Path Planning: An Application of the Static Hamilton Jacobi Equation

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Joint work with
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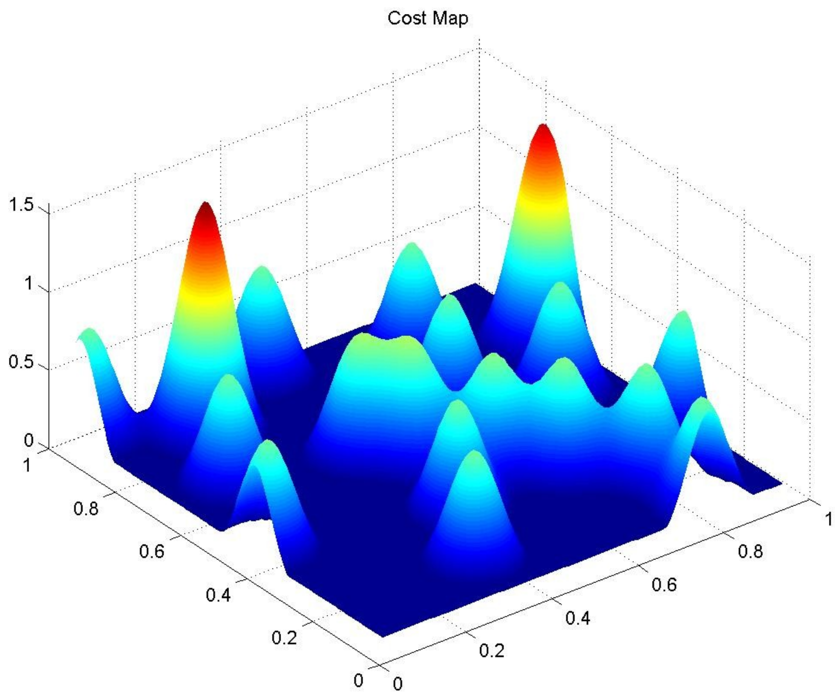
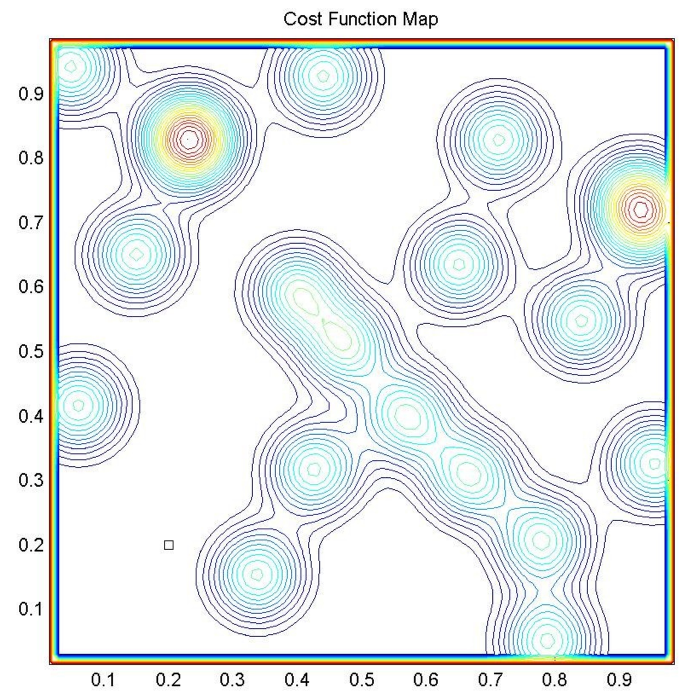
Basic Path Planning

- Find the optimal path $p(s)$ to a target (or from a source)
  – No constraints on the path
- Problem data
  – Cost $c(x)$ to pass through each state in the state space
  – Set of targets or sources (provides boundary conditions)





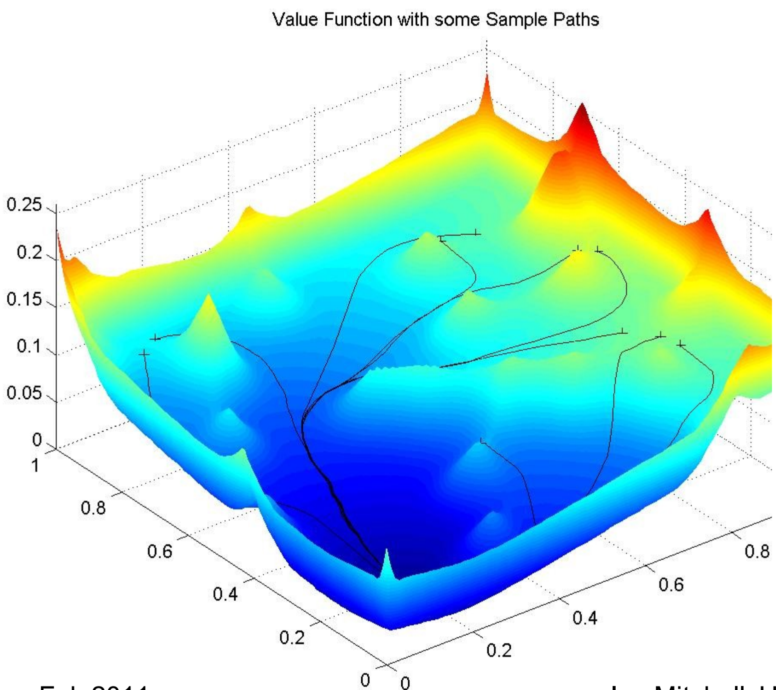
Value Function for Path Planning

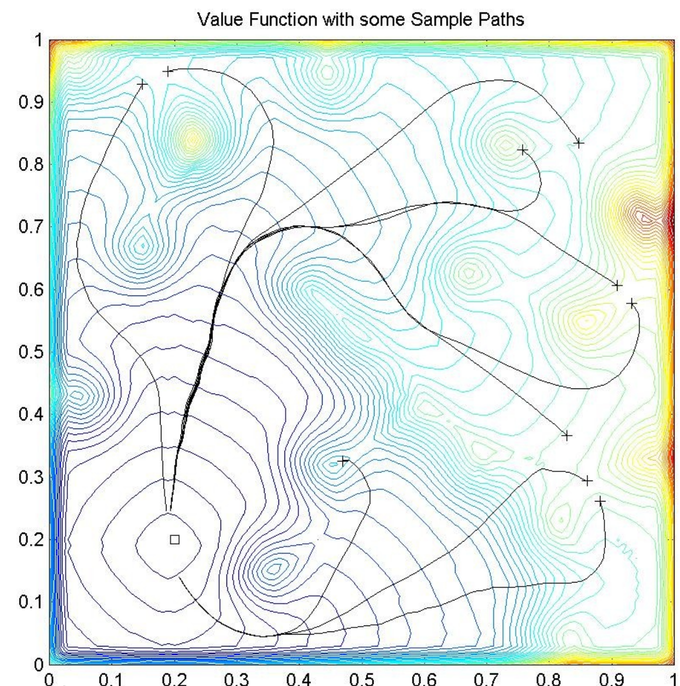
- Value function solves Eikonal equation

$$\|D_x V(x)\| = c(x)$$

- Optimal paths found by gradient descent

$$\frac{d}{ds}p(s) = \frac{D_x V(x)}{\|D_x V(x)\|}$$





Continuous Dynamic Programming

- Continuous DPP for path $p(\cdot)$

$$\vartheta(p(s)) = \min_{p(\cdot)} \left[\vartheta(p(s + \Delta s)) + \int_s^{s+\Delta s} c(p(\sigma)) d\sigma \right]$$

- Rearrange

$$\min_{p(\cdot)} \left[\frac{\vartheta(p(s)) - \vartheta(p(s + \Delta s))}{\Delta s} - \frac{\int_s^{s+\Delta s} c(p(\sigma)) d\sigma}{\Delta s} \right] = 0$$

- Take limit $\Delta s \to 0$

$$\min_{p(\cdot)} \left[-\frac{d\vartheta(p(s))}{ds} - c(p(s)) \right] = 0$$

- Set $x = p(s)$ and chain rule

$$\min_{p(\cdot)} \left[\frac{\partial \vartheta(x)}{\partial x} \frac{dp(s)}{ds} + c(x) \right] = 0$$

Static Hamilton-Jacobi PDE

- Let control be $u(s) = \frac{dp(s)}{ds}$ and observe that the only dependence on $p(\cdot)$ is u to arrive at the static HJ PDE

$$\min_u \left[D_x\vartheta(x) \cdot u + c(x)\right] = H(x, D_x\vartheta(x)) = 0$$

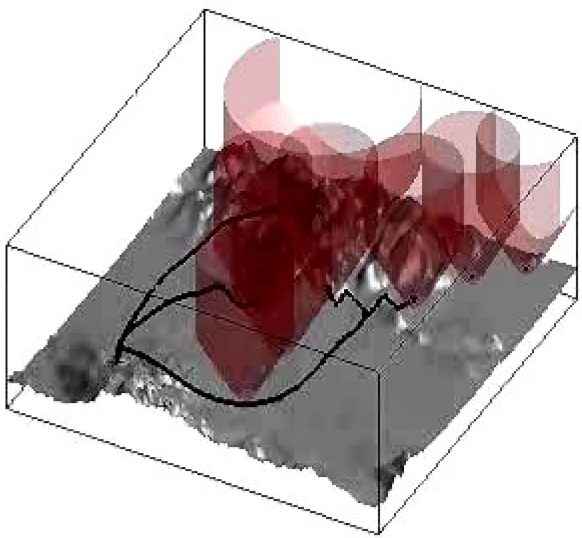
- From original problem for $x \in \mathcal{T}$ we get boundary conditions $\vartheta(x) = 0$
- If constraint on u is isotropic (eg: $\|u\|_2 \leq 1$), choose optimal control

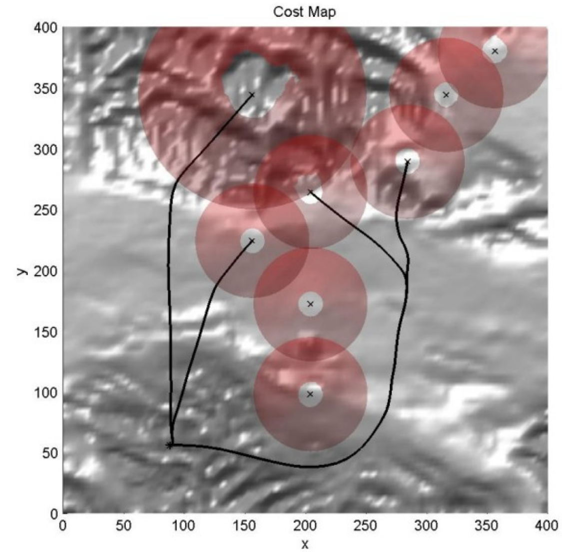
$$u(s) = \frac{D_x\vartheta(x)}{\|D_x\vartheta(x)\|_2}$$

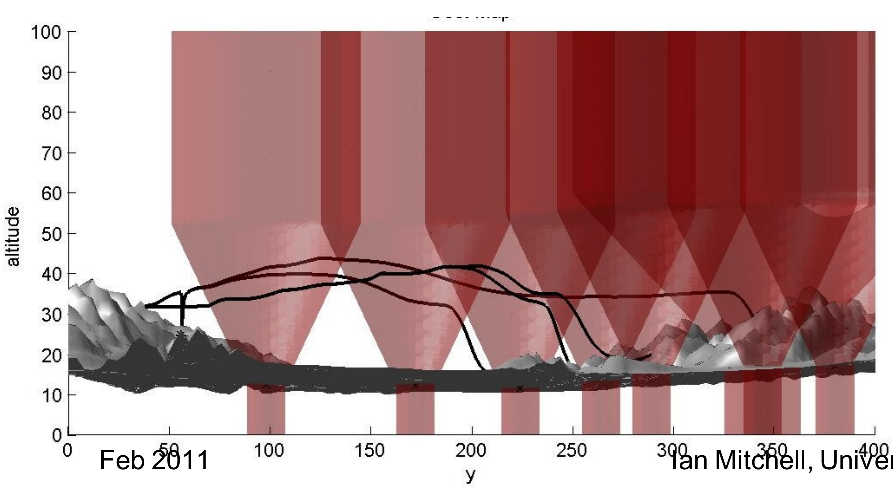
and PDE becomes the Eikonal equation

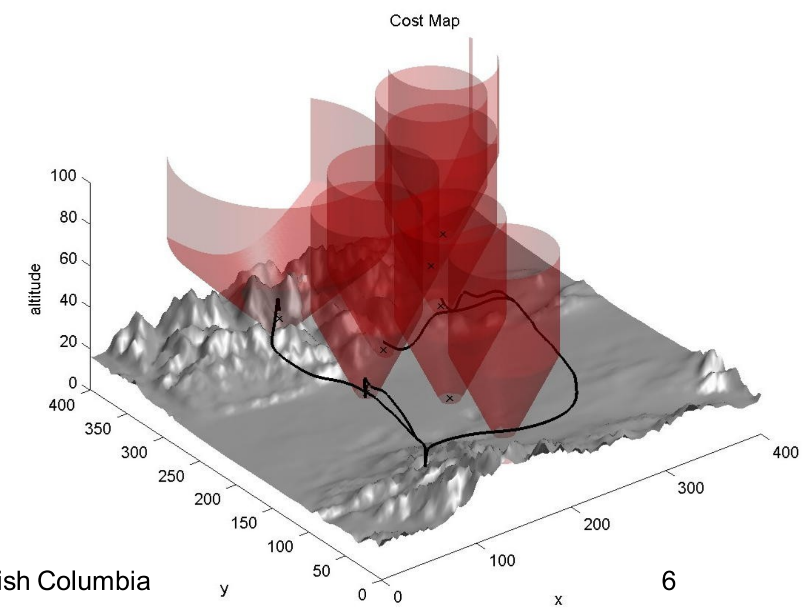
$$\begin{aligned} \|D_x\vartheta(x)\|_2 &= c(x) \quad \text{for } x \in \mathbb{R}^2 \setminus \mathcal{T} \\ \vartheta(x) &= 0 \qquad \text{for } x \in \partial\mathcal{T} \end{aligned}$$

Demanding Example? No!





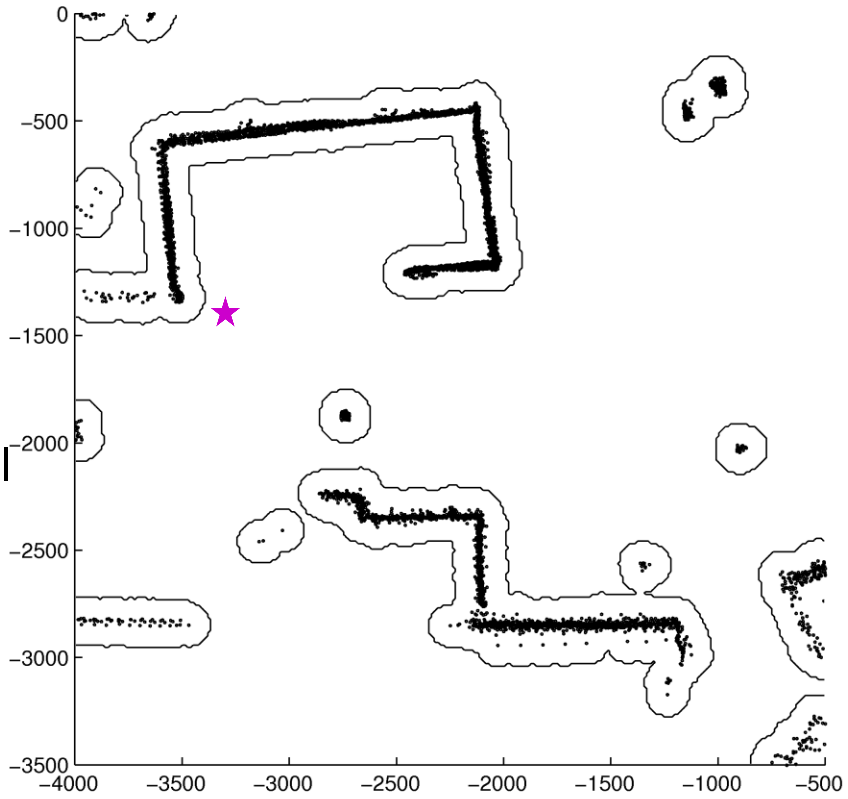




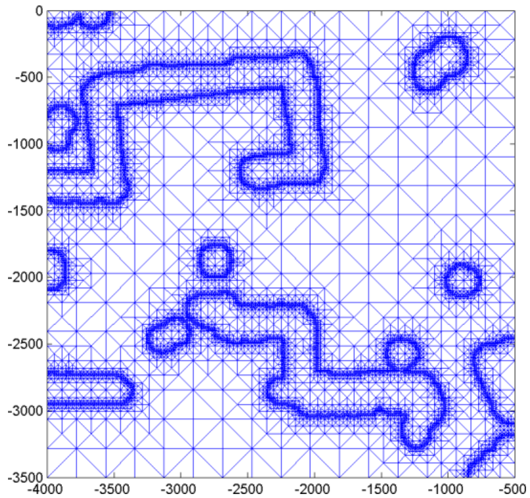
Robot Path Planning

- Find shortest path to objective while avoiding obstacles
  - Obstacle maps from laser scanner
  - Configuration space accounts for robot shape
  - Cost function essentially binary
- Value function measures cost to go
  - Solution of Eikonal equation
  - Gradient determines optimal control

typical laser scan with configuration space obstacles



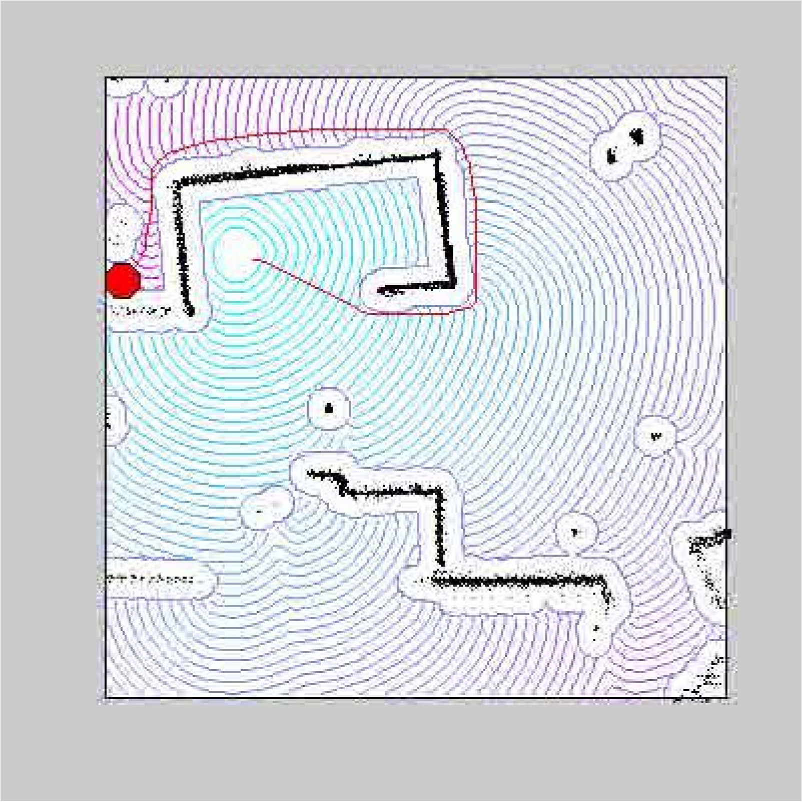
adaptive grid



Alton & Mitchell, "Optimal Path Planning under Different Norms in Continuous State Spaces," ICRA 2006

Continuous Value Function Approximation

- Contours are value function
  - Constant unit cost in free space, very high cost near obstacles
- Gradient descent to generate the path



Hamilton-Jacobi Flavours

- Time-dependent Hamilton-Jacobi used for dynamic implicit surfaces and finite horizon optimal control / differential games

$$D_t\phi(x,t) + H(x, D_x\phi(x,t)) = 0$$

  - Solution continuous but not necessarily differentiable
  - Time stepping approximation with high order accurate schemes
  - Numerical schemes have conservation law analogues
- Stationary (static) Hamilton-Jacobi used for target based cost to go and time to reach problems

$$H(x, D_x\vartheta(x)) = 0 \qquad \|D_x\vartheta(x)\| = c(x)$$

  - Solution may be discontinuous
  - Many competing algorithms, variety of speed & accuracy
  - Numerical schemes use characteristics (trajectories) of solution

Solving Static HJ PDEs

- Two methods available for using time-dependent techniques to solve the static problem
  - Iterate time-dependent version until Hamiltonian is zero
  - Transform into a front propagation problem
- Schemes designed specifically for static HJ PDEs are essentially continuous versions of value iteration from dynamic programming
  - Approximate the value at each node in terms of the values at its neighbours (in a consistent manner)
  - Details of this process define the “local update”
  - Eulerian schemes, plus a variety of semi-Lagrangian
- Result is a collection of coupled nonlinear equations for the values of all nodes in terms of all the other nodes
- Two value iteration methods for solving this collection of equations: marching and sweeping
  - Correspond to label setting and label correcting in graph algorithms

Cost Depends on…

- So far assumed that cost depends only on position
- More generally, cost could depend on position and direction of motion (eg action / input)
  - Variable dependence on position: inhomogenous cost
  - Variable dependence on direction: anisotropic cost
- Discrete graph
  - Cost is associated with edges instead of nodes
  - Dijkstra's algorithm is essentially unchanged
- Continuous space
  - Static HJ PDE no longer reduces to the Eikonal equation

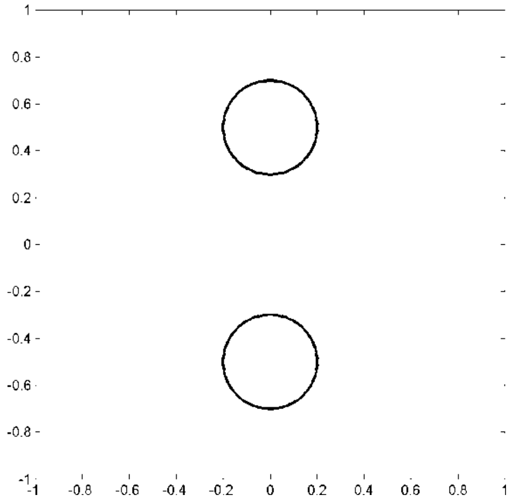
$$\min_{u \in U} \left[D_x \vartheta(x) \cdot u + c(x) \right] = 0 \quad \not\Leftrightarrow \quad \| D_x \vartheta(x) \| = c(x)$$

$$\text{when } U \text{ is not a circle / sphere}$$

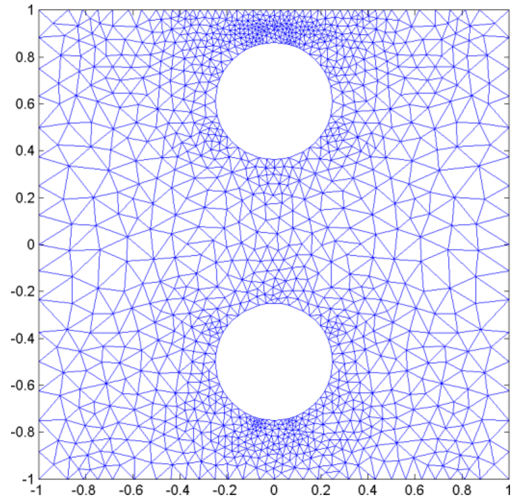
  - Gradient of ϑ may not be the optimal direction of motion
  - Isotropy is related to but stronger than holonomicity or small time local controllability

Other Static HJ Issues: Obstacles

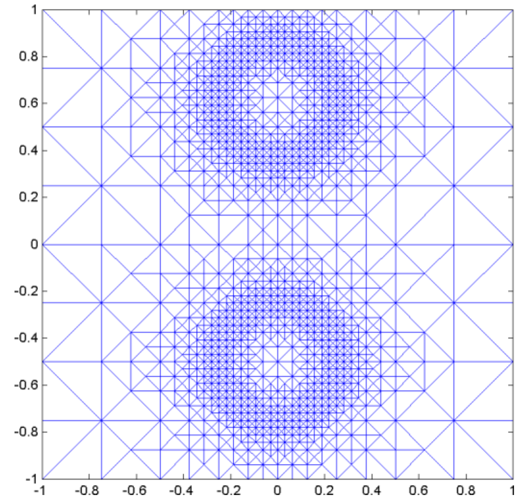
- Computational domain should not include (hard) obstacles
  - Requires “body-fitted” and often non-acute grid: straightforward in 2D, challenging in 3D, open problem in 4D+
- Alternative is to give nodes inside the obstacle a very high cost
  - Side effect: the obstacle boundary is blurred by interpolation
- Improved resolution around obstacles is possible with semi-structured adaptive meshes
  - Not trivial in higher dimensions; acute meshes may not be possible



original obstacles



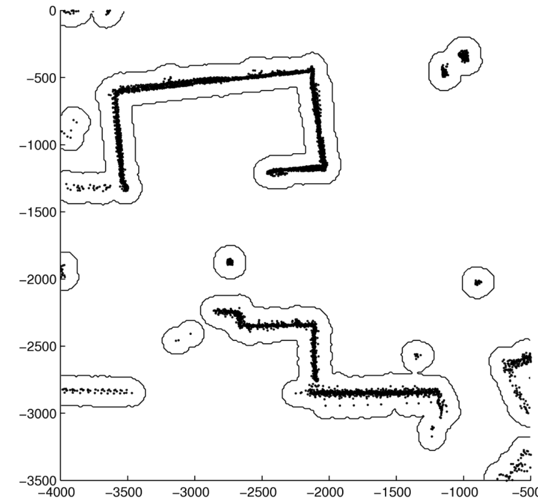
body fitted mesh



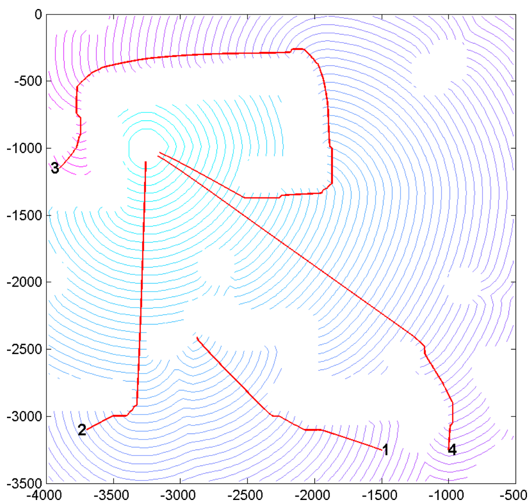
semi-structured mesh

Adaptive Meshing is Practically Important

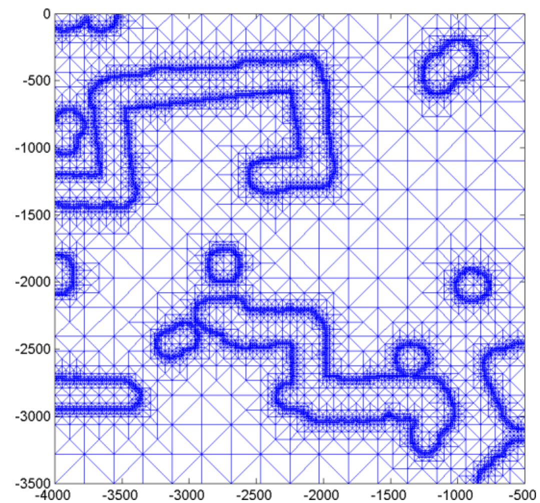
- Much of the static HJ literature involves only 2D and/or fixed Cartesian meshes with square aspect ratios
  - “Extension to variably spaced or unstructured meshes is straightforward…”
- Nontrivial path planning demands adaptive meshes
  - And configuration space meshing, and dynamic meshing, and …



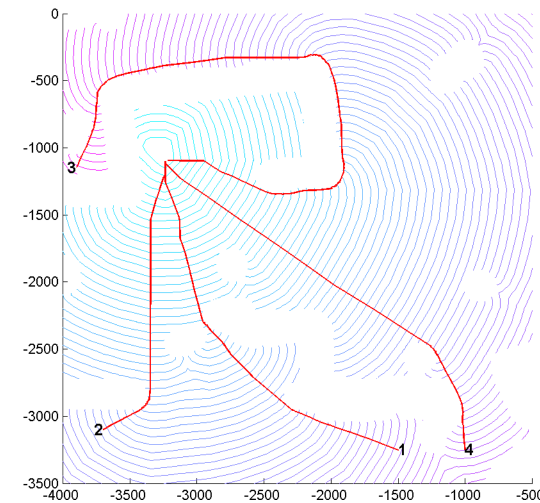
original obstacles



Cartesian mesh’s paths



adaptive mesh



adaptive mesh’s paths

Methods: Direct Time-Dependent Version

$$H(x, D_x\vartheta(x)) = 0 \text{ for } x \in \Omega \setminus \mathcal{T}$$
$$\vartheta(x) = 0 \text{ for } x \in \partial\mathcal{T}$$

- Time-dependent version: replace $\vartheta(x) \to \vartheta(t,x)$ and add temporal derivative

$$D_t\vartheta(t, x) + H(x, D_x\vartheta(t, x)) = 0$$

  - Solve until $D_t\vartheta(t,x) = 0$, so that $\vartheta(t,x) = \vartheta(x)$
- Not a good idea
  - No reason to believe that $D_t\vartheta(t,x) \to 0$ in general
  - In limit $t \to \infty$, there is no guarantee that $\vartheta(t,x)$ remains continuous, so numerical methods may fail

Transform Static to Time-Dependent HJ

Create implicit surface definition of T

$$\phi(x,0) \begin{cases} \leq 0, x \in T; \\ = 0, x \in \partial T; \\ \geq 0, x \in \mathbb{R}^d \setminus T. \end{cases}$$

Under assumption $D_x\phi(x,0) \cdot p \neq 0$ on ∂T, make change of variables

$$D_x\vartheta(x) \leftarrow \frac{D_x\phi(x,t)}{D_t\phi(x,t)}$$

and get toolbox appropriate PDE

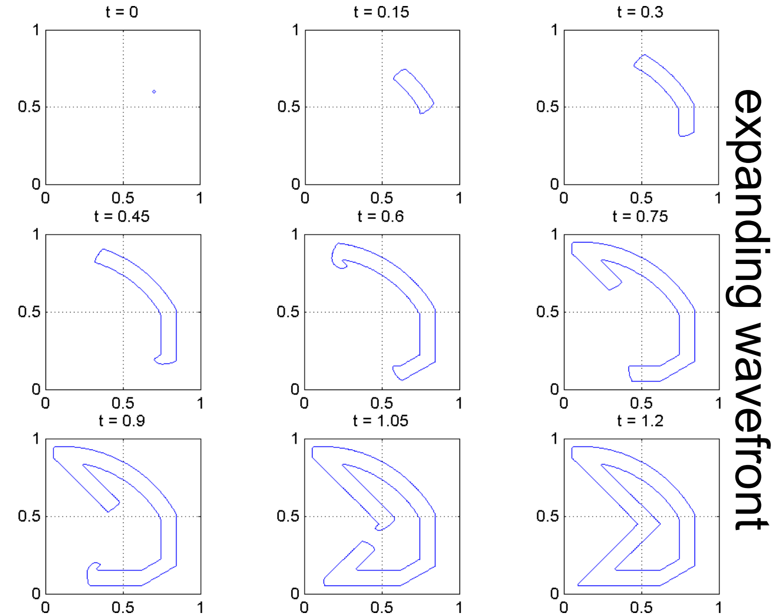
$$D_t\phi(x,t) + \min_{p \in \mathbb{S}^1} \frac{D_x\phi(x,t) \cdot p}{\ell(x,p)} = 0.$$

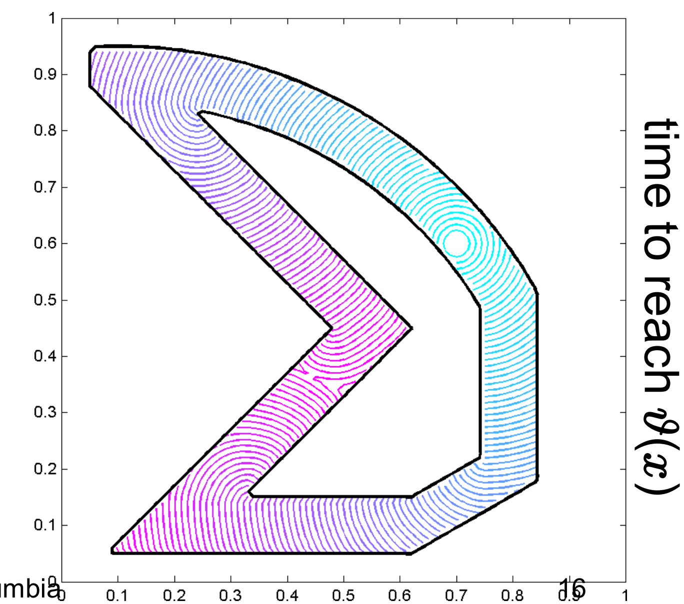
After solving, set ϑ to be crossing time

$$\vartheta(x) = \{t \mid \phi(x,t) = 0\}.$$

Methods: Time-Dependent Transform

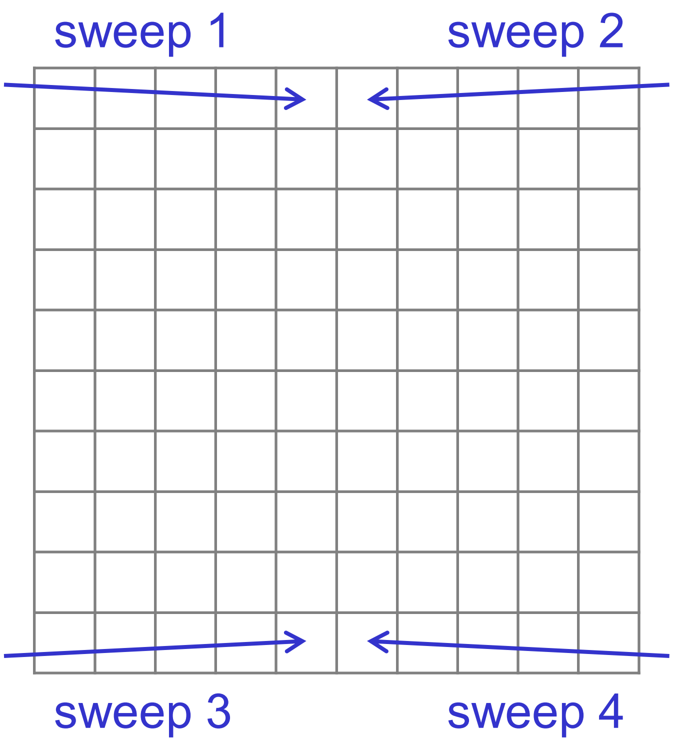
- Equivalent wavefront propagation problem [Osher 93]
- Pros:
  - Implicit surface function for wavefront is always continuous
  - Handles anisotropy
  - High order accuracy schemes available on uniform Cartesian grid
  - Subgrid resolution of obstacles through implicit surface representation
  - ToolboxLS code is available
- Cons:
  - CFL requires many timesteps
  - Computation over entire grid at each timestep





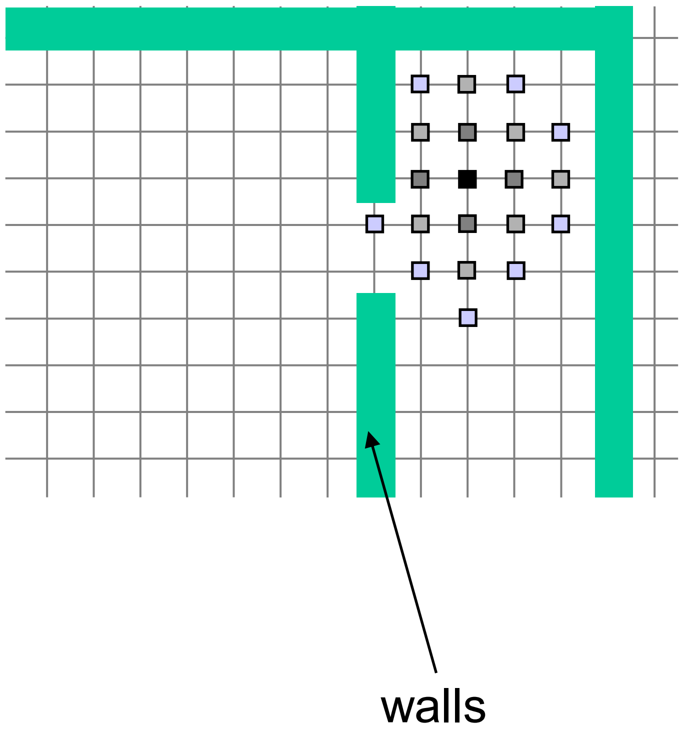
Methods: Fast Sweeping

- Gauss-Seidel iteration through the grid
  - For a particular node, use a consistent update (same as fast marching)
  - Several different node orderings are used in the hope of quickly propagating information along characteristics
  - Zhao, Qian, Zhang, Tsai, Osher, Chang, Kao, …
- Pros:
  - Easy to implement
  - handles anisotropy, nonconvexity, obtuse unstructured grids
- Cons:
  - Multiple sweeps required for convergence



Methods: Fast Marching / Ordered Upwind

- Dijkstra's algorithm with a consistent node update formula
  - Tsitsiklis, Sethian, Kimmel, Vladimirsky, …
- Pros:
  - Efficient, single pass
  - Isotropic case relatively easy to implement
- Cons:
  - Random memory access pattern
  - No advantage from accurate initial guess
  - Requires causality relationship between node values
  - Anisotropic case trickier to implement



More General Anisotropic Cost / Speed

- Dirichlet problem for a static Hamilton-Jacobi PDE:

$$H(x, Du(x)) = 0, \qquad x \in \Omega$$

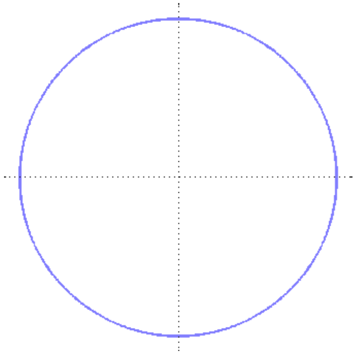
$$u(x) = g(x), \quad x \in \partial\Omega$$

- Control-theoretic Hamiltonian:

$$H(x, q) = \max_{a \in \mathcal{A}} [(-q \cdot a) f(x, a)] - 1$$

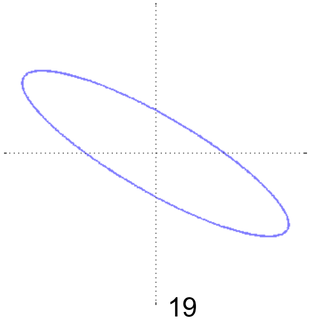
- Unit vector controls:

$$\mathcal{A} = \{a \in \mathbb{R}^d \mid \|a\| = 1\}$$



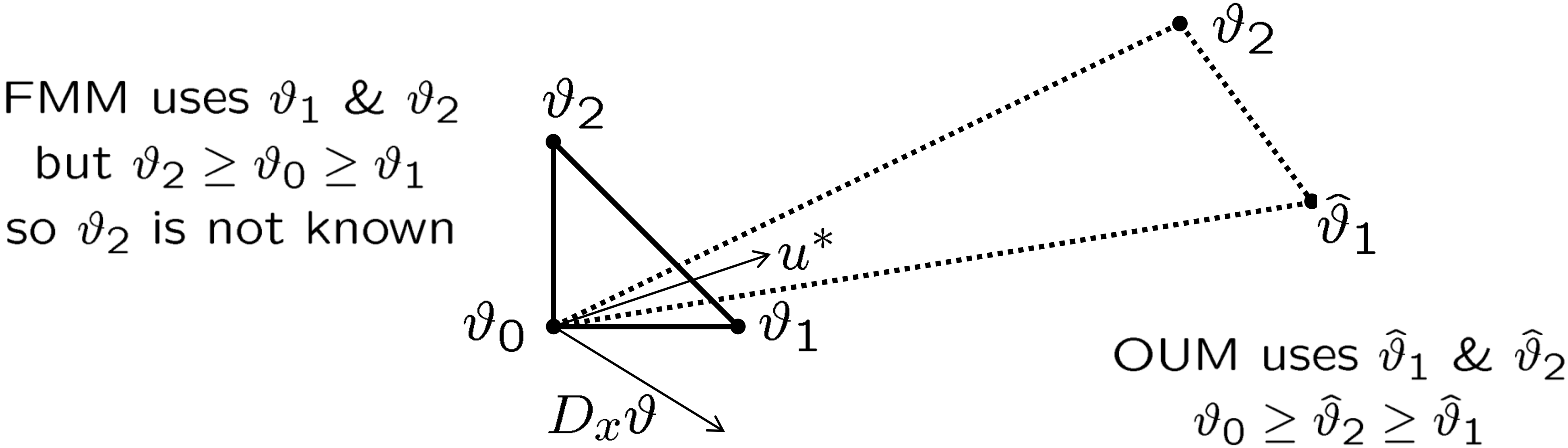
- Speed profile:

$$\mathcal{A}_f(x) = \{a f(x, a) \mid a \in \mathbb{R}^d \text{ and } \|a\| \leq 1\}$$



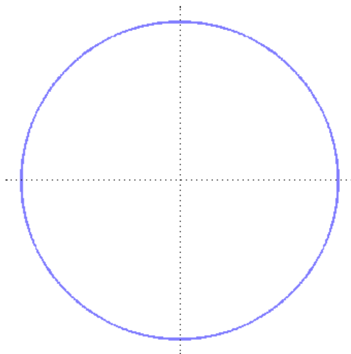
Anisotropy Leads to Causality Problems

- To compute the value at a node, we look back along the optimal trajectory ("characteristic"), which may not be the gradient
- Nodes in the simplex containing the characteristic may have value greater than the current node
  - Under Dijkstra's algorithm / FMM, only values less than the current node are known to be correct
- Ordered upwind extension of FMM searches a larger set of simplices to find one whose values are all known
- However, for some anisotropies and grids, regular FMM works

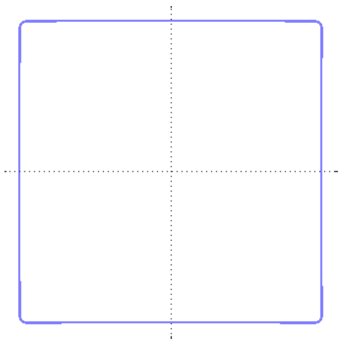


Speed Profiles

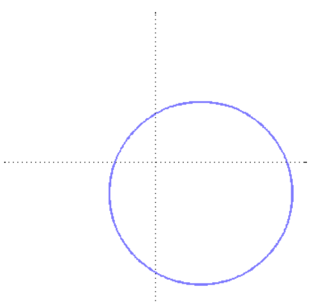
- To ensure continuity of the value function, origin must be in the interior (small time locally controllable)

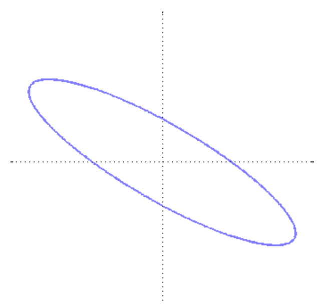


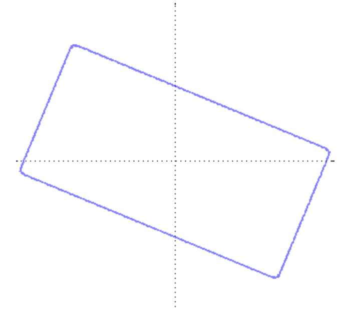
- To use Eikonal solvers, speed profile must be a circle / sphere at each point



- On an orthogonal grid, FMM will still work for axis-aligned anisotropies
- For more general anisotropies, OUM or fast sweeping methods are required



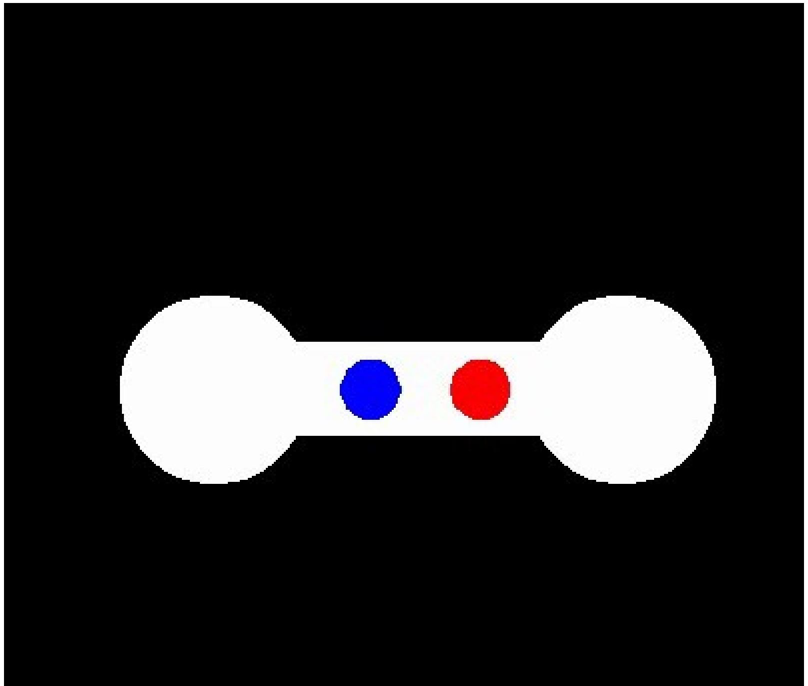




FMM for Axis-Aligned Anisotropies

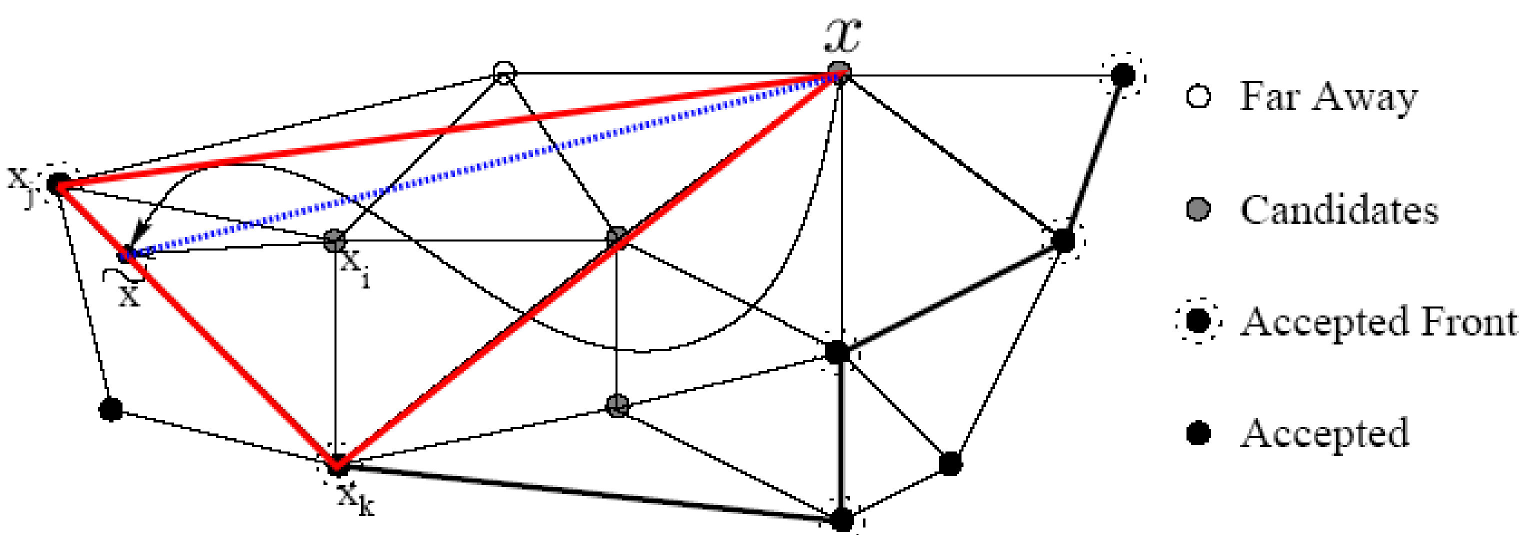
- FMM can be used on an orthogonal grid for Hamiltonians satisfying *strict one-sided monotonicity*
  - Related to "Osher's criterion" but does not require differentiability
- Alton & Mitchell, SINUM 2008
- Example: two robots moving in the plane

$$\left\| \left(\left\| \left(\frac{\partial \vartheta(x)}{\partial x_1}, \frac{\partial \vartheta(x)}{\partial x_2} \right) \right\|_2, \left(\frac{\partial \vartheta(x)}{\partial x_3}, \frac{\partial \vartheta(x)}{\partial x_4} \right) \right\|_2 \right\|_1 = c(x).$$



Ordered Upwind Method (OUM)

- Extension of FMM to solve problems with general convex speed profiles in O(N log N)
- Update() looks beyond immediate neighbors to use virtual simplices that include nodes within $h\Upsilon$,
  - Anisotropy coefficient Υ is ratio of fastest to slowest speed
- Search for such neighbours occurs only on the front of newly accepted nodes (Accepted Front OUM / AFOUM)

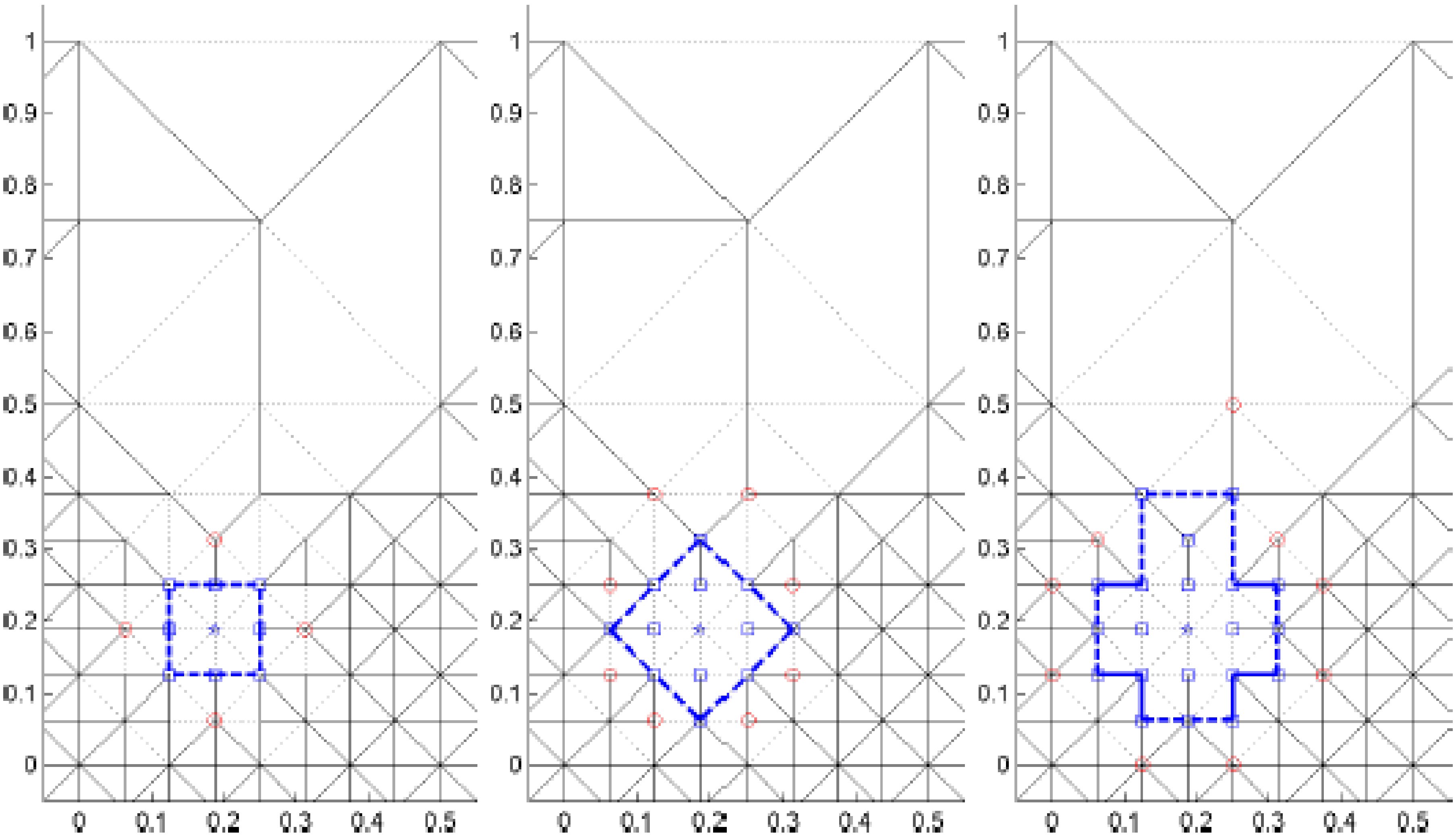


[Sethian & Vladimirsky, SINUM, 2003]

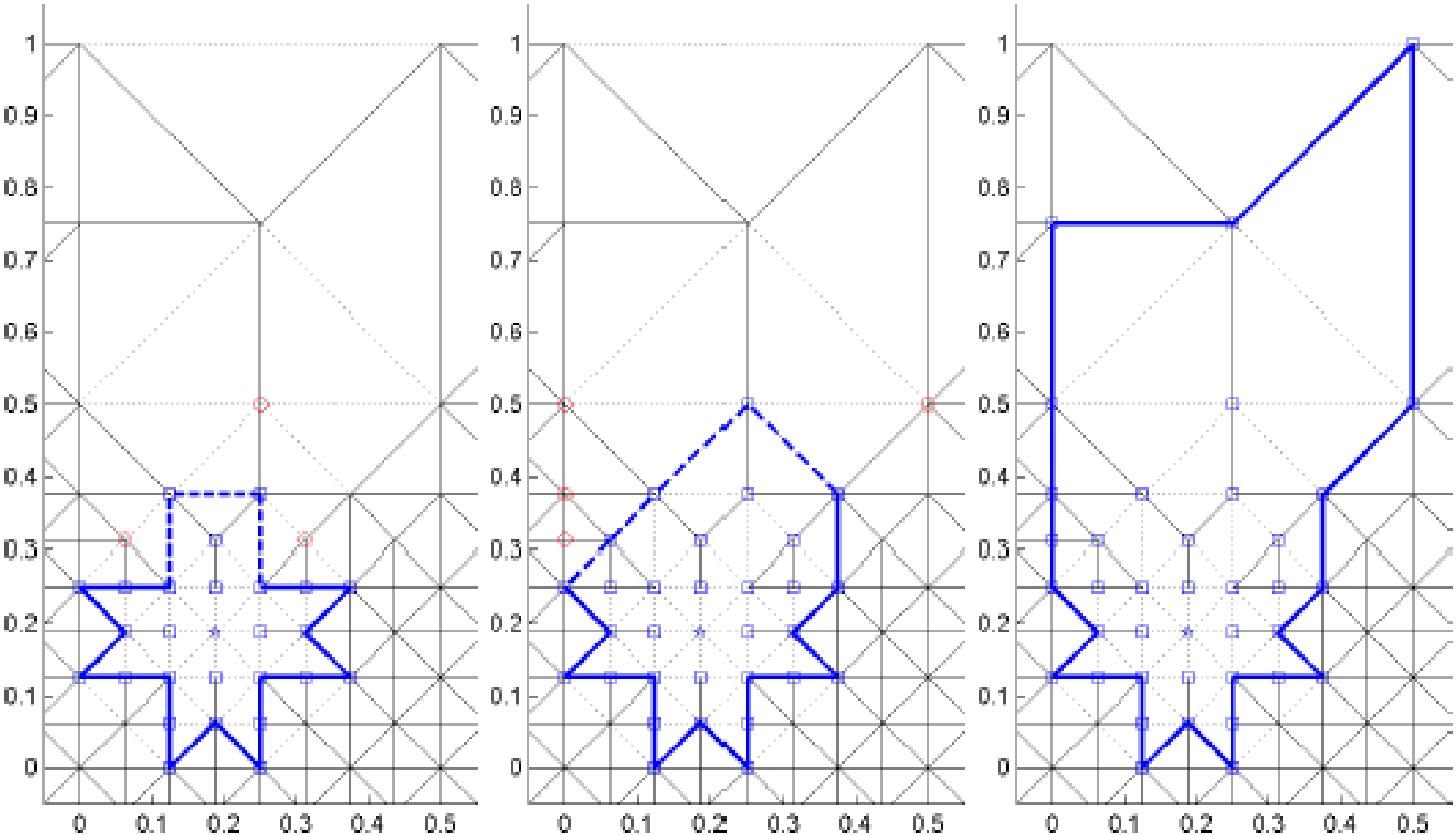
Monotone Acceptance OUM (MAOUM)

- Like AFOUM
  - extension of FMM to solve problems with general convex speed profiles in $\mathcal{O}$(N log N)
- Unlike AFOUM
  - Dijkstra-like algorithm: computes solution in order of nondecreasing value
  - Standard convergence proof [Barles & Souganidis, 1991]
  - Simple conversion to a Dial-like algorithm that sorts and accepts solution values using buckets
  - Stencil size adjusts to the local level of grid refinement
  - No accepted front
  - Initial pass through grid to generate stencils based on tests that can be applied to each potential face of the stencil
  - Must store stencils
- Alton & Mitchell, submitted to J. Scientific Computing

Stencil Generation Algorithm

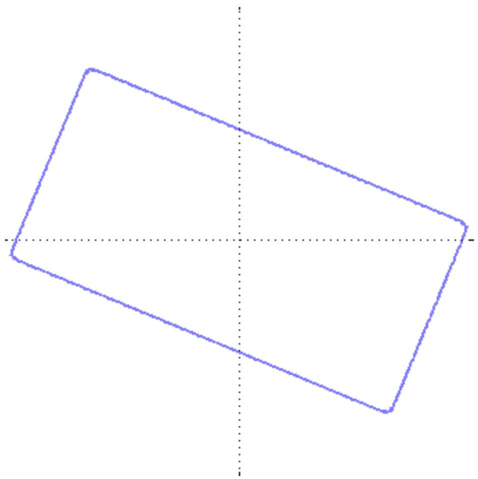


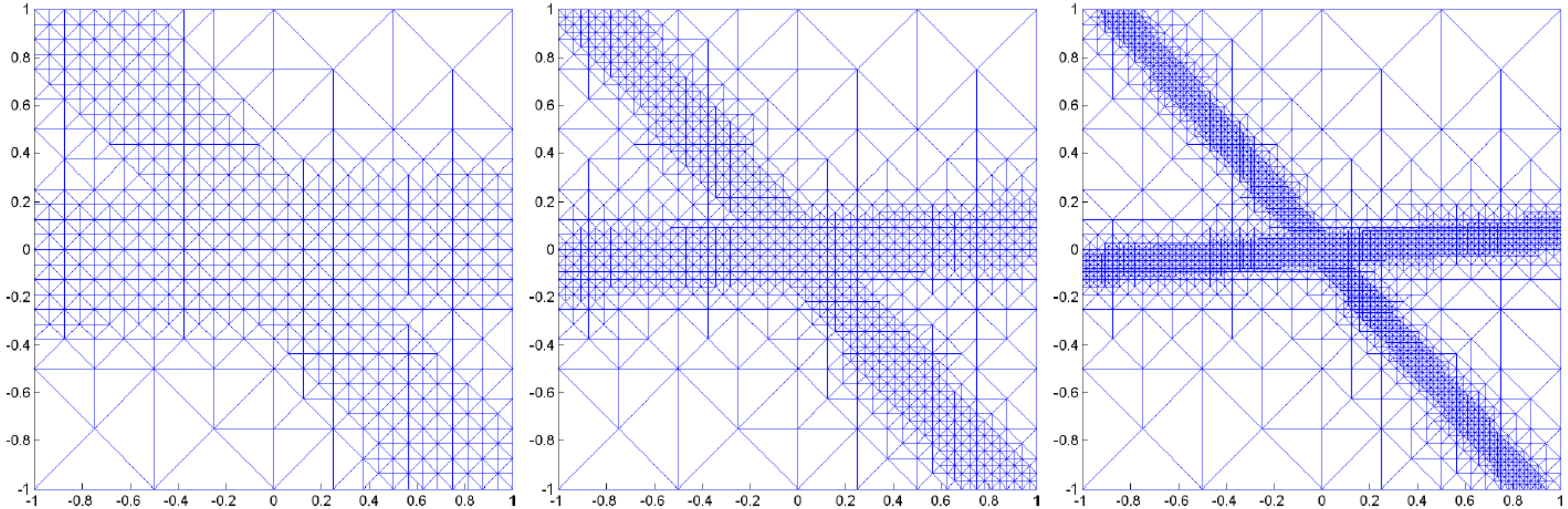
Stencil Generation (continued)



Experiment: Rectangular Speed Profile

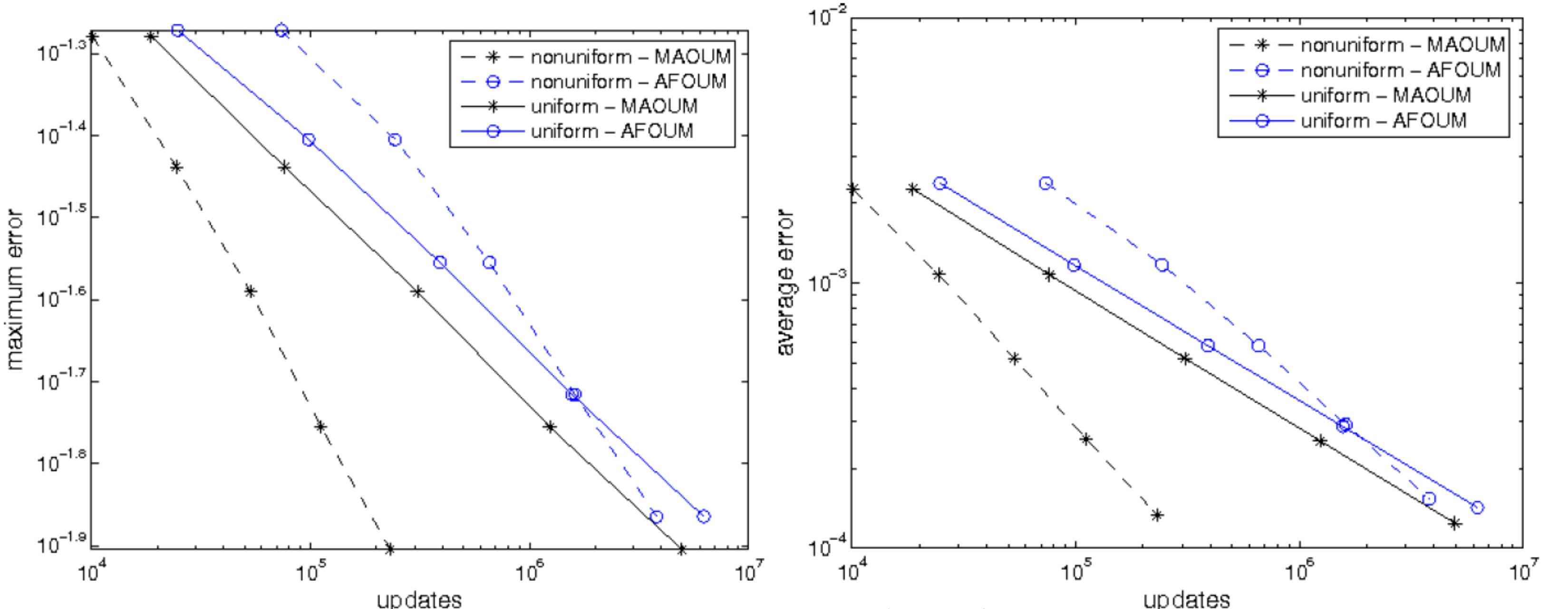
- Homogeneous speed profile
- Boundary condition specified at origin
- Grid refined where solution and characteristics are highly curved





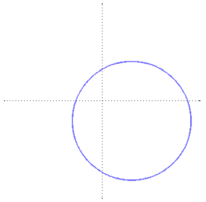
Results: Rectangular Speed Profile

- MAOUM and AFOUM on uniform and nonuniform grids
- Maximum and average error versus updates
- Nonuniform grid has better error convergence rate for both algorithms than nonuniform grid
- MAOUM on nonuniform grid has smallest error



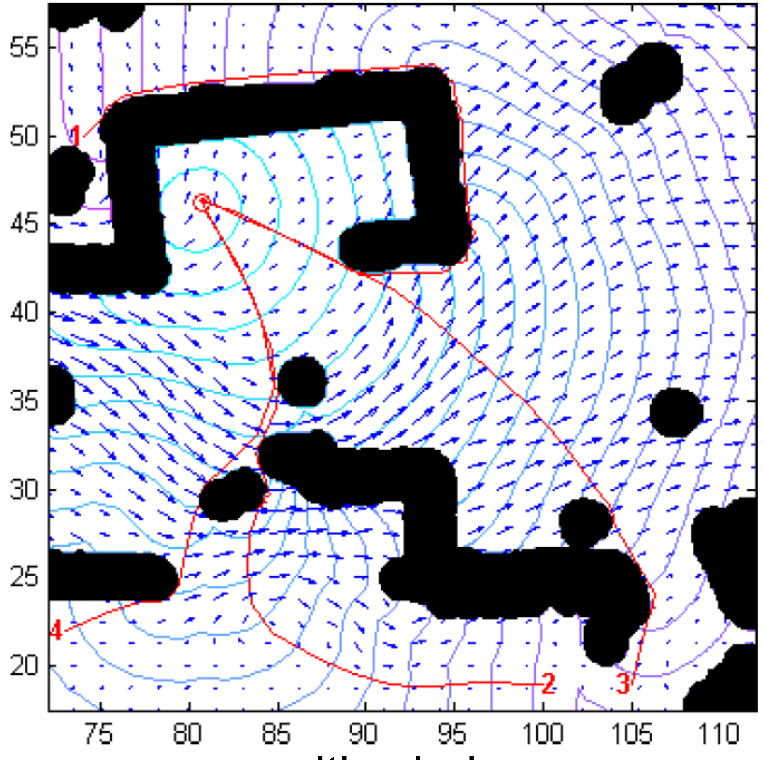
Example: Robot Path Planning

- Robot wants to reach goal in minimal time avoiding obstacles and fighting a fierce wind
- Solved with new ordered upwind scheme: Monotone Acceptance OUM
  - Alton & Mitchell, submitted J. Scientific Computing

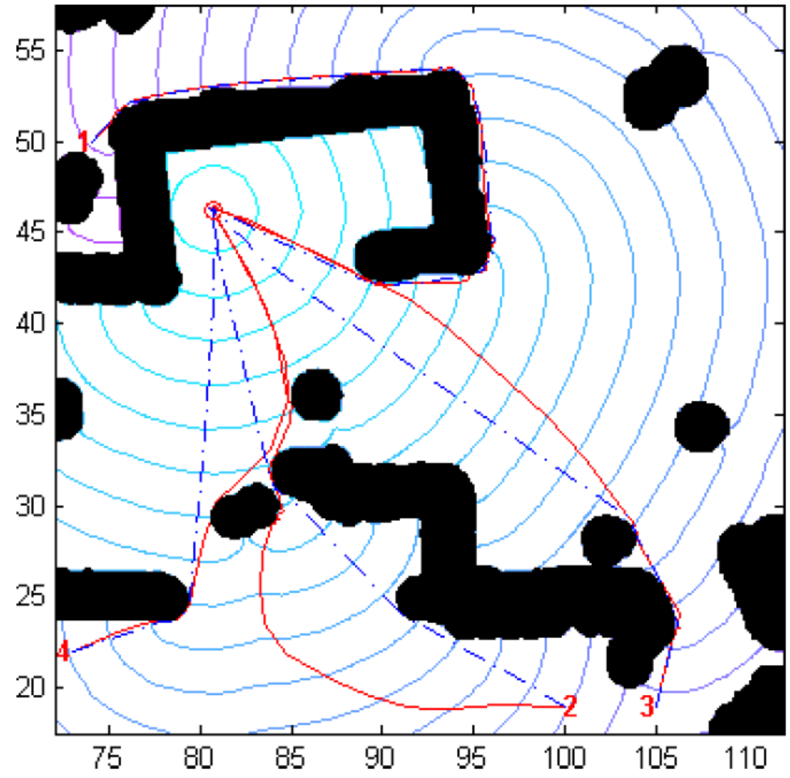


speed profile

$$\mathcal{A}_f(x) = \{y \mid \|y - \vec{f}_w(x)\| \le f_r\}$$



with wind



with and without wind

Path Planning: An Application of the Static Hamilton Jacobi Equation

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