Open Problems Concerning Automorphism Groups of Projective Planes

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Projective Planes

A projective plane is a point-line incidence structure such that

- every pair of distinct points lies on a common line;
- every pair of distinct lines meets in a common point;
- there exists a quadrangle (four points, no three of which are collinear).

There exists a cardinal number n (finite or infinite), called the order of the plane, such that

- every line has $n+1$ points;
- every point is on $n+1$ lines;
- there are n^2+n+1 points and the same number of lines.

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Known planes of small order

Number of planes up to isomorphism (i.e. collineations):

n	number of planes of order n
2	1
3	1
4	1
5	1
7	1
8	1
9	4
11	$\geqslant 1$
13	$\geqslant 1$

n	number of planes of order n
16	$\geqslant 22$
17	$\geqslant 1$
19	$\geqslant 1$
23	$\geqslant 1$
25	$\geqslant 193$
27	$\geqslant 13$
29	$\geqslant 1$
$\dots$	$\dots$
49	$> 280{,}000$

`pzip`: A compression utility for finite planes

Storage requirements for a projective plane of order *n*:

n	size of line sets	size of MOLS	`gzipped` MOLS	`pzip`
11	5 KB	1.3 KB	0.2 KB	0.06 KB
25	63 KB	15 KB	9 KB	0.9 KB
49	550 KB	110 KB	81 KB	6 KB

See http://www.uwyo.edu/moorhouse/pzip.html

The Classical Planes

Let F be a field. Denote by F^3 a 3-dimensional vector space over F.

The classical projective plane $P^2(F)$ has as its points and lines the subspaces of F^3 of dimension 1 and 2, respectively. Incidence is inclusion. The order of the plane is $|F|$, finite or infinite.

The automorphism group of $P^2(F)$ is $P\Gamma L_3(F)$, which acts 2-transitively on points, and transitively on ordered quadrangles. No known planes have as much symmetry as the classical planes.

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Let Π be a projective plane, and let $G = \mathrm{Aut}(\Pi)$.

Theorem (Ostrom-Dembowski-Wagner)

In the finite case, Π is classical iff G is 2-transitive on points.

In the infinite case, there exist nonclassical planes whose automorphism group is 2-transitive on points (even transitive on ordered quadrangles).

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Subplanes

Consider a classical projective plane $\Pi = P^2(F)$.

Every quadrangle in Π generates a subplane isomorphic to $P^2(K)$ where K is the prime subfield of F (i.e. $\mathbb{F}_p$ or $\mathbb{Q}$, according to the characteristic of F).

Such a subplane is proper iff $[F : K] > 1$.

Subplanes

Open Question

Let Π be a finite projective plane in which every quadrangle generates a proper subplane. Must Π be classical? (necessarily of order p^r with $r \geqslant 2$)

The answer is known only in special cases:

If Π is a finite projective plane in which every quadrangle generates a subplane of order 2, then $\Pi \cong P^2(\mathbb{F}_{2^r})$ (Gleason, 1956).

If Π is a finite projective plane of order n^2 in which every quadrangle generates a subplane of order n, then $n = p$ and $\Pi \cong P^2(\mathbb{F}_{p^2})$ (Blokhuis and Sziklai, 2001 for n prime; Kantor and Penttila, 2010 in general).

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Point Orbits and Line Orbits

Consider a projective plane Π with automorphism group $G = \mathrm{Aut}(\Pi)$.

Theorem (Brauer, 1941)

In the finite case, G has equally many orbits on points and on lines.

Open Problem (attributed to Kantor)

In the general case, must G have equally many orbits on points and on lines?

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Orbits on n-tuples of Points

In the classical case $\Pi = P^2(F)$, G has

- 1 orbit on points;
- 1 orbit on ordered pairs of distinct points;
- 2 orbits on ordered triples of distinct points;
- $O(|F|)$ orbits on ordered 4-tuples of distinct points. (In the case of collinear 4-tuples, consider the cross-ratio.)

Open Problem

Does there exist an infinite plane with only finitely many orbits on k-tuples of distinct points for every $k \geqslant 1$?

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$\aleph_0$-categorical planes

A permutation group G on X is oligomorphic if G has finitely many orbits on X^k for each $k \geqslant 1$. See Cameron (1990).

(Taking k-tuples of points in X, or k-tuples of distinct points, doesn't matter.)

Open Question

Does there exist an infinite projective plane Π admitting a group $G \leqslant \mathrm{Aut}(\Pi)$ which is oligomorphic on points? (equivalently, on lines).

If such a plane exists, we may assume (by the Löwenheim-Skolem Theorem) that its order is $\aleph_0$ (countably infinite). Such a plane is called $\aleph_0$-categorical.

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$\aleph_0$-categorical planes

From now on, assume Π is an $\aleph_0$-categorical projective plane, and let $G \leqslant \mathrm{Aut}(\Pi)$ be oligomorphic on points.

Useful fact: In an oligomorphic group G, the stabilizer of any finite point set is also oligomorphic.

Lemma

Every finite substructure $S \subset \Pi$ lies in a finite subplane.

Proof.

Let $G_{(S)} \leqslant G$ be the pointwise stabilizer of S. Then $G_{(S)}$ fixes pointwise the substructure $\langle S \rangle$ generated by S. This substructure must be finite, otherwise $G_{(S)}$ has infinitely many fixed points, hence infinitely many orbits. □

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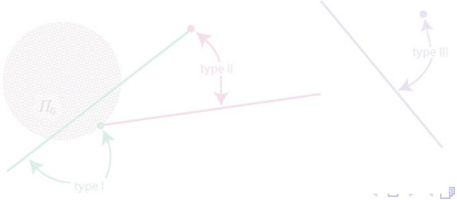
Π an $\aleph_0$-categorical projective plane, $G \leq \mathrm{Aut}(\Pi)$ oligomorphic

Without loss of generality, G fixes pointwise a finite subplane $\Pi_0 \subset \Pi$. (Otherwise replace G by the oligomorphic subgroup $G_{(S)}$ where S is a quadrangle.)

Consider a point $P \in \Pi$. We say

- P is of type I if $P \in \Pi_0$;
- P is of type II if $P \notin \Pi_0$ but P lies on a line of Π_0;
- P is of type III if P lies on no line of Π_0.

Dually classify lines of Π as type I, II or III.



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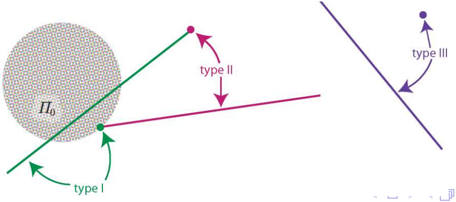

The Burnside Ring $\mathfrak{B}(G)$

Two G-sets X and Y are equivalent if there exists a G-equivariant bijection $\theta : X \to Y$, i.e. $\theta(x^g) = \theta(x)^g$ for all $x \in X$, $g \in G$.

The equivalence class of a G-set X is denoted $[X]$.

Given G-sets X and Y, the disjoint union $X \uplus Y$ and Cartesian product $X \times Y$ are G-sets.

The Burnside ring $\mathfrak{B}(G)$ is the $\mathbb{Z}$-algebra consisting of formal sums $\sum_{[X]} c_{[X]}[X]$, $c_{[X]} \in \mathbb{Z}$ (almost all zero), where

$$[X] + [Y] = [X \uplus Y], \qquad [X][Y] = [X \times Y].$$

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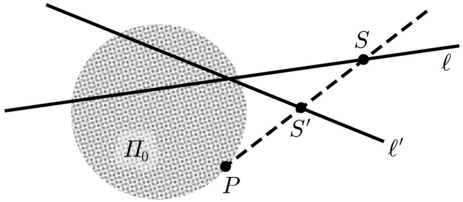
Let P and ℓ be a point and line of Π_0.

The set II_ℓ of type II points of ℓ is a G-set; as is the set II_P of type II lines through P.



Lemma

$[II_P] = [II_\ell]$, independent of the choice of point P and line ℓ of Π_0.

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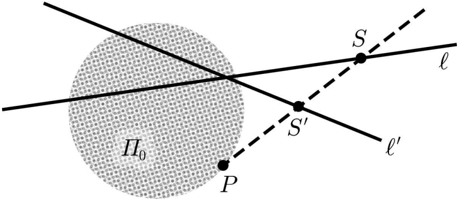

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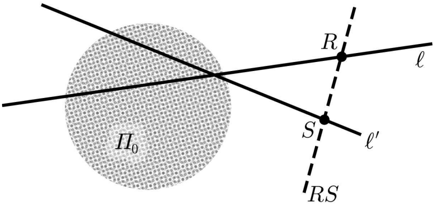
Denote by III the G-set consisting of all type III points. Dually, $\widetilde{III}$ is the G-set consisting of all type III lines.

Lemma

Let ℓ be a line of Π_0. Then $[II_\ell]^2 = [\widetilde{III}] + c[II_\ell]$ where $c = n_0(n_0 - 1)$, n_0 = order of Π_0.



$$(R, S) \mapsto RS$$

$$II_\ell \times II_{\ell'} \to \widetilde{III} \uplus \Big(\biguplus_{\substack{O \in \Pi_0;\\ O \notin \ell \cup \ell'}} II_O \Big)$$

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Corollary

$[\widetilde{III}] = [III]$ *and* $[II_\ell]^2 = [III] + c[II_\ell]$

Proof.

Dualising the previous lemma,

$$[III] + c[II_\ell] = [II_\ell]^2 = [\widetilde{III}] + c[II_\ell].$$

Cancellation of the $c[II_\ell]$ terms is justified in $\mathfrak{B}(G)$. □

Π an $\aleph_0$-categorical projective plane, $G \le \mathrm{Aut}(\Pi)$ oligomorphic

Let $\nu_{m,n}$ = number of G-orbits on $II_\ell^m \times III^n$.

Lemma

For all $m, n \geqslant 0$, we have $\nu_{m+2,n} = \nu_{m,n+1} + c\nu_{m+1,n}$.

Proof.

$$\begin{aligned}[II_\ell]^{m+2}[III]^n &= [II_\ell]^m\big([III] + c[II_\ell]\big)[III]^n \\ &= [II_\ell]^m[III]^{n+1} + c[II_\ell]^{m+1}[III]^n.\end{aligned}$$

□

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The previous recurrence for

$$\nu_{m,n} = \text{number of } G\text{-orbits on } II_\ell^m \times III^n$$

is rephrased in terms of the generating function

$$F(s,t) = \sum_{m,n\geqslant 0} \nu_{m,n} s^m t^n$$

as follows.

Lemma

$$F(s,t) = \sum_{k\geqslant 0} (a_k + b_k s) F_k(s,t) \text{ where}$$

$$F_k(s,t) = \frac{1}{(1-cs)t - s^2}\left[t^{k+1} - \frac{s^{2(k+1)}}{(1-cs)^{k+1}}\right].$$

Π an $\aleph_0$-categorical projective plane, $G \leq \mathrm{Aut}(\Pi)$ oligomorphic

Theorem

Under our assumption (existence of an $\aleph_0$-categorical projective plane), there exist (infinitely many) finite nonclassical projective planes, in which every quadrangle generates a proper subplane.

Proof (Sketch).

Without loss of generality, the subplane $\Pi_0 \subset \Pi$ is nonclassical. Let M be the maximum order of a subplane of the form $\langle \Pi_0, P, Q, R, S \rangle$ where (P, Q, R, S) is a quadrangle of Π. Any subplane of Π containing Π_0 of order exceeding M, has the required property. □

Subplanes of known planes

In all known cases of a finite projective plane of order n with a subplane of order n_0, we have

- $n = n_0^r$ for some $r \geqslant 1$; or
- $n_0 \in \{2, 3\}$.

Moreover, subplanes of order 3 are rare unless $n = 3^r$.

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Thank You!

Questions?