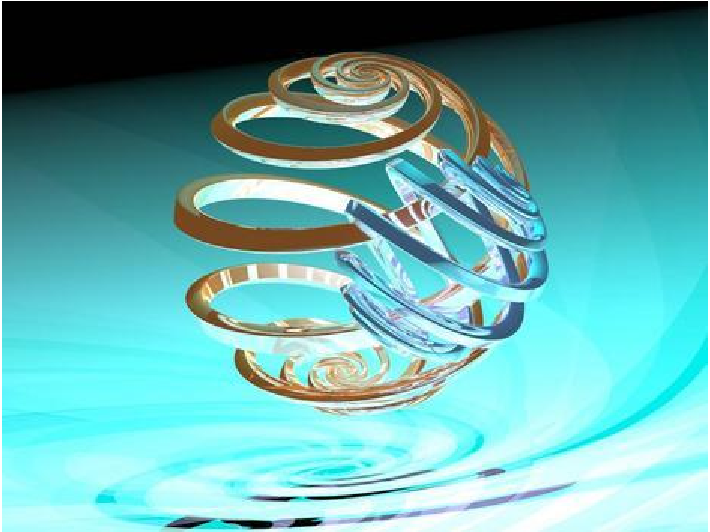
Let A be the adjacency matrix of some graph X. We work with a quantum system whose evolution is specified by the matrix $H_X(t)$, where

$$H_X(t) := \exp(itA)$$

This matrix arises in the theory of continuous quantum walks. It is unitary and symmetric (so $\overline{H_X(t)} = H_X(-t)$).

Grover, not Shor

Entanglement



Example

Example

If $X = K_2$, then

$$H_X(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}.$$

Some Probability Distributions

Since $H(t)$ is unitary the Schur product

$$H(t) \circ \overline{H(t)} = H(t) \circ H(-t)$$

is a doubly stochastic matrix.

The Average Mixing Matrix

Definition

The average mixing matrix is

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T H(t) \circ H(-t)\, dt.$$

Using the Spectral Decomposition

If A has spectral decomposition

$$A = \sum_r \theta_r E_r$$

then $H_X(t)$ has spectral decomposition

$$H_X(t) = \sum_r e^{it\theta_r} E_r$$

and

$$H_X(t) \circ H_X(-t) = \sum_r E_r^{\circ 2} + 2 \sum_{r<s} \cos((\theta_r - \theta_s)t) E_r \circ E_s.$$

A Sum of Squares

Theorem

The average mixing matrix is equal to

$$\sum_r E_r^{\circ 2}.$$

Average Mixing is not Uniform

Theorem (Godsil)

If X is a graph on n vertices and its average mixing matrix is $n^{-1}J$, then $n \leq 2$.

Paths

Theorem (Godsil)

Let $T = T_n$ be the permutation matrix such that $Te_i = e_{n+1-i}$ for all i. The average mixing matrix for the path P_n is

$$\frac{1}{2n+2}(2J + I + T).$$

Rationality

Theorem (Godsil)

The average mixing matrix of a graph is rational.

An Issue

- If D is the discriminant of the minimal polynomial of A then $D^2\widehat{M}_X$ is integral. If the eigenvalues of A are simple then $D\widehat{M}_X$ is integral.

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- If D is the discriminant of the minimal polynomial of A then $D^2\widehat{M}_X$ is integral. If the eigenvalues of A are simple then $D\widehat{M}_X$ is integral.
- There's a graph on seven vertices with discriminant

$$540034607936 = 2^6 \times 8438040749.$$

Odd Cycles are Almost Uniform

Theorem (Godsil)

If n is odd then the average mixing matrix for the cycle C_n is

$$\frac{n-1}{n^2}J + \frac{1}{n}I.$$

From Tensor to Schur

Since

$$\frac{1}{v}I + \frac{1}{vm}\sum_{i=0}^{d} A_i \otimes A_i = \frac{1}{v}J + \frac{1}{m}\sum_{j=0}^{d} E_j \otimes E_j$$

it follows that

$$\frac{1}{v}I + \frac{1}{vm}\sum_{i=0}^{d} A_i \circ A_i = \frac{1}{v}J + \frac{1}{m}\sum_{j=0}^{d} E_j \circ E_j$$

Average Mixing on Pseudocyclic Graphs

If X is a pseudocyclic graph on v vertices with valency $m = (v-1)/d$ then

$$\sum_{i=0}^{d} \frac{1}{vv_i} A_i \circ A_i = \frac{1}{v}\left(I + \frac{1}{m}(J-I)\right)$$

and

$$\sum_{j=0}^{d} \frac{1}{m_j} E_j \circ E_j = \frac{1}{v^2} J + \frac{1}{m} \sum_{r=1}^{d} E_r^{\circ 2}$$

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Theorem (Godsil)

The average mixing matrix of a pseudocyclic graph X with valency m on n vertices is:

$$\frac{n-m+1}{n^2} J + \frac{m-1}{n} I.$$

The End(s)