Lower bounds for width-restricted clause learning

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partially based on joint work with

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Outline

Resolution

Clause: disjunction $a_1 \vee \ldots \vee a_k$ of literals $a_i = x$ or $a_i = \bar{x}$.

The width of C is $w(C) := k$.

Formula (in CNF): conjunction $C_1 \wedge \ldots \wedge C_m$ of clauses.

Resolution rule

If C, D are clauses with $x \in C$ and $\bar{x} \in D$, then

$$Res_x(C, D) := (C \setminus x) \vee (D \setminus \bar{x})$$

Resolution proofs

Definition

A Resolution derivation R of clause C from formula F is a dag labelled with clauses s.t.

- there is exactly one sink labelled C
- If v has 2 predecessors u and u', then

$$C_v = Res_x(C_u, C_{u'})$$

  for some variable x
- if v is a source, then $C_v \in F$

The width of R is the maximal width of a clause in R

If the dag is a tree, we call R tree-like

A Resolution refutation of F is a derivation of the empty clause $\Box$ from F.

DLL and Tree Resolution

Algorithm DLL (Davis, Logemann, Loveland 1962)

$DLL(F, \alpha)$
test if $\alpha \models F$ or $\Box \in F\alpha$
pick variable x in $F\alpha$
recursively solve
$DLL(F, \alpha[x := 0])$ and
$DLL(F, \alpha[x := 1])$

Theorem

If unsatisfiable formula F is refuted by DLL in s steps, then F has a tree-like resolution refutation R of size s.

Clause Learning

In the case $\Box \in F\alpha$: (conflict)

- find $\alpha' \subseteq \alpha$ implying conflict (conflict analysis)
- add clause $\bigvee_{\alpha'(a)=0} a$ to F (learning)

Learning too many clauses $\rightsquigarrow$ memory explosion $\rightsquigarrow$

Heuristic to decide which clauses to learn.

We show: Learning only short clauses does not help!

Resolution Trees with Lemmas

A Resolution tree with lemmas (RTL) for formula F is an ordered binary tree labelled with clauses s.t.

- $C_{\mathrm{root}} = \Box$
- if v has 2 children u and u', then

  $C_v = Res_x(C_u, C_{u'})$ for some variable x
- if v has 1 child u, then

  $C_v \supseteq C_u$
- if v is a leaf, then

  $C_v \in F$ or $C_v = C_u$ for some $u \prec v$ (lemma)

$\prec$ is the post-order on trees.

Clause learning and *RTL*

Theorem (Buss, Hoffmann, JJ)

If unsatisfiable formula F is refuted by DLL+CL in s steps, then F has an RTL-refutation R of size $s \cdot n^{O(1)}$.

Moreover, the lemmas used in R are among the clauses learned by the algorithm.

In fact, the paper defines a subsystem $WRTI < RTL$ for which also the converse holds.

Here: lower bounds for $RTL(k)$:

A refutation R in RTL is in $RTL(k)$, if every lemma C used in R is of width $w(C) \leq k$.

The Pigeonhole Principle

...says: There is no injective map $[n+1] \to [n]$

The formula PHP_n:

- variables $x_{i,j}$ for $i \leq n+1$ and $j \leq n$
- pigeon clauses $x_{i,1} \vee \ldots \vee x_{i,n}$ for every i
- hole clauses $\bar{x}_{i,j} \vee \bar{x}_{i',j}$ for $i < i'$

Complexity of the Pigeonhole Principle

Theorem (Haken 1985)

Resolution proofs of PHP_n require size $2^{\Omega(n)}$.

Theorem (Buss, Pitassi 1997)

There are regular resolution proofs of PHP_n of size $n^3 2^n$.

Theorem (Iwama, Miyazaki 1999)

Tree-like resolution proofs of PHP_n require size $2^{\Omega(n \log n)}$.

The lower bound

Goal: solving PHP_n takes long when learning only short clauses.

To this end: lower bound for $RTL(k)$-refutations of PHP_n:

Theorem

Every $RTL(n/2)$-refutation of PHP_n is of size $2^{\Omega(n \log n)}$.

Matching restrictions

A restriction ρ is a partial truth assignment.

Notation: $F\lceil\rho$ for ρ applied to F.

Property: Let R be a derivation of C from F.

There is a derivation R' of $C\lceil\rho$ from $F\lceil\rho$ of size $|R'| \leq |R|$.
We denote R' by $R\lceil\rho$.

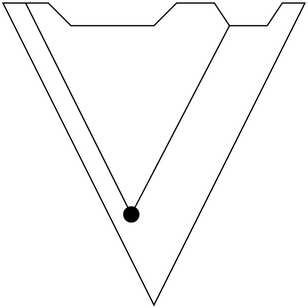
Matching restriction: defined by $\{(i_1, j_1), \ldots, (i_k, j_k)\}$:

$$\rho(x_{i,j}) = \begin{cases} 1 & \text{if } (i,j) \in \rho \\ 0 & \text{if } (i,j') \in \rho \text{ or } (i',j) \in \rho \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Property: $PHP_n\lceil\rho \equiv PHP_{n-|\rho|}$.

Proof of the lower bound

- Let R be a refutation of PHP_n
- Find first C with $w(C) \leq k$
- Subtree R_C is tree-like derivation of C
- Pick ρ with $C\lceil\rho = 0$
- $R_C\lceil\rho$ is refutation of $PHP_n\lceil\rho$
- ρ matching restriction $\rightarrow$
  $$PHP_n\lceil\rho = PHP_{n-|\rho|}$$
- lower bound by IWAMA/MIYAZAKI



Main Lemma: For C in R with $w(C) \leq k$, there is a matching restriction ρ with $C\lceil\rho = 0$ and $|\rho| \leq k$

The Ordering Principle

...says: An ordering of $[n]$ has a maximum

The formula Ord_n:

- variables $x_{i,j}$ for $i, j \leq n$ and $i \neq j$
- totality clauses $x_{i,j} \vee x_{j,i}$ for all i, j
- asymmetry clauses $\bar{x}_{i,j} \vee \bar{x}_{j,i}$ for all i, j
- transitivity clauses $\bar{x}_{i,j} \vee \bar{x}_{j,k} \vee \bar{x}_{k,i}$ for all i, j, k
- maximum clauses $\bigvee_{j \neq i} x_{i,j}$ for all i

Complexity of the Ordering Principle

Theorem (Stålmarck 1997)

There are regular resolution proofs of Ord_n of size $O(n^3)$.

Theorem (Bonet, Galesi 1999)

Tree-like resolution proofs of Ord_n require size $2^{\Omega(n)}$.

Ordering restrictions

Ordering restriction: defined by $S \subseteq [n]$ and an ordering $\prec$ on S.

$$\sigma(x_{i,j}) = \begin{cases} 1 & \text{if } i,j \in S \text{ and } i \prec j \\ 0 & \text{if } i,j \in S \text{ and } j \prec i \\ x_{s,j} & \text{if } i \in S \text{ and } j \notin S \\ x_{i,s} & \text{if } i \notin S \text{ and } j \in S \\ x_{i,j} & \text{otherwise,} \end{cases}$$

where $s \in S$ is fixed.

Property: $Ord_n\lceil\sigma \;\equiv\; Ord_{n-|S|+1}$.

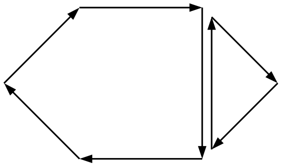
Cyclic clauses

For clause C, the graph $G(C)$ has edges

$$(i,j) \quad \text{for } \bar{x}_{i,j} \in C \quad \text{and} \quad (j,i) \quad \text{for } x_{i,j} \in C$$

Definition: C is *cyclic*, if $G(C)$ contains a cycle.

Lemma: A cyclic clause C has a tree-like resolution derivation from Ord_n of size $O(w(C))$.



The main lemmas

Lemma

If there is an $RTL(k)$-refutation of Ord_n of size s, then there is another one using no cyclic lemmas of size $O(sk)$.

Proof: Replace each cyclic lemma by its derivation of size $O(k)$.

Lemma

If C is acyclic with $w(C) \leq k$, then there is an ordering restriction σ with $|\sigma| \leq 2k$ such that $C\lceil\sigma = 0$.

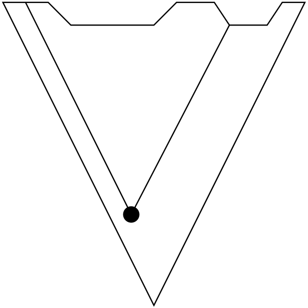
Proof: For C acyclic $G(C)$ is a dag

$\rightsquigarrow$ obtain σ as a topological ordering of $G(C)$.

The lower bound

Theorem

For $k < n/4$, every $RTL(k)$-refutation of Ord_n is of size $2^{\Omega(n)}$.



- Let R be a refutation of Ord_n
- Remove cyclic lemmas
- Find first C with $w(C) \leq k$
- Subtree R_C is tree-like derivation of C
- Pick σ with $C\lceil\sigma = 0$
- $R_C\lceil\sigma$ is refutation of $Ord_n\lceil\sigma$
- $Ord_n\lceil\sigma = Ord_{n-|\sigma|+1}$
- lower bound by Bonet/Galesi

A Game

Let X be a set of variables, and $w \leq |X|$.

A *w-system of restrictions* over X is $\mathcal{H} \neq \emptyset$ with

- $|\rho| \leq w$ for $\rho \in \mathcal{H}$,
- downward closure:
  if $\rho' \subseteq \rho \in \mathcal{H}$, then $\rho' \in \mathcal{H}$
- extension property:
  if $\rho \in \mathcal{H}$ with $|\rho| < w$, and $v \in X \setminus \mathrm{dom}\,\rho$,
  then there is $\rho' \supseteq \rho$ in $\mathcal{H}$ that sets v.

$\mathcal{H}$ avoids C if $C\lceil\rho \neq 0$ for all $\rho \in \mathcal{H}$

$\mathcal{H}$ avoids F if $\mathcal{H}$ avoids all $C \in F$

Resolution width and systems of restrictions

Theorem (Atserias & Dalmau)

*F requires resolution width w iff
there is a w-system of restrictions that avoids F.*

Theorem (Ben-Sasson & Wigderson)

*If a d-CNF formula F requires resolution width w,
then tree-like resolution proofs of F require size 2^{w-d}.*

Restricted systems

Lemma

Let $\mathcal{H}$ be a w-system of restrictions over X, and $\rho \in \mathcal{H}$.

$$\mathcal{H}\lceil\rho := \{\, \sigma \,;\ \operatorname{dom}\sigma \subseteq X \setminus \operatorname{dom}\rho \ \textit{ and } \ \sigma \cup \rho \in \mathcal{H} \ \textit{ and } \ |\sigma| \leq w - |\rho| \,\}$$

is a $w - |\rho|$ system of restrictions over $X \setminus \operatorname{dom}\rho$

Lemma

If $\mathcal{H}$ avoids F, then $\mathcal{H}\lceil\rho$ avoids $F\lceil\rho$.

The general lower bound

Theorem

If F requires resolution width w, then every $RTL(k)$-refutation of F is of size 2^{w-2k}.

- Let R be a refutation of F.
- Find first C with $w(C) \leq k$ not avoided by $\mathcal{H}$
- Let $G :=$ lemmas in subtree R_C. Note that $\mathcal{H}$ avoids G, and $w(G) \leq k$
- Pick $\rho \in \mathcal{H}$ with $C\lceil\rho = 0$ and $|\rho| \leq k$
- $R_C\lceil\rho$ is refutation of $F' := F \wedge G\lceil\rho$
- $\mathcal{H}\lceil\rho$ avoids F', thus F' requires width $w - k$
- $R_C\lceil\rho$ is of size 2^{w-2k} by Ben-Sasson & Wigderson

Application

$E_3(F) :=$ 3-CNF expansion of F

Theorem (Bonet, Galesi, JJ)

$E_3(Ord_n)$ requires resolution width $n/2$.

Corollary

Every $RTL(n/6)$-refutation of $E_3(Ord_n)$ is of size $2^{n/6}$.

Corollary

Every $RTL(n/6)$-refutation of Ord_n is of size $2^{n/6-\log n}$.

Newsflash!

Theorem

For every k, there is a family of formulas $F_n^{(k)}$ such that

- $F_n^{(k)}$ *have* $RTL(k+1)$*-refutations of size* $n^{O(1)}$.

  Even regular, without weakening.

- $F_n^{(k)}$ *requires* $RTL(k)$*-refutations of size* $2^{\Omega(n/\log n)}$.

This even holds for $k = k(n)$ *when* $k(n) = O(\log n)$.