Stable discretisations for sparse FFTs

Stefan Kunis[1]

UNIVERSITÄT OSNABRÜCK

joint work with T. Görner, L. Kämmerer, I. Melzer, and D. Potts

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Outline

1. Motivation - DFT, Compressed sensing, Prony's method
2. Hyperbolic cross FFT
3. Butterfly sparse FFT
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Motivation - DFT, Compressed sensing, Prony's method

- torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \cong \left[-\frac{1}{2}, \frac{1}{2}\right)^d$, index set

$$I_N = \mathbb{Z}^d \cap \left[-\frac{N}{2}, \frac{N}{2}\right)^d$$

- trigonometric polynomials

$$f\left(\mathbf{x}\right) = \sum_{\mathbf{k} \in I_N} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{x}}$$

- (inverse) discrete Fourier transform (DFT)

$$\mathbf{f} = \mathbf{F} \hat{\mathbf{f}}, \qquad f_{\mathbf{j}} = \sum_{\mathbf{k} \in I_N} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{j} / N}, \quad \mathbf{j} \in I_N$$

- FFT (Gauß; Cooley, Tukey; Frigo, Johnson)

$$\mathcal{O}\left(N^d \log N\right)$$

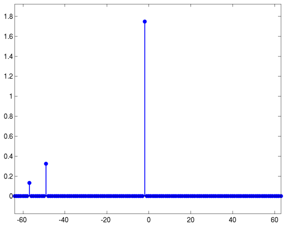
Motivation - DFT, Compressed sensing, Prony's method

- recover $T \subset I_N$ and $\hat{f}_{\mathbf{k}} \in \mathbb{C}$, $\mathbf{k} \in T$, from

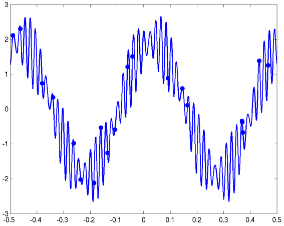
$$f_{\mathbf{x}} = \sum_{k \in T} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{x}}, \quad \mathbf{x} \in X \subset \frac{1}{N} I_N$$

- underdetermined linear system $S = |T| \leq |X| = M \ll N^d$

$$\mathbf{F}_{X \times I_N} \hat{\mathbf{f}} = \mathbf{f}, \qquad \mathbf{F}_{X \times I_N} \in \mathbb{C}^{M \times N^d}$$



$\hat{\mathbf{f}}$



$f(x)$ and $\mathbf{f}$

Motivation - DFT, Compressed sensing, Prony's method

- random sampling, recovery with probability $1-\eta$

$$X \subset \frac{1}{N} I_N \quad \text{or} \quad X \subset \mathbb{T}^d$$

- for each $\hat{\mathbf{f}} \in \mathbb{C}^{N^d}$, $\operatorname{supp} \hat{\mathbf{f}} = T$, if (Thresholding; Rauhut, K.)

$$M \geq C \cdot \frac{\max_{\mathbf{k} \in T} |\hat{f}_{\mathbf{k}}|^2}{\min_{\mathbf{k} \in T} |\hat{f}_{\mathbf{k}}|^2} \cdot S \cdot \log(N^d/\eta)$$

- for every $\hat{\mathbf{f}} \in \mathbb{C}^{N^d}$, $|\operatorname{supp} \hat{\mathbf{f}}| \leq S$, if (OMP; Tropp; Rauhut, K.)

$$M \geq C \cdot S^2 \cdot \log(N^d/\eta)$$

(ℓ^1, ROMP; Candes, Romberg, Tao; Rauhut; Needell, Vershynin)

$$M \geq C \cdot S \cdot \log^4(N^d) \log(1/\eta)$$

Motivation - DFT, Compressed sensing, Prony's method

- deterministic sampling, $d = 2$, quadratic chirp

  (Weil; Strohmer, Heath; Pfander, Rauhut; Applebaum, Calderbank, Howard, Jafarpour, Searle; Xu)

  $$X = \{\frac{1}{M}(j, j^2)^\top \mod 1 : j = 1, \ldots, M\}$$

  result based on coherence: for every $\hat{\mathbf{f}} \in \mathbb{C}^{N^2}$, $|\operatorname{supp} \hat{\mathbf{f}}| \leq S$, if

  $$M \geq \max\{N, (2S-1)^2 + 1\}, \quad M \text{ prime}$$

- X, $T_0 \subset I_N$, $|I_N| = N^2$ (Bourgain, Dilworth, Ford, Konyagin, Kutzarova)

  $$M = N \leq |T_0| \leq M^{1+\varepsilon}, \quad |T_0| \text{ prime}$$

  for every $\hat{\mathbf{f}} \in \mathbb{C}^{|T_0|}$, $|\operatorname{supp} \hat{\mathbf{f}}| \leq S$, if

  $$M \geq S^{2-\varepsilon}$$

Motivation - DFT, Compressed sensing, Prony's method

- sublinear-time (Gilbert, Guha, Indyk, Muthukrishnan, Strauss; Zou, Daubechies; Iwen)

$$X = \bigcup_{\substack{p \leq \ldots \\ \text{prime}}} \{\frac{j}{p} : j = 1, \ldots, p\}$$

result based on aliasing: for every $\hat{\mathbf{f}} \in \mathbb{C}^N$, $|\operatorname{supp} \hat{\mathbf{f}}| \leq S$, if

$$M \geq C \cdot S^2 \cdot \log^4 N, \quad \text{with linear runtime in } M$$

- stable Prony-type method $T \subset [-\frac{N}{2}, \frac{N}{2}]$ (Prony; ...; Potts, Tasche)

$$X = \{\frac{j}{N} : j = 1, \ldots, M\}$$

for each $\hat{\mathbf{f}} \in \mathbb{C}^S$ with well separated frequencies

$$M \geq C \cdot N \cdot \max_{\substack{k,k' \in T \\ k \neq k'}} |k - k'|^{-1}$$

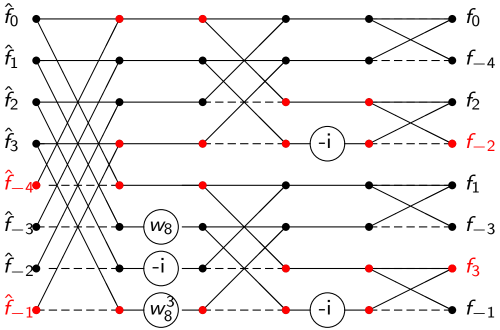
Motivation - DFT, Compressed sensing, Prony's method

- sparse DFT, $T, X \subset I_N$, $S = |T| = |X| \ll N^d$

$$f_{\mathbf{j}} = \sum_{k \in T} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{j}/N}, \quad \mathbf{j} \in X$$

$$\mathcal{O}\left(S^2\right)$$

- divide and conquer, compute on nonzeros - pruning



$$\mathcal{O}\left(N \log \frac{S^2}{N}\right)$$

for $S \geq \sqrt{N}$

Hyperbolic cross FFT

- $d \in \mathbb{N}$, discrete Fourier transform

$$
\begin{aligned}
f(\mathbf{x}) &= \sum_{\mathbf{k}\in\hat{G}_n^d} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}} \\
\hat{G}_n^d &= (-2^{n-1}, 2^{n-1}]^d \cap \mathbb{Z}^d \\
\mathbf{x} &= \left(\frac{j_1}{2^n}, \ldots, \frac{j_d}{2^n}\right)^T \in \mathbb{T}^d, \quad j_1, \ldots, j_d \in \{0, \ldots, 2^n - 1\}
\end{aligned}
$$

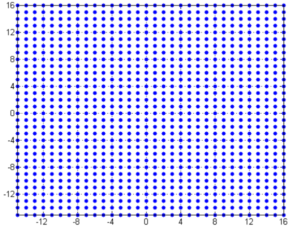
- unitary up to a scaling factor
- problem size $|\hat{G}_n^d| = 2^{nd}$, complexity
  - DFT: $\mathcal{O}(2^{2nd})$ or $\mathcal{O}(2^{n(d+1)})$
  - FFT: $\mathcal{O}(2^{nd} nd)$
- problem size and complexity increase strongly with dimension d

Hyperbolic cross FFT

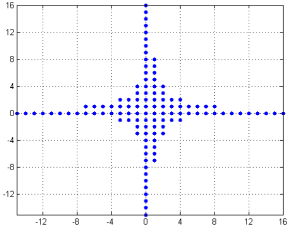
- evaluate trigonometric polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in H_n^d} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{x}}$$

- aim
  - stable spatial discretisation
  - fast algorithm



full frequency grid $\hat{G}_5^2$



hyperbolic cross H_5^2

Hyperbolic cross FFT

- one dimensional frequency grid

$$\hat{G}_n = \{-2^{n-1}+1, \ldots, 2^{n-1}\}, \quad \hat{G}_0 = \{0\}$$

- dimension $d \in \mathbb{N}$, refinement $n \in \mathbb{N}_0$

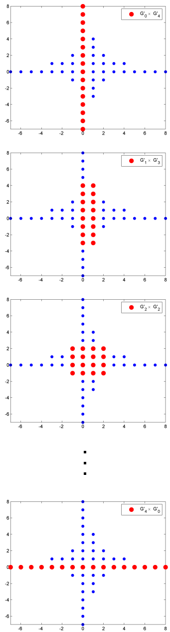
$$H_n^d = \bigcup_{\substack{\mathbf{q} \in \mathbb{N}_0^d \\ \|\mathbf{q}\|_1 = n}} \hat{G}_{q_1} \times \ldots \times \hat{G}_{q_d}$$

- hyperbolic cross

$$\mathbf{k} \in H_n^d \;\Rightarrow\; |k_1 \cdots k_d| \leq 2^{n-d}$$

- problem size

$$\left|H_n^d\right| = C_d 2^n n^{d-1} \ll 2^{nd}$$



Hyperbolic cross FFT

- $G_n = 2^{-n}([0, 2^n) \cap \mathbb{Z})$, sparse grid

$$S_n^d = \bigcup_{\substack{\mathbf{q} \in \mathbb{N}_0^d \\ \|\mathbf{q}\|_1 = n}} G_{q_1} \times \ldots \times G_{q_d}$$

- hyperbolic cross FFT

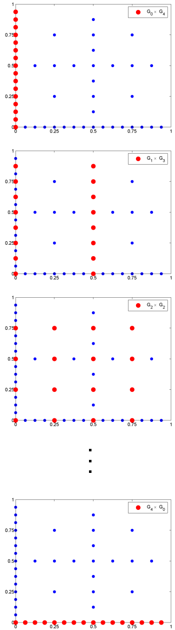
$$f(\mathbf{x}) = \sum_{\mathbf{k} \in H_n^d} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{x}}, \qquad \mathbf{x} \in S_n^d$$

- problem size

$$\left| H_n^d \right| = \left| S_n^d \right| = \mathcal{O}(2^n n^{d-1})$$

- complexity (Baszenski, Delvos 1989; Hallatschek 1992)

$$\mathcal{O}(2^n n^d)$$



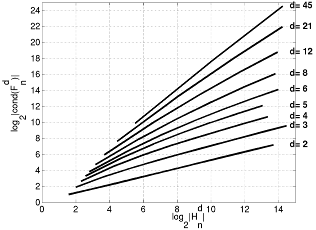
Hyperbolic cross FFT

- Fourier matrix

$$\mathbf{F} = \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}\right)_{\mathbf{x}\in S_n^d,\,\mathbf{k}\in H_n^d}$$

- stability (Kämmerer, K.)

$$c_d 2^{\frac{n}{2}} n^{\frac{2d-3}{2}} \le \kappa(\mathbf{F}) \le C_d 2^{\frac{n}{2}} n^{2d-2}$$



- Boolean sum decomposition, $d = 2$, $n = 1$

$$\mathbf{F}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hyperbolic cross FFT

- random sampling

$$X = \{\mathbf{x}_j \in \mathbb{T}^d : j = 1, \ldots, M\}$$

- Fourier matrix

$$\mathbf{A} = \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{kx}}\right)_{\mathbf{x}\in X,\,\mathbf{k}\in H_n^d} \in \mathbb{C}^{M\times|H_n^d|}, \quad \left|H_n^d\right| = \mathcal{O}(2^n n^{d-1})$$

- w.h.p bounded condition number (Gröchenig, Pötscher, Rauhut)

$$M \geq C|H_n^d| \log |H_n^d| = C_d 2^n n^d$$

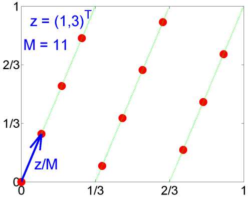
- nonequispaced hyperbolic cross FFT (Döhler, Potts, K.)

$$\mathcal{O}(2^n n^{2d-1})$$

- but ...

Hyperbolic cross FFT

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$



$$\mathbf{x}_j = \frac{j\mathbf{z}}{M} \bmod \mathbf{1};\ j = 1, \ldots, M$$

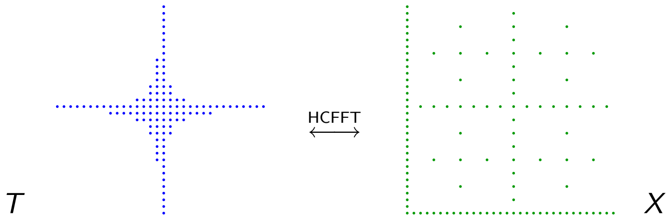
- reformulation as 1-d DFT

$$f(\mathbf{x}_j) = \sum_{\mathbf{k} \in H_n^d} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}_j} = \sum_{l=1}^{M} \left(\sum_{\mathbf{k}\mathbf{z} \equiv l \bmod M} \hat{f}_{\mathbf{k}} \right) \mathrm{e}^{2\pi\mathrm{i}\frac{jl}{M}}$$

- complexity $\mathcal{O}(M \log M + |H_n^d|)$
- if stable, then (Kämmerer, K.)

$$M \geq 2^{2n-2} = C_d |H_n^d|^2 / \log^{2d-2} |H_n^d|$$

Hyperbolic cross FFT



- HCFFT, $S = |T| = |X| = \mathcal{O}(N \log^{d-1} N)$ (Baszenski, Delvos; Hallatschek)

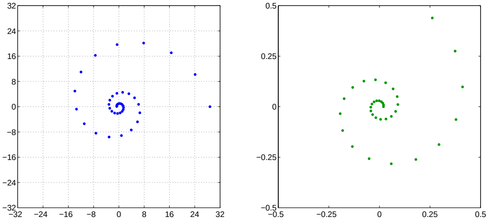
  $\mathcal{O}(S \log N)$

- for arbitrary $X \subset \mathbb{T}^d$, $S = |T| = |X|$ (Döhler, Fenn, Kämmerer, Potts, K.)

  $\mathcal{O}(S \log^{d-1} N \, |\log \varepsilon|^d)$

Butterfly sparse FFT

- smooth and sparse $T \subset [-\frac{N}{2}, \frac{N}{2})^d$, $X \subset [-\frac{N}{2}, \frac{N}{2})^d$



- $|T| = |X| = N^{d-1}$ nodes time frequency and spatial domain
  - ℓth dyadic subdivision of $[-\frac{N}{2}, \frac{N}{2})^d$ into $2^{d\ell}$ boxes has only $2^{(d-1)\ell}$ nonempty ones
  - ℓth dyadic subdivision of $[-\frac{1}{2}, \frac{1}{2})^d$ into $2^{d\ell}$ boxes has only $2^{(d-1)\ell}$ nonempty ones

$$\mathcal{O}(N^{d-1} \log N \; p^{d+1}), \; p = |\log \varepsilon|?$$

(Edelman; Michielsen, Boag; Chew, Song; **Ying**; O'Neil, Woolfe, Rokhlin; Candes, Demanet; Tygert)

Butterfly sparse FFT

- considered model problem, nonsparse, univariate, given

$$T_* = \{k_\ell \in [0, N] : \ell = 0, \ldots, N\}$$
$$X_* = \{x_j \in [0, N] : j = 0, \ldots, N\}$$
$$\hat{\mathbf{f}} = (\hat{f}_k)_{k \in T} \in \mathbb{C}^N$$

- evaluate almost periodic function for $x \in X_*$

$$f(x) = \sum_{k \in T_*} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} k x / N}$$

- FFT for nonequispaced nodes in time and frequency domain

(nnFFT, Elbel, Steidl; Keiner, Potts, K.; type-3 nuFFT, Greengard, Lee)

Butterfly sparse FFT

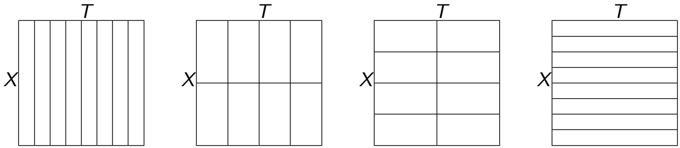
- well known low rank property, $p \geq \lceil \max(2e\pi, |\log_2 \varepsilon|) \rceil$

$$\left| e^{2\pi i kx/N} - \sum_{s=0}^{p-1} \frac{(2\pi i)^s}{N^s s!} k^s x^s \right| \leq \varepsilon, \qquad |kx| \leq \frac{N}{2}$$

- admissible partitions of $T \times X = [0, N]^2$

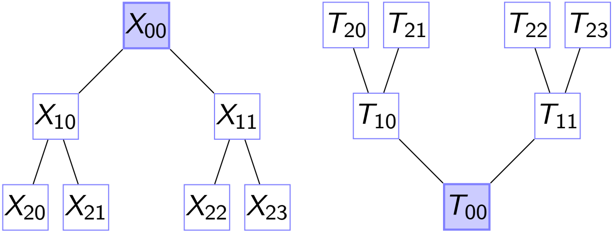
$$\mathrm{diam}(T)\mathrm{diam}(X) \leq N$$

- dyadic decompositions of T and X, examples for $N = 8$

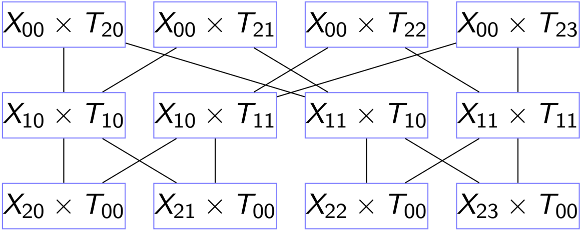


Butterfly sparse FFT

- dyadic decompositions of T and X, examples for $N = 4$



- butterfly graph, nodes are admissible pairs



Butterfly sparse FFT

- local in T, global in X: start with

$$f^{T_{30}}(x) = \sum_{k \in T_{30} \cap T_*} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} kx}$$

- approximate $f^{T_{30}}$ on X_{00}

$$f^{X_{00} T_{30}}$$

- approximate $f^{X_{00} T_{30}} + f^{X_{00} T_{31}}$ on X_{10}

$$f^{X_{10} T_{20}}$$

- ... go on
- local in X, global in T: finally

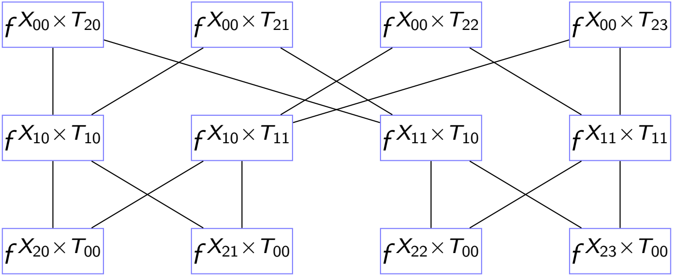
$$f^{X_{30} T_{00}}$$

is an approximation to $f^{T_{00}} = f$ on X_{30}

Butterfly sparse FFT

- approximations

local in T, global in X



local in X, global in T

Butterfly sparse FFT

- frequency band $T = [k_{min}, k_{\max}]$, admissible $X = [x_{\min}, x_{\max}]$
- almost periodic function $g \in \Pi_T$, $T' \subset T$, $g : X \to \mathbb{C}$,

$$g(x) = \sum_{k \in T'} \hat{g}_k \mathrm{e}^{2\pi \mathrm{i} kx/N}$$

- p equispaced frequencies $T_p \subset T$
- p Chebyshev nodes $x_s \in X$
- interpolation operator $\mathcal{J}_p^{XT} : \Pi_{T'} \to \Pi_{T_p}$

$$\mathcal{J}_p^{XT} g(x_s) = g(x_s), \qquad s = 0, \dots, p-1$$

Butterfly sparse FFT

- local error, admissible T, X, $g \in \Pi_T$, and $p \geq 3$ (Melzer, K.)

$$\|g - \mathcal{J}_p^{XT} g\|_{C(X)} \leq C_p \|g\|_{C(X)}$$

with $C_p = \frac{4\pi^p}{4^p p! - 2\pi^p} \leq c_0 c_1^p$

- global error, $N = 2^L$, $T, X = [0, N]^d$, $f \in \Pi_T$, $\varepsilon > 0$, and

$$p \approx |\log \varepsilon| + \log \log N + \log d$$

then (Melzer, K.)

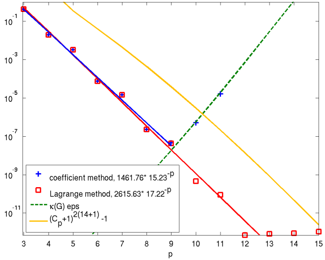
$$\|f - \tilde{f}\|_{C(X)} \leq \varepsilon \|\hat{f}\|_1$$

Butterfly sparse FFT

- local stability, admissible T, X, $p \geq 3$ odd (Melzer, K.)

$$\kappa_{\mathrm{Y}}(\mathcal{J}_p^{XT}) \geq \frac{1}{\sqrt{p}} \left(\frac{2(p-1)}{\pi} \right)^{p-1}$$

$$\kappa_{\mathrm{L}}(\mathcal{J}_p^{XT}) \leq \frac{\sqrt{2p}}{4} 6^{p+1}$$



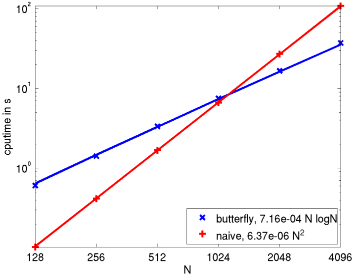
Numerics, application & summary

- smooth and sparse $T, X \subset [0, N]^d$, $|T| = |X| = N^{d-1}$

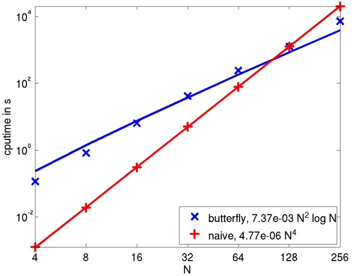
$$f(\mathbf{x}) = \sum_{\mathbf{k} \in T} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{x}/N}$$

- computation time

naive	$\mathcal{O}(N^{2d-2})$
butterfly	$\mathcal{O}(N^{d-1} \log N \, (\lvert \log \varepsilon \rvert + \log \log N)^{d+1})$



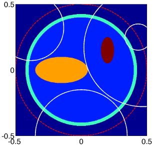
$d = 2$

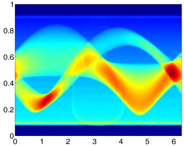


$d = 3$

Numerics, application & summary

- $2d$ spherical mean values, forward simulation





N^2 image data $\longrightarrow$ N acoustic sensors, N times/radii

N^3 volume data $\longrightarrow$ N^2 acoustic sensors, N times/radii

- $3d$ photoacoustic imaging, forward simulation

naive	$\mathcal{O}(N^5)$
nonequispaced FFTs	$\mathcal{O}(N^4 \log N + N^3 \lvert \log \varepsilon \rvert^3)$
butterfly sparse FFT	$\mathcal{O}(N^3 \log N \, (\lvert \log \varepsilon \rvert + \log \log N)^5)$

Numerics, application & summary

- sparse DFT, $T, X \subset I_N$, $S = |T| = |X| \ll N^d$

$$f_{\mathbf{j}} = \sum_{k \in T} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{kj}/N}, \quad \mathbf{j} \in X$$

- direct computation, pruned FFT

$$\mathcal{O}(S^2), \quad \mathcal{O}(N^d \log S)$$

- hyperbolic cross FFT, butterfly sparse FFT

$$\mathcal{O}(S \log N), \quad \mathcal{O}(S \log N (\log\log N)^{d+1})$$

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