Dual polar graphs and the quantum algebra $U_q(\mathfrak{sl}_2)$

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BIRS, April 28, 2011

Overview

1. The quantum algebra $U_q(\mathfrak{sl}_2)$
2. Distance-regular graphs
3. Near polygons
4. Dual polar graphs
5. A $U_q(\mathfrak{sl}_2)$-module structure for dual polar graphs

The quantum algebra $U_q(\mathfrak{sl}_2)$

Let $q \in \mathbb{C}$ such that q is not a root of 1.

Definition

Let $U_q(\mathfrak{sl}_2)$ denote the unital associative $\mathbb{C}$-algebra with generators $k^{\pm 1}, e, f$ and the following relations:

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1 \\ ke &= q^2 ek \\ kf &= q^{-2} fk \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}} \end{aligned}$$

Distance-regular graphs

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges. Let D denote the diameter of Γ. Γ is called **distance-regular** whenever for all integers $h, i, j (0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X | \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of x and y. The p^h_{ij} are called the **intersection numbers** of Γ.

Standard module

Let $V = \mathbb{C}^X$.
Observe $Mat_X(\mathbb{C})$ acts on V by left multiplication.
We call V the **standard module**.
For $y \in X$, let $\hat{y}$ denote the element of V with 1 in the y-coordinate and 0 in all other coordinates.

Bose-Mesner algebra

For $0 \leq i \leq D$ let A_i denote the ith distance matrix of Γ.
We abbreviate $A = A_1$.
Observe

1. $A_0 = I$
2. $\sum_{i=0}^{D} A_i = J$
3. $\bar{A}_i = A_i \qquad (0 \leq i \leq D)$
4. $A_i^t = A_i \qquad (0 \leq i \leq D)$
5. $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \qquad (0 \leq i, j \leq D)$

Using these facts $\{A_i\}_{i=0}^{D}$ form a basis for a commutative subalgebra M of $Mat_X(\mathbb{C})$, called the **Bose-Mesner algebra** of Γ.
It turns out A generates M.

Bose-Mesner algebra, cont.

M has a second basis $\{E_i\}_{i=0}^D$ such that

1. $E_0 = |X|^{-1} J$
2. $\sum_{i=0}^D E_i = I$
3. $\bar{E}_i = E_i \qquad (0 \leq i \leq D)$
4. $E_i^t = E_i \qquad (0 \leq i \leq D)$
5. $E_i E_j = \delta_{ij} E_i \qquad (0 \leq i, j \leq D)$

We call $\{E_i\}_{i=0}^D$ the **primitive idempotents** of Γ.

Eigenvalues

Since $\{E_i\}_{i=0}^D$ form a basis for M there exists complex scalars $\{\theta_i\}_{i=0}^D$ such that $A = \sum_{i=0}^D \theta_i E_i$.
Observe $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$.
The scalars $\{\theta_i\}_{i=0}^D$ are mutually distinct since A generates M.
We call θ_i the **eigenvalue** of Γ associated with E_i.

Krein parameters

Since $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$, M is closed under $\circ$. There exists complex scalars q_{ij}^h $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \qquad (0 \leq i, j \leq D)$$

We call q_{ij}^h the **Krein parameters** of Γ.

Q-polynomial distance-regular graphs

The graph Γ is said to be Q-**polynomial** (with respect to given ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents) whenever for $0 \leq h, i, j \leq D$, $q_{ij}^{h} = 0$ (resp. $q_{ij}^{h} \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two.

Dual primitive idempotent

Assume Γ is Q-polynomial with respect to $\{E_i\}_{i=0}^{D}$.
Fix a vertex $x \in X$.
For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x,y) = i \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call $\{E_i^*\}_{i=0}^{D}$ the **dual primitive idempotents** of Γ with respect to x.

Observe $E_i^* V = \mathbb{C}\text{-span}\{\hat{z} | \partial(x,z) = i\}$.

Dual Bose-Mesner algebra

Observe

1. $\sum_{i=0}^{D} E_i^* = I$
2. $\bar{E}_i^* = E_i^* \qquad (0 \leq i \leq D)$
3. $E_i^{*t} = E_i^* \qquad (0 \leq i \leq D)$
4. $E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \leq i, j \leq D)$

From these facts $\{E_i^*\}_{i=0}^{D}$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $Mat_X(\mathbb{C})$. Call M^* the **dual Bose-Mesner algebra** of Γ with respect to x.

Dual adjacency matrix

For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with (y, y)-entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy}, \qquad y \in X.$$

Then $\{A_i^*\}_{i=0}^D$ is a basis for M^* such that

1. $A_0^* = I$
2. $\bar{A}_i^* = A_i^* \qquad (0 \leq i \leq D)$
3. $A_i^{*t} = A_i^* \qquad (0 \leq i \leq D)$
4. $A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \qquad (0 \leq i, j \leq D)$

Abbreviate $A^* = A_1^*$ and call it the **dual adjacency matrix** of Γ with respect to x.

It turns out A^* generates M^*.

Dual eigenvalues

Since $\{E_i^*\}_{i=0}^D$ form a basis for M^*, there exists complex scalars $\{\theta_i^*\}_{i=0}^D$ such that $A^* = \sum_{i=0}^D \theta_i^* E_i^*$.
Observe $A^* E_i^* = E_i^* A^* = \theta_i^* E_i^*$ for $0 \leq i \leq D$.
The scalars $\{\theta_i^*\}_{i=0}^D$ are mutually distinct since A^* generates M^*.
We call θ_i^* the **dual eigenvalue** of Γ associated with E_i^*.

Subconstituent algebra T

Let $T = T(x)$ denote the subalgebra of $Mat_X(\mathbb{C})$ generated by M and M^*. We call T the **subconstituent algebra** or **Terwilliger algebra** of Γ with respect to x.
Observe that A, A^* generate T.
Fact: V is a direct sum of irreducible T-modules.

Irreducible T-modules

Let W denote an irreducible T-module.
Observe $W = \sum_{i=0}^{D} E_i^* W = \sum_{i=0}^{D} E_i W$ (d.s.).
Define

- $r = min\{i | 0 \leq i \leq D, E_i^* W \neq 0\}$, endpoint of W
- $t = min\{i | 0 \leq i \leq D, E_i W \neq 0\}$, dual endpoint of W
- $d = |\{i | 0 \leq i \leq D, E_i^* W \neq 0\}| - 1$, diameter of W
- $d' = |\{i | 0 \leq i \leq D, E_i W \neq 0\}| - 1$

It turns out that $d = d'$.

Near polygons

Definition

A connected graph $\Gamma = (X, R)$ of diameter $D \geq 2$ is called a **near polygon** if the following two axioms hold.

(NP1) There are no induced subgraphs of shape $K_{1,2,1}$.

(NP2) If $y \in X$ and M is a maximal clique of Γ with $\partial(y, M) < D$, then there exists a unique vertex in M nearest to y.

Regular near polygons

Lemma

A distance-regular graph Γ *is a near polygon if and only if the axiom (NP1) holds and* $a_i = a_1 c_i$ *for* $1 \leq i \leq D$.

In this case we call Γ a **regular near polygon**.

Weak-geodetically closed subgraphs

Definition

Let $\Gamma = (X, R)$ denote a regular near polygon. A subgraph G of Γ is called **weak-geodetically closed** whenever for all vertices x, y in G and for all vertices z in X

$$\partial(x,z) + \partial(z,y) \leq \partial(x,y) + 1 \qquad \rightarrow \qquad z \in G.$$

Quads

Definition

Let Γ denote a regular near polygon. A subgraph Q of Γ is called a **quad** whenever Q has diameter 2 and Q is weak-geodetically closed.

Dual polar graphs

- Let b denote a prime power.
- Let $\mathbb{F}_b$ denote a finite field of order b.
- Let U denote a finite dimensional vector space over $\mathbb{F}_b$ endowed with a symplectic form, a quadratic form, or a Hermitean form.
- A subspace W of U is called **isotropic** whenever the form vanishes completely on W.
- Each maximal isotropic subspace of U has same dimension, say D.

Dual polar graphs

We define a graph $\Gamma = (X, R)$ where

- X is the set of all maximal isotropic subspaces of U
- $R = \{yz \in X^2 | dim(y \cap z) = D - 1\}$

Γ is distance-transitive so Γ is distance-regular.
For $y, z \in X$, $\partial(y, z) = i$ if the only if $dim(y \cap z) = D - i$.

We call Γ a **dual polar graph**.

From now on, fix a dual polar graph $\Gamma = (X, R)$. Fix a vertex x and $T = T(x)$.

Dual polar graphs

Γ is a Q-polynomial with respect to the ordering $\theta_0 > \theta_1 > \ldots > \theta_D$ of eigenvalues. Moreover, the dual eigenvalues are given by

$$\theta_i^* = \zeta + \xi b^{-i} \qquad \text{for } 0 \leq i \leq D,$$

where

$$\begin{aligned} \zeta &= -\frac{b(b^{D+e-2}+1)}{b-1}, \\ \xi &= \frac{b^2(b^{D+e-2}+1)(b^{D+e-1}+1)}{(b-1)(b^e+b)}. \end{aligned}$$

Raising, flattening and lowering maps

Definition

$$R = \sum_{i=0}^{D-1} E^*_{i+1} A E^*_i \qquad \text{raising map}$$

$$F = \sum_{i=0}^{D} E^*_i A E^*_i \qquad \text{flattening map}$$

$$L = \sum_{i=1}^{D} E^*_{i-1} A E^*_i \qquad \text{lowering map}$$

Observe $F^t = F$ and $R^t = L$.

Raising, flattening and lowering maps

Let $y \in X$ such that $\partial(x, y) = i$.

$$
\begin{aligned}
R\hat{y} &= \sum_{z \in \Gamma_{i+1}(x) \cap \Gamma(y)} \hat{z} \\
F\hat{y} &= \sum_{z \in \Gamma_{i}(x) \cap \Gamma(y)} \hat{z} \\
L\hat{y} &= \sum_{z \in \Gamma_{i-1}(x) \cap \Gamma(y)} \hat{z}
\end{aligned}
$$

Observe that $A = R + F + L$.

The map K

Pick $q \in \mathbb{C}$ such that $b = q^2$.

Definition

$$K = \sum_{i=0}^{D} q^{-2i} E_i^*.$$

Observe K is invertible and

$$A^* = \zeta I + \xi K.$$

R, F, L, K together generate T.

R, F, L, K

Lemma

1. $KR = q^{-2}RK$.
2. $KF = FK$.
3. $KL = q^{2}LK$.

Reminiscent of the defining relations of $U_q(\mathfrak{sl}_2)$.
It's almost as $k \approx K, e \approx L, f \approx R$ but not quite.

Relations involving R, F, L

Lemma

1. $LF - q^2 FL = (q^{2e} - 1)L.$
2. $FR - q^2 RF = (q^{2e} - 1)R.$

Pf: Γ is regular near polygon, its quads are classical, has constant line size $a_1 + 2$.

Relations involving R, L

Lemma

1. $$-\frac{q^4}{q^2+1}RL^2 + LRL - \frac{q^{-2}}{q^2+1}L^2R = q^{2e+2D-2}L,$$
2. $$-\frac{q^4}{q^2+1}R^2L + RLR - \frac{q^{-2}}{q^2+1}LR^2 = q^{2e+2D-2}R.$$

Pf: A, A^* satisfy the tridiagonal relation

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] = 0.$$

Summary of relations in T

1. $KK^{-1} = K^{-1}K = 1$
2. $KR = q^{-2}RK$
3. $KF = FK$
4. $KL = q^2 LK$
5. $LF - q^2 FL = (q^{2e} - 1)L$
6. $FR - q^2 RF = (q^{2e} - 1)R$
7. $-\frac{q^4}{q^2+1}RL^2 + LRL - \frac{q^{-2}}{q^2+1}L^2R = q^{2e+2D-2}L$
8. $-\frac{q^4}{q^2+1}R^2L + RLR - \frac{q^{-2}}{q^2+1}LR^2 = q^{2e+2D-2}R$

The central elements C_0, C_1, C_2 of T

Definition

1. $$C_0 = KF + \frac{q^{2e}-1}{q^2-1}K,$$
2. $$C_1 = -\frac{q^{-2}}{q^2+1}KLR + \frac{q^2}{q^2+1}KRL + \frac{q^{2e+2D-2}}{q^2-1}K,$$
3. $$C_2 = -\frac{q^{-2}}{q^2+1}K^2LR + \frac{1}{q^2+1}K^2RL + \frac{q^{2e+2D-2}}{q^4-1}K^2.$$

Theorem

C_0, C_1, C_2 generate the center of T.

Actions of C_0, C_1, C_2 on an irreducible T-module W

Lemma

Let W denote an irreducible T-module with diameter d, endpoint r and dual endpoint t. Then on W

1. C_0 *acts as the scalar* $\dfrac{1}{q^2-1}(q^{2e+2D-2d-2r-2t} - q^{2t-2r})$,
2. C_1 *acts as the scalar* $\dfrac{q^{2e+2D-1}}{q^4-1} q^{-d-2r}(q^{d+1} + q^{-d-1})$,
3. C_2 *acts as the scalar* $\dfrac{q^{2e+2D-2}}{q^4-1} q^{-2d-4r}$.

Central elements Φ, Ψ of T

Lemma

There exist central elements Φ, Ψ of T with the following property. For all irreducible T-module W with endpoint r, dual endpoint t and diameter d, Φ, Ψ act on W as follows:

$$\Phi = q^{r+t+d-D} 1$$
$$\Psi = q^{r-t} 1$$

C_2 and Φ, Ψ

Lemma

$$C_2 = \frac{q^{2e-2}}{q^4 - 1}(\Phi\Psi)^{-2}$$

$U_q(\mathfrak{sl}_2)$-module structure on the standard T-module V

Recall the standard T-module $V = \mathbb{C}^X$.

Theorem

There exists a unique $U_q(\mathfrak{sl}_2)$-module structure on V such that on V

$$\begin{aligned} k &= q^D \Phi \Psi K, \\ k^{-1} &= q^{-D} (\Phi \Psi K)^{-1}, \\ e &= \Phi \Psi K L, \\ f &= q^{1-2e-D} R. \end{aligned}$$