The convex intersection body of a convex body

Mathieu Meyer and Shlomo Reisner

Université de Paris Est Marne-la-Vallée, University of Haifa

Banff, May 2011

Introduction

Let L be a convex body in $\mathbb{R}^n$ containing 0 in its interior.

Introduction

Let L be a convex body in $\mathbb{R}^n$ containing 0 in its interior.

The **intersection body $I(L)$ of L** is defined by its radial function:

$$\rho_{I(L)}(u) = \mathrm{vol}(L \cap u^{\perp}), \ u \in S^{n-1}$$

Introduction

Let L be a convex body in $\mathbb{R}^n$ containing 0 in its interior.

The **intersection body $I(L)$ of L** is defined by its radial function:

$$\rho_{I(L)}(u) = \mathrm{vol}(L \cap u^{\perp}),\ u \in S^{n-1}$$

The **cross-section body $C(L)$ of L**, is defined by

$$\rho_{C(L)}(u) = \max_{t} \mathrm{vol}\big((L \cap (tu + u^{\perp})\big).$$

Introduction

Let L be a convex body in $\mathbb{R}^n$ containing 0 in its interior.

The **intersection body $I(L)$ of L** is defined by its radial function:

$$\rho_{I(L)}(u) = \mathrm{vol}(\mathrm{L} \cap \mathrm{u}^{\perp}), \ \mathrm{u} \in \mathrm{S}^{\mathrm{n}-1}$$

The **cross-section body $C(L)$ of L**, is defined by

$$\rho_{C(L)}(u) = \max_t \mathrm{vol}\big((\mathrm{L} \cap (\mathrm{tu} + \mathrm{u}^{\perp})\big).$$

In general, $I(L)$ and $C(L)$ are not convex bodies, although they are identical and convex when L is centrally symmetric. This follows from Brunn-Minkowski theorem and Busemann's theorem (1950).

Introduction

Let L be a convex body in $\mathbb{R}^n$ containing 0 in its interior.

The **intersection body $I(L)$ of L** is defined by its radial function:

$$\rho_{I(L)}(u) = \mathrm{vol}(L \cap u^{\perp}), \; u \in S^{n-1}$$

The **cross-section body $C(L)$ of L**, is defined by

$$\rho_{C(L)}(u) = \max_t \mathrm{vol}\big((L \cap (tu + u^{\perp})\big).$$

In general, $I(L)$ and $C(L)$ are not convex bodies, although they are identical and convex when L is centrally symmetric. This follows from Brunn-Minkowski theorem and Busemann's theorem (1950).

We define a new convex body associated with L, generalizing $I(L)$ and $C(L)$, the **convex intersection body $CI(L)$ of L** by its radial function

$$\rho_{CI(L)}(u) = \min_{z \in P_u(L^{*g(L)})} \mathrm{vol}_{n-1}\Big(\big[P_u(L^{*g(L)})\big]^{*z}\Big). \qquad (1)$$

Introduction

In the formula : $\rho_{CI(L)}(u) = \min_{z \in P_u(L^{*g(L)})} \mathrm{vol_{n-1}}\Big(\big[\mathrm{P_u}(\mathrm{L}^{*g(L)})\big]^{*z}\Big)$,
$g(L)$ is the centroid of L, P_u denotes the orthogonal projection from $\mathbb{R}^n$ onto $u^{\perp}$, and if $E \subset R^n$ is an affine subspace, $M \subset E$ and $z \in E$, $M^{*z} = \{y \in E; \langle y - z, x - z\rangle \leq 1$ for every $x \in M\}$.

Introduction

In the formula : $\rho_{CI(L)}(u) = \min_{z\in P_u(L^{*g(L)})} \mathrm{vol_{n-1}}\Big(\big[\mathrm{P_u(L^{*g(L)})}\big]^{*z}\Big)$,
$g(L)$ is the centroid of L, P_u denotes the orthogonal projection from $\mathbb{R}^n$ onto $u^{\perp}$, and if $E \subset R^n$ is an affine subspace, $M \subset E$ and $z \in E$, $M^{*z} = \{y \in E; \langle y - z, x - z\rangle \leq 1 \text{ for every } x \in M\}$.

This means thus : First apply duality with respect to $g(L)$, then project onto $u^{\perp}$, finally apply duality with respect to z and minimize the $(n-1)$-dimensional volume over z.

Introduction

In the formula : $\rho_{CI(L)}(u) = \min_{z \in P_u(L^{*g(L)})} \mathrm{vol_{n-1}}\Big(\big[\mathrm{P_u}(\mathrm{L}^{*\mathrm{g(L)}})\big]^{*\mathrm{z}}\Big)$,
$g(L)$ is the centroid of L, P_u denotes the orthogonal projection from $\mathbb{R}^n$ onto $u^\perp$, and if $E \subset R^n$ is an affine subspace, $M \subset E$ and $z \in E$, $M^{*z} = \{y \in E; \langle y - z, x - z\rangle \leq 1$ for every $x \in M\}$.

This means thus : First apply duality with respect to $g(L)$, then project onto $u^\perp$, finally apply duality with respect to z and minimize the $(n-1)$-dimensional volume over z.

As we shall see, this convex intersection body of L is actually convex !

Notation

If L a convex set in $\mathbb{R}^n$, let $[L]$ be the affine space spanned by L and $z \in \mathrm{relint}(L)$, **the polar body of L with respect to z** is

$$L^{*z} = \{y \in [L]; \langle y-z, x-z\rangle \le 1 \text{ for all } x \in L\} = \big((L-z)^* + z\big) \cap [L],$$

where $M^{=}M^{*0}$.

Notation

If L a convex set in $\mathbb{R}^n$, let $[L]$ be the affine space spanned by L and $z \in \mathrm{relint}(L)$, **the polar body of L with respect to z** is

$$L^{*z} = \{y \in [L]; \langle y-z, x-z\rangle \leq 1 \text{ for all } x \in L\} = \big((L-z)^* + z\big) \cap [L],$$

where $M^{=}M^{*0}$. The function $z \to \mathrm{vol(L^{*z})}$ is strictly convex on the relative interior of L and tends to $+\infty$ as z approaches the relative boundary of L in $[L]$. So it reaches a minimum at a unique point $s(L) \in \mathrm{int}(L)$, the **Santaló point of L**.

Notation

If L a convex set in $\mathbb{R}^n$, let $[L]$ be the affine space spanned by L and $z \in \mathrm{relint}(L)$, **the polar body of L with respect to z** is

$$L^{*z} = \{y \in [L]; \langle y - z, x - z \rangle \leq 1 \text{ for all } x \in L\} = \big((L - z)^* + z\big) \cap [L],$$

where $M^{=}M^{*0}$. The function $z \to \mathrm{vol(L^{*z})}$ is strictly convex on the relative interior of L and tends to $+\infty$ as z approaches the relative boundary of L in $[L]$. So it reaches a minimum at a unique point $s(L) \in \mathrm{int}(L)$, the **Santaló point of L**. We denote $L^{*s} := L^{*s(L)}$.

Notation

If L a convex set in $\mathbb{R}^n$, let $[L]$ be the affine space spanned by L and $z \in \mathrm{relint}(L)$, **the polar body of L with respect to z** is

$$L^{*z} = \{y \in [L]; \langle y - z, x - z\rangle \leq 1 \text{ for all } x \in L\} = \big((L - z)^* + z\big) \cap [L],$$

where $M^{=}M^{*0}$. The function $z \to \mathrm{vol(L^{*z})}$ is strictly convex on the relative interior of L and tends to $+\infty$ as z approaches the relative boundary of L in $[L]$. So it reaches a minimum at a unique point $s(L) \in \mathrm{int}(L)$, the **Santaló point of L**. We denote $L^{*s} := L^{*s(L)}$.

The Santaló point $s(L)$ is the unique point $z \in \mathrm{int}(L)$ which is the centroid of L^{*z} : one has $z = s(L)$ iff $g(L^{*z}) = z$, where $g(M)$ denotes the centroid of M in $|M]$.

Notation

If L a convex set in $\mathbb{R}^n$, let $[L]$ be the affine space spanned by L and $z \in \mathrm{relint}(L)$, **the polar body of L with respect to z** is

$$L^{*z} = \{y \in [L]; \langle y-z, x-z\rangle \leq 1 \text{ for all } x \in L\} = \left((L-z)^* + z\right) \cap [L],$$

where $M^{=}M^{*0}$. The function $z \to \mathrm{vol(L^{*z})}$ is strictly convex on the relative interior of L and tends to $+\infty$ as z approaches the relative boundary of L in $[L]$. So it reaches a minimum at a unique point $s(L) \in \mathrm{int}(L)$, the **Santaló point of L**. We denote $L^{*s} := L^{*s(L)}$. The Santaló point $s(L)$ is the unique point $z \in \mathrm{int}(L)$ which is the centroid of L^{*z} : one has $z = s(L)$ iff $g(L^{*z}) = z$, where $g(M)$ denotes the centroid of M in $|M]$. In general, $s(L) \neq g(L)$.

The body $J(K)$

We define first define a body $J(K)$ attached to any body K. For, $x \neq 0$, let $P_x : \mathbb{R}^n \to x^\perp$ be the orthogonal projection onto $x^\perp$.

The body $J(K)$

We define first define a body $J(K)$ attached to any body K. For, $x \neq 0$, let $P_x : \mathbb{R}^n \to x^\perp$ be the orthogonal projection onto $x^\perp$.

Theorem

If K is a convex body in $\mathbb{R}^n$. Define $N_K : \mathbb{R}^n \to \mathbb{R}_+$ by :

$$N_K(u) = \frac{1}{\mathrm{vol}\big((\mathrm{P_u K})^{*s}\big)} = \frac{1}{\min_{z \in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big)} \text{ for } u \in S^{n-1},$$

and $N_K(ru) = rN_K(u)$ for $r \geq 0$. Then N_K is a norm on $\mathbb{R}^n$.

The body $J(K)$

We define first define a body $J(K)$ attached to any body K. For, $x \neq 0$, let $P_x : \mathbb{R}^n \to x^\perp$ be the orthogonal projection onto $x^\perp$.

Theorem

If K is a convex body in $\mathbb{R}^n$. Define $N_K : \mathbb{R}^n \to \mathbb{R}_+$ by :

$$N_K(u) = \frac{1}{\mathrm{vol}\big((\mathrm{P_u K})^{*s}\big)} = \frac{1}{\min_{z \in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big)} \text{ for } u \in S^{n-1},$$

and $N_K(ru) = rN_K(u)$ for $r \geq 0$. Then N_K is a norm on $\mathbb{R}^n$.

Definition. The preceding theorem associates to any convex body K a centrally symmetric convex body $J(K)$ in $\mathbb{R}^n$ defined by

$$J(K) = \{x \in \mathbb{R}^n; N_K(x) \leq 1\}.$$

Its radial function is $r_{J(K)}(u) = \mathrm{vol}\big((\mathrm{P_u K})^{*s}\big)$.

Why is $J(K)$ convex ?

I recall some facts.

Definition. Let $v \in S^{n-1}$, $B \subset \mathbb{R}^n$ bounded and $V : B \to \mathbb{R}$ bounded. The **shadow system (L_t), $t \in [a, b]$, of convex bodies in $\mathbb{R}^n$, with direction v, basis B and speed V**, is the family of convex bodies

$$L_t = \mathrm{conv}(\{b + tV(b)v; b \in B\}, \text{ for } t \in [a, b]).$$

Why is $J(K)$ convex ?

I recall some facts.

Definition. Let $v \in S^{n-1}$, $B \subset \mathbb{R}^n$ bounded and $V : B \to \mathbb{R}$ bounded. The **shadow system (L_t), $t \in [a, b]$, of convex bodies in $\mathbb{R}^n$, with direction v, basis B and speed V**, is the family of convex bodies

$$L_t = \mathrm{conv}(\{b + tV(b)v; b \in B\}, \text{ for } t \in [a, b]).$$

The following, due to Shephard, was also used by Campi-Gronchi.

Proposition Let K be a convex body in $\mathbb{R}^n$. Then, for $u, v \in S^{n-1}$, such that $\langle u, v\rangle = 0$, the family $L_t = \Pi_{u+tv,u^\perp} K$, $t \in \mathbb{R}$, is a shadow system of convex bodies in $u^\perp$, in the direction v.

Why is $J(K)$ convex ?

I recall some facts.

Definition. Let $v \in S^{n-1}$, $B \subset \mathbb{R}^n$ bounded and $V : B \to \mathbb{R}$ bounded. The **shadow system (L_t), $t \in [a,b]$, of convex bodies in $\mathbb{R}^n$, with direction v, basis B and speed V**, is the family of convex bodies

$$L_t = \mathrm{conv}(\{b + tV(b)v; b \in B\}, \text{ for } t \in [a,b]).$$

The following, due to Shephard, was also used by Campi-Gronchi.

Proposition Let K be a convex body in $\mathbb{R}^n$. Then, for $u, v \in S^{n-1}$, such that $\langle u, v\rangle = 0$, the family $L_t = \Pi_{u+tv,u^\perp} K$, $t \in \mathbb{R}$, is a shadow system of convex bodies in $u^\perp$, in the direction v.

Here $\Pi_{x,y^\perp} : \mathbb{R}^n \to y^\perp$ denotes *the linear projection onto $y^\perp$ with direction parallel to $x \notin y^\perp := \{z \in \mathbb{R}^n; \langle z, y\rangle = 0\}$*

Proof of the proposition

Proof of the proposition

We set $u = e_n$ and $v = e_{n-1}$, where $e_1, \dots, e_n$ is an orthonormal basis of $\mathbb{R}^n$ ($\mathbb{R}^j = [e_1, \dots, e_j]$).

Proof of the proposition

We set $u = e_n$ and $v = e_{n-1}$, where $e_1, \dots, e_n$ is an orthonormal basis of $\mathbb{R}^n$ ($\mathbb{R}^j = [e_1, \dots, e_j]$). For $t \in \mathbb{R}$, one has

$$\Pi_{u+tv,u^\perp} K = \{U - rte_{n-1}\,;\,(U,r) \in P_u K \times \mathbb{R} \text{ such that } U + re_n \in K\}.$$

Proof of the proposition

We set $u = e_n$ and $v = e_{n-1}$, where $e_1, \dots, e_n$ is an orthonormal basis of $\mathbb{R}^n$ ($\mathbb{R}^j = [e_1, \dots, e_j]$). For $t \in \mathbb{R}$, one has

$$\Pi_{u+tv,u^\perp} K = \{U - rte_{n-1}\,;\, (U, r) \in P_u K \times \mathbb{R} \text{ such that } U + re_n \in K\}.$$

For $U \in P_u K$, let $\mathcal{I}(U) = \{r \in \mathbb{R}\,;\, U + re_n \in K\} = [a(U), b(U)]$.
Define $x(U) \in \mathbb{R}$ by $\langle U - x(U)e_{n-1}, e_{n-1}\rangle = 0$.

Proof of the proposition

We set $u = e_n$ and $v = e_{n-1}$, where $e_1, \dots, e_n$ is an orthonormal basis of $\mathbb{R}^n$ ($\mathbb{R}^j = [e_1, \dots, e_j]$). For $t \in \mathbb{R}$, one has

$$\Pi_{u+tv,u^\perp} K = \{U - rte_{n-1}\,;\,(U,r) \in P_u K \times \mathbb{R} \text{ such that } U + re_n \in K\}.$$

For $U \in P_u K$, let $\mathcal{I}(U) = \{r \in \mathbb{R}\,;\, U + re_n \in K\} = [a(U), b(U)]$.
Define $x(U) \in \mathbb{R}$ by $\langle U - x(U)e_{n-1}, e_{n-1}\rangle = 0$. Let

$$D_1 = \{U \in P_u K; x(U) \in \mathbf{Q}\} \text{ and } D_2 = \{U \in P_u K; x(U) \in \mathbb{R} \setminus \mathbf{Q}\}.$$

Define $V : P_u K \to \mathbb{R}$ by

$$v(U) = -b(U) \text{ if } U \in D_1 \text{ and } v(U) = -a(U) \text{ if } U \in D_2.$$

By the continuity of the concave functions $-a, b : P_u(K) \to \mathbb{R}$, we get

$$\Pi_{u+tv,u^\perp} K = \mathrm{conv}\{U + tV(U)e_n; U \in P_u K\} \text{ for all } t \in \mathbb{R}$$

The converse statement

To understand better, observe that the converse statement of the last proposition is true :

Every shadow system L_t in $\mathbb{R}^n$ can be seen as $L_t = \Pi_{u+tv,u^\perp}(K)$ for some convex body $K \subset \mathbb{R}^{n+1}$ and $u, v \in S^n$.

The converse statement

To understand better, observe that the converse statement of the last proposition is true :

Every shadow system L_t in $\mathbb{R}^n$ can be seen as $L_t = \Pi_{u+tv,u^\perp}(K)$ for some convex body $K \subset \mathbb{R}^{n+1}$ and $u, v \in S^n$.

This is very simple : embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ with $v = e_n$ and $u = e_{n+1}$.

The converse statement

To understand better, observe that the converse statement of the last proposition is true :

Every shadow system L_t in $\mathbb{R}^n$ can be seen as $L_t = \Pi_{u+tv,u^\perp}(K)$ for some convex body $K \subset \mathbb{R}^{n+1}$ and $u, v \in S^n$.

This is very simple : embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ with $v = e_n$ and $u = e_{n+1}$. Let $M = \mathrm{conv}(\{b - V(b)u; b \in B\}$. Then

$$\Pi_{u+tv,u^\perp}(b - V(b)u) = b + tV(b)v \in u^\perp$$

because $b - V(b)u - (b + tV(b)v) = -V(b)(u + tv)$ is parallel to $u + tv$.

The converse statement

To understand better, observe that the converse statement of the last proposition is true :

Every shadow system L_t in $\mathbb{R}^n$ can be seen as $L_t = \Pi_{u+tv,u^\perp}(K)$ for some convex body $K \subset \mathbb{R}^{n+1}$ and $u, v \in S^n$.

This is very simple : embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ with $v = e_n$ and $u = e_{n+1}$. Let $M = \mathrm{conv}(\{b - V(b)u; b \in B\}$. Then

$$\Pi_{u+tv,u^\perp}(b - V(b)u) = b + tV(b)v \in u^\perp$$

because $b - V(b)u - (b + tV(b)v) = -V(b)(u + tv)$ is parallel to $u + tv$. It follows that

$$\Pi_{u+tv,u^\perp} M = L_t.$$

The volume of the polar in shadow systems

The volume of the polar in shadow systems

The following result was proved by Reisner and MM (07):

Theorem

Let $t \in [a,b] \to L_t$ be a shadow system in $\mathbb{R}^n$; define

$$\phi(t) = \frac{1}{\mathrm{vol}\big((\mathrm{L_t})^{*\mathrm{s}}\big)} = \frac{1}{\min_z \mathrm{vol}\big((\mathrm{L_t})^{*\mathrm{z}}\big)} \ .$$

Then ϕ is a convex function on $[a,b]$.

The volume of the polar in shadow systems

The following result was proved by Reisner and MM (07):

Theorem

Let $t \in [a,b] \to L_t$ be a shadow system in $\mathbb{R}^n$; define

$$\phi(t) = \frac{1}{\mathrm{vol}\big((\mathrm{L_t})^{*\mathrm{s}}\big)} = \frac{1}{\min_z \mathrm{vol}\big((\mathrm{L_t})^{*\mathrm{z}}\big)} .$$

Then ϕ is a convex function on $[a,b]$.

We use also :

Lemma Suppose that $N : \mathbb{R}^n \to \mathbb{R}$ satisfies $N(x) > 0$ for $x \neq 0$, $N(\alpha x) = |\alpha| N(x)$ for all $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$ and that the following simple lemma for all $u, v \in S^{n-1}$ with $\langle u, v\rangle = 0$, $t \mapsto N(u + tv)$ is convex. Then N is a norm on $\mathbb{R}^n$.

Proof of the convexity of $J(K)$.

We want to prove the following :

Theorem For a convex body K, $r_{J(K)}(u) = \min_{z \in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big)$ is the radial function of a centrally symmetric convex body.

Proof of the convexity of $J(K)$.

We want to prove the following :

Theorem For a convex body K, $r_{J(K)}(u) = \min_{z \in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big)$ is the radial function of a centrally symmetric convex body.

Proof. By the lemma, we need only to check that if $u, v \in S_{n-1}$ satisfy $\langle u, v \rangle = 0$, then $t \to g_{u,v}(t) = N_K(u + tv)$ is convex.

Proof of the convexity of $J(K)$.

We want to prove the following :

Theorem For a convex body K, $r_{J(K)}(u) = \min_{z\in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big)$ is the radial function of a centrally symmetric convex body.

Proof. By the lemma, we need only to check that if $u, v \in S_{n-1}$ satisfy $\langle u, v\rangle = 0$, then $t \to g_{u,v}(t) = N_K(u + tv)$ is convex. The projection $P_{u+tv}K$ of K on $\{u + tv\}^\perp$ in the direction $u + tv$ is an affine image of $\Pi_{u+tv,u^\perp}K$ and

$$\mathrm{vol(P_{u+tv}K)} = \frac{1}{\sqrt{1+t^2}}\mathrm{vol(\Pi_{u+tv,u^\perp}K)}.$$

Proof of the convexity of $J(K)$.

We want to prove the following :

Theorem For a convex body K, $r_{J(K)}(u) = \min_{z\in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big)$ is the radial function of a centrally symmetric convex body.

Proof. By the lemma, we need only to check that if $u, v \in S_{n-1}$ satisfy $\langle u, v\rangle = 0$, then $t \to g_{u,v}(t) = N_K(u + tv)$ is convex. The projection $P_{u+tv}K$ of K on $\{u + tv\}^\perp$ in the direction $u + tv$ is an affine image of $\Pi_{u+tv,u^\perp}K$ and

$$\mathrm{vol(P_{u+tv}K)} = \frac{1}{\sqrt{1+t^2}}\mathrm{vol}(\Pi_{u+tv,u^\perp}K).$$

Hence

$$\min_{z\in\{u+tv\}^\perp} \mathrm{vol}\big((\mathrm{P_{u+tv}K})^{*z}\big) = \sqrt{1+t^2}\min_{z\in u^\perp} \mathrm{vol}\big((\Pi_{u+tv,u^\perp}\mathrm{K})^{*z}\big)\big)$$

It follows that

$$N(u+tv) = \frac{|u+tv|}{\min_{z\in\{u+tv\}^\perp} \mathrm{vol}\big((\mathrm{P_{u+tv}K})^{*z}\big)}$$

$$= \frac{1}{\min_{z\in u^\perp} \mathrm{vol}\big((\Pi_{\mathrm{u+tv,u^\perp}}\mathrm{K})^{*z}\big)} \ .$$

Proof of the theorem about $J(K)$.

It follows that

$$N(u+tv) = \frac{|u+tv|}{\min_{z\in\{u+tv\}^\perp} \mathrm{vol}\big((\mathrm{P}_{u+tv}K)^{*z}\big)\big)}$$

$$= \frac{1}{\min_{z\in u^\perp} \mathrm{vol}\big((\Pi_{u+tv,u^\perp}K)^{*z}\big)}\ .$$

By the proposition, $t \to \Pi_{u+tv,u^\perp}K$ is a shadow system. Thus by the last theorem, $g_{u,v}$ is convex.

Remarks

Remarks.

1) If K is centrally symmetric (centered at 0), then all its projections P_uK are centrally symmetric (centered at 0) so that

$$\min_{z\in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big) = \mathrm{vol}\big((\mathrm{P_u K})^{*0}\big) = \mathrm{vol}(\mathrm{K}^{*0} \cap \mathrm{u}^\perp).$$

Remarks.

1) If K is centrally symmetric (centered at 0), then all its projections $P_u K$ are centrally symmetric (centered at 0) so that

$$\min_{z \in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big) = \mathrm{vol}\big((\mathrm{P_u K})^{*0}\big) = \mathrm{vol}(\mathrm{K}^{*0} \cap \mathrm{u}^\perp).$$

We get that $u \to \mathrm{vol}(\mathrm{K}^{*0} \cap \mathrm{u}^\perp)$ is the radial function of a convex body. This is Busemann theorem for the sections of K^*.

Remarks.

1) If K is centrally symmetric (centered at 0), then all its projections $P_u K$ are centrally symmetric (centered at 0) so that

$$\min_{z\in u^\perp} \mathrm{vol}\big(\mathrm{P_u K})^{*z}\big) = \mathrm{vol}\big(\mathrm{P_u K})^{*0}\big) = \mathrm{vol}(\mathrm{K}^{*0} \cap \mathrm{u}^\perp).$$

We get that $u \to \mathrm{vol}(\mathrm{K}^{*0} \cap \mathrm{u}^\perp)$ is the radial function of a convex body. This is Busemann theorem for the sections of K^*.

2) Let $J(K) = \{x \in \mathbb{R}^n; N_K(x) \leq 1\}$. One has $J(K+x) = J(K)$ and for all linear isomorphism A, $J\big((AK)\big) = |\det(A)|\,(A^*)^{-1}\big(J(K)\big)$.

Remarks.

1) If K is centrally symmetric (centered at 0), then all its projections P_uK are centrally symmetric (centered at 0) so that

$$\min_{z\in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big) = \mathrm{vol}\big((\mathrm{P_u K})^{*0}\big) = \mathrm{vol}(\mathrm{K}^{*0} \cap \mathrm{u}^\perp).$$

We get that $u \to \mathrm{vol}(\mathrm{K}^{*0} \cap \mathrm{u}^\perp)$ is the radial function of a convex body. This is Busemann theorem for the sections of K^*.

2) Let $J(K) = \{x \in \mathbb{R}^n; N_K(x) \leq 1\}$. One has $J(K + x) = J(K)$ and for all linear isomorphism A, $J\big((AK)\big) = |\det(A)|\,(A^*)^{-1}\big(J(K)\big)$.

3) If $n = 2$ and if R is the rotation by angle $\pi/2$ in $\mathbb{R}^2$, then

$$\mathrm{vol}(\mathrm{P_u K}) = \mathrm{h_K}(\mathrm{Ru}) + \mathrm{h_K}(-\mathrm{Ru}) = \mathrm{h_K}(\mathrm{Ru}) + \mathrm{h_{-K}}(\mathrm{Ru}),$$

so that $J(K) = \frac{1}{4}R(K - K)$.

The convex intersection bodies $IC(L, z)$ of a convex body L.

Let L be a convex body in $\mathbb{R}^n$.

The convex intersection bodies $IC(L,z)$ of a convex body L.

Let L be a convex body in $\mathbb{R}^n$. For $z \in \mathrm{int}(L)$, the **intersection body $I(L,z)$ of L with respect to z** is star-body whose radial function $\rho_{I(L,z)}$ is given by

$$\rho_{I(L,z)}(u) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x}-\mathrm{z}, \mathrm{u}\rangle = 0\}\big) = \mathrm{vol}\big(\mathrm{L} \cap (\mathrm{z}+\mathrm{u}^{\perp})\big).$$

The convex intersection bodies $IC(L, z)$ of a convex body L.

Let L be a convex body in $\mathbb{R}^n$. For $z \in \mathrm{int}(L)$, the **intersection body $I(L, z)$ of L with respect to z** is star-body whose radial function $\rho_{I(L,z)}$ is given by

$$\rho_{I(L,z)}(u) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x} - \mathrm{z}, \mathrm{u}\rangle = 0\}\big) = \mathrm{vol}\big(\mathrm{L} \cap (\mathrm{z} + \mathrm{u}^{\perp})\big).$$

The body **cross-section body $C(L)$ of L** was already defined by its radial function :

$$\rho_{C(L)}(u) = \max_{x \in L} \mathrm{vol}\big(\mathrm{L} \cap (\mathrm{x} + \mathrm{u}^{\perp})\big).$$

The convex intersection bodies $IC(L, z)$ of a convex body L.

Let L be a convex body in $\mathbb{R}^n$. For $z \in \mathrm{int}(L)$, the **intersection body $I(L, z)$ of L with respect to z** is star-body whose radial function $\rho_{I(L,z)}$ is given by

$$\rho_{I(L,z)}(u) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x} - \mathrm{z}, \mathrm{u}\rangle = 0\}\big) = \mathrm{vol}\big(\mathrm{L} \cap (\mathrm{z} + \mathrm{u}^{\perp})\big).$$

The body **cross-section body $C(L)$ of L** was already defined by its radial function :

$$\rho_{C(L)}(u) = \max_{x \in L} \mathrm{vol}\big(\mathrm{L} \cap (\mathrm{x} + \mathrm{u}^{\perp})\big).$$

Of course, $I(L, z) \subset C(L)$. Makai, Martini and Odor proved that $I(L, z) = C(L)$ iff L is centrally symmetric about z.

The convex intersection bodies $IC(L,z)$ of a convex body L.

Let L be a convex body in $\mathbb{R}^n$. For $z \in \mathrm{int}(L)$, the **intersection body $I(L,z)$ of L with respect to z** is star-body whose radial function $\rho_{I(L,z)}$ is given by

$$\rho_{I(L,z)}(u) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x}-\mathrm{z}, \mathrm{u}\rangle = 0\}\big) = \mathrm{vol}\big(\mathrm{L} \cap (\mathrm{z}+\mathrm{u}^\perp)\big).$$

The body **cross-section body $C(L)$ of L** was already defined by its radial function :

$$\rho_{C(L)}(u) = \max_{x \in L} \mathrm{vol}\big(\mathrm{L} \cap (\mathrm{x}+\mathrm{u}^\perp)\big).$$

Of course, $I(L,z) \subset C(L)$. Makai, Martini and Odor proved that $I(L,z) = C(L)$ iff L is centrally symmetric about z. We define the **convex intersection body $CI(L,z)$ of L with respect to z** by

$$CI(L,z) = J(L^{*z}) ,$$

and if $z = g(L)$, the centroid of L, we set $CI(L) = CI\big(L, g(L)\big)$.

The radial function of $CI(L, z)$ is thus given for $u \in S^{n-1}$ by

$$\rho_{CI(L,z)}(u) = \min_{x \in u^\perp} \mathrm{vol}\Big(\big(\mathrm{P_u}(\mathrm{L^{*z}})\big)^{*\mathrm{x}}\Big) = \mathrm{vol}\Big(\big(\mathrm{P_u}(\mathrm{L^{*z}})\big)^{*\mathrm{s}}\Big).$$

The radial function of $CI(L, z)$ is thus given for $u \in S^{n-1}$ by

$$\rho_{CI(L,z)}(u) = \min_{x \in u^\perp} \mathrm{vol}\Big(\big(\mathrm{P_u}(\mathrm{L}^{*z})\big)^{*x} \Big) = \mathrm{vol}\Big(\big(\mathrm{P_u}(\mathrm{L}^{*z})\big)^{*s} \Big).$$

In view of the first theorem, one has

Theorem

Let L be a convex body. Then for every $z \in \mathrm{int}(L)$, the convex intersection body $CI(L, z)$ of L with respect to z is a centrally symmetric convex body such that $CI(L, z) \subset I(L, z)$.

Remarks.

Remarks.

1) For every one-to-one affine map $A : \mathbb{R}^n \to \mathbb{R}^n$, $I(AL, Az) = |\det(A)| A^{*-1}\big(I(L, z)\big)$, as well as

$$CI(AL, Az) = |\det(A)|\, A^{*-1}\big(CI(L)\big).$$

Remarks.

1) For every one-to-one affine map $A : \mathbb{R}^n \to \mathbb{R}^n$, $I(AL, Az) = |\det(A)| A^{*-1}\big(I(L, z)\big)$, as well as

$$CI(AL, Az) = |\det(A)|\, A^{*-1}\big(CI(L)\big).$$

2) The inclusion $CI(L, z) \subset I(L, z)$ is exact: there exists u such that

$$\mathrm{vol}\big(L \cap (z + u^{\perp})\big) = \mathrm{vol}\Big(\big(P_u(L^{*z})\big)^{*s}\Big),$$

Remarks.

1) For every one-to-one affine map $A : \mathbb{R}^n \to \mathbb{R}^n$,
$I(AL, Az) = |\det(A)| A^{*-1}\big(I(L,z)\big)$, as well as

$$CI(AL, Az) = |\det(A)|\, A^{*-1}\big(CI(L)\big).$$

2) The inclusion $CI(L,z) \subset I(L,z)$ is exact: there exists u such that

$$\mathrm{vol}\big(\mathrm{L} \cap (\mathrm{z} + \mathrm{u}^\perp)\big) = \mathrm{vol}\Big(\big(\mathrm{P_u}(\mathrm{L}^{*\mathrm{z}})\big)^{*\mathrm{s}}\Big),$$

because this equality means that the z is centroid of $L \cap (z + u^\perp)$. To see that such a u exists, define $\phi : S^{n-1} \to \mathbb{R}$ by
$\phi(v) = \mathrm{vol}(\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x} - \mathrm{z}; \mathrm{v}\rangle \geq 0\})$.

Remarks.

1) For every one-to-one affine map $A : \mathbb{R}^n \to \mathbb{R}^n$,
$I(AL, Az) = |\det(A)| A^{*-1}\big(I(L,z)\big)$, as well as

$$CI(AL, Az) = |\det(A)|\, A^{*-1}\big(CI(L)\big).$$

2) The inclusion $CI(L,z) \subset I(L,z)$ is exact: there exists u such that

$$\mathrm{vol}\big(\mathrm{L} \cap (\mathrm{z} + \mathrm{u}^{\perp})\big) = \mathrm{vol}\Big(\big(\mathrm{P_u}(\mathrm{L}^{*\mathrm{z}})\big)^{*\mathrm{s}}\Big),$$

because this equality means that the z is centroid of $L \cap (z + u^{\perp})$. To see that such a u exists, define $\phi : S^{n-1} \to \mathbb{R}$ by $\phi(v) = \mathrm{vol}(\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x} - \mathrm{z}; \mathrm{v}\rangle \geq 0\})$. Since ϕ is continuous, it reaches its maximum at some $u \in S^{n-1}$, and we know that z is then the centroid of $L \cap (z + u^{\perp})$.

Remarks.

Remarks.

3) It was proved by Grünbaum that for every convex body $L \in \mathbb{R}^n$, there exists some $z_0 \in \mathrm{int}(L)$ such that z_0 is the centroid of $L \cap (z + u_i^\perp)$ for $(n+1)$ different hyperplanes through z_0, with normals $u_1, \ldots, u_{n+1}$. For this z_0, the boundaries of $CI(L, z_0)$ and of $I(L, z_0)$ have at least $2(n+1)$ contact points.

Remarks.

3) It was proved by Grünbaum that for every convex body $L \in \mathbb{R}^n$, there exists some $z_0 \in \mathrm{int}(L)$ such that z_0 is the centroid of $L \cap (z + u_i^\perp)$ for $(n+1)$ different hyperplanes through z_0, with normals $u_1, \dots, u_{n+1}$. For this z_0, the boundaries of $CI(L, z_0)$ and of $I(L, z_0)$ have at least $2(n+1)$ contact points.

4) When L is centrally symmetric about z, $CI(L, z) = I(L, z)$ and the theorem reduces to the classical Busemann's theorem.

Remarks.

3) It was proved by Grünbaum that for every convex body $L \in \mathbb{R}^n$, there exists some $z_0 \in \mathrm{int}(L)$ such that z_0 is the centroid of $L \cap (z + u_i^\perp)$ for $(n+1)$ different hyperplanes through z_0, with normals $u_1, \dots, u_{n+1}$. For this z_0, the boundaries of $CI(L, z_0)$ and of $I(L, z_0)$ have at least $2(n+1)$ contact points.

4) When L is centrally symmetric about z, $CI(L,z) = I(L,z)$ and the theorem reduces to the classical Busemann's theorem. Conversely, :

Proposition $CI(L,z) = I(L,z)$ iff L is centrally symmetric about z.

Remarks.

3) It was proved by Grünbaum that for every convex body $L \in \mathbb{R}^n$, there exists some $z_0 \in \mathrm{int}(L)$ such that z_0 is the centroid of $L \cap (z + u_i^\perp)$ for $(n+1)$ different hyperplanes through z_0, with normals $u_1, \ldots, u_{n+1}$. For this z_0, the boundaries of $CI(L, z_0)$ and of $I(L, z_0)$ have at least $2(n+1)$ contact points.

4) When L is centrally symmetric about z, $CI(L,z) = I(L,z)$ and the theorem reduces to the classical Busemann's theorem. Conversely, :

Proposition $CI(L,z) = I(L,z)$ iff L is centrally symmetric about z.

This follows from the following lemma:

Lemma

Let L be a convex body and $z \in L$. Then z is the centroid of every hyperplane section of L through itself iff $L - z$ is centrally symmetric.

Proof of the lemma.

Proof of the lemma.

Fix some $z_0 \in \mathrm{int}(L)$, $z_0 \neq z$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(y) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L} - \mathrm{z}_0; \langle \mathrm{x}, \mathrm{y} \rangle \geq 1\}\big).$$

By Meyer-Reisner (89), F is C^1 on $\{F > 0\} = \mathbb{R}^n \setminus \{0\}$ and for $y \neq 0$

$$\nabla F(y) = \langle \nabla F(y), y \rangle \big(g(\{x \in L; \langle x - z_0, y \rangle = 1\}) - z_0\big).$$

Proof of the lemma.

Fix some $z_0 \in \mathrm{int}(L)$, $z_0 \neq z$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(y) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L} - \mathrm{z}_0; \langle \mathrm{x}, \mathrm{y} \rangle \geq 1\}\big).$$

By Meyer-Reisner (89), F is C^1 on $\{F > 0\} = \mathbb{R}^n \setminus \{0\}$ and for $y \neq 0$

$$\nabla F(y) = \langle \nabla F(y), y \rangle \big(g(\{x \in L; \langle x - z_0, y \rangle = 1\}) - z_0\big).$$

Let $H = \{y \in \mathbb{R}^n; \langle z - z_0, y \rangle = 1\}$.

Proof of the lemma.

Fix some $z_0 \in \mathrm{int}(L)$, $z_0 \neq z$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(y) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L} - \mathrm{z}_0; \langle \mathrm{x}, \mathrm{y} \rangle \geq 1\}\big).$$

By Meyer-Reisner (89), F is C^1 on $\{F > 0\} = \mathbb{R}^n \setminus \{0\}$ and for $y \neq 0$

$$\nabla F(y) = \langle \nabla F(y), y \rangle \big(g(\{x \in L; \langle x - z_0, y \rangle = 1\}) - z_0\big).$$

Let $H = \{y \in \mathbb{R}^n; \langle z - z_0, y \rangle = 1\}$. For $y \in H$, the hyperplane $\{x \in \mathbb{R}^n; \langle x - z_0, y \rangle = 1\}$ passes through z, so that by the hypothesis, $g(\{w \in L; \langle x - z_0, y \rangle = 1\}) = z$,

Proof of the lemma.

Fix some $z_0 \in \mathrm{int}(L)$, $z_0 \neq z$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(y) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L} - \mathrm{z}_0; \langle \mathrm{x}, \mathrm{y} \rangle \geq 1\}\big).$$

By Meyer-Reisner (89), F is C^1 on $\{F > 0\} = \mathbb{R}^n \setminus \{0\}$ and for $y \neq 0$

$$\nabla F(y) = \langle \nabla F(y), y \rangle \big(g(\{x \in L; \langle x - z_0, y \rangle = 1\}) - z_0 \big).$$

Let $H = \{y \in \mathbb{R}^n; \langle z - z_0, y \rangle = 1\}$. For $y \in H$, the hyperplane $\{x \in \mathbb{R}^n; \langle x - z_0, y \rangle = 1\}$ passes through z, so that by the hypothesis, $g(\{w \in L; \langle x - z_0, y \rangle = 1\}) = z$, and thus

$$\nabla F(y) = \langle \nabla F(y), y \rangle (z - z_0).$$

Proof of the lemma.

Fix some $z_0 \in \mathrm{int}(L)$, $z_0 \neq z$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(y) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L} - \mathrm{z}_0; \langle \mathrm{x}, \mathrm{y} \rangle \geq 1\}\big).$$

By Meyer-Reisner (89), F is C^1 on $\{F > 0\} = \mathbb{R}^n \setminus \{0\}$ and for $y \neq 0$

$$\nabla F(y) = \langle \nabla F(y), y \rangle \big(g(\{x \in L; \langle x - z_0, y \rangle = 1\}) - z_0\big).$$

Let $H = \{y \in \mathbb{R}^n; \langle z - z_0, y \rangle = 1\}$. For $y \in H$, the hyperplane $\{x \in \mathbb{R}^n; \langle x - z_0, y \rangle = 1\}$ passes through z, so that by the hypothesis, $g(\{w \in L; \langle x - z_0, y \rangle = 1\}) = z$, and thus

$$\nabla F(y) = \langle \nabla F(y), y \rangle (z - z_0).$$

Now if $y, y' \in H$, one has $\langle y' - y, z - z_0 \rangle = 0$ and for every $t \in \mathbb{R}$, $(1 - t)y' + ty \in H$,

Proof of the lemma.

Fix some $z_0 \in \mathrm{int}(L)$, $z_0 \neq z$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(y) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L} - \mathrm{z}_0; \langle \mathrm{x}, \mathrm{y} \rangle \geq 1\}\big).$$

By Meyer-Reisner (89), F is C^1 on $\{F > 0\} = \mathbb{R}^n \setminus \{0\}$ and for $y \neq 0$

$$\nabla F(y) = \langle \nabla F(y), y \rangle \big(g(\{x \in L; \langle x - z_0, y \rangle = 1\}) - z_0\big).$$

Let $H = \{y \in \mathbb{R}^n; \langle z - z_0, y \rangle = 1\}$. For $y \in H$, the hyperplane $\{x \in \mathbb{R}^n; \langle x - z_0, y \rangle = 1\}$ passes through z, so that by the hypothesis, $g(\{w \in L; \langle x - z_0, y \rangle = 1\}) = z$, and thus

$$\nabla F(y) = \langle \nabla F(y), y \rangle (z - z_0).$$

Now if $y, y' \in H$, one has $\langle y' - y, z - z_0 \rangle = 0$ and for every $t \in \mathbb{R}$, $(1 - t)y' + ty \in H$, so that

$$F(y') - F(y) = \int_0^1 \langle y' - y, \nabla F\big((1 - t)y + ty'\big)\rangle dt = 0.$$

Proof of the lemma.

Thus F is equal to some constant c on H.

Proof of the lemma.

Thus F is equal to some constant c on H. Define a function $G : S^{n-1} \to \mathbb{R}$ by

$$G(u) = \mathrm{vol}\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x} - \mathrm{z}, \mathrm{u} \rangle \geq 0\}.$$

and let

$$U = \{u \in S^{n-1}; \langle u, z - z_0 \rangle > 0\}.$$

Proof of the lemma.

Thus F is equal to some constant c on H. Define a function $G : S^{n-1} \to \mathbb{R}$ by

$$G(u) = \mathrm{vol}\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x} - \mathrm{z}, \mathrm{u} \rangle \geq 0\}.$$

and let

$$U = \{u \in S^{n-1}; \langle u, z - z_0 \rangle > 0\}.$$

Then $u \to y(u) := \frac{u}{\langle u, z - z_0 \rangle}$ is a one-to-one mapping from U onto H, and it is easy to check that

$$G(u) = F\big(y(u)\big) \text{ for every } u \in U.$$

Proof of the lemma.

Thus F is equal to some constant c on H. Define a function $G : S^{n-1} \to \mathbb{R}$ by

$$G(u) = \mathrm{vol}\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x} - \mathrm{z}, \mathrm{u}\rangle \geq 0\}.$$

and let

$$U = \{u \in S^{n-1}; \langle u, z - z_0\rangle > 0\}.$$

Then $u \to y(u) := \frac{u}{\langle u, z - z_0\rangle}$ is a one-to-one mapping from U onto H, and it is easy to check that

$$G(u) = F\big(y(u)\big) \text{ for every } u \in U.$$

It follows that $G = c$ on U, and since $G(u) + G(-u) = \mathrm{vol(L)}$ for every $u \in S^{n-1}$, $G = \mathrm{vol(L)} - \mathrm{c}$ on $-U$. Now, $S^{n-1} \cap (z - z_0)^{\perp}$ is contained in the closures of both U and of $-U$ in S^{n-1}.

Proof of the lemma.

Thus F is equal to some constant c on H. Define a function $G : S^{n-1} \to \mathbb{R}$ by

$$G(u) = \mathrm{vol}\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x} - \mathrm{z}, \mathrm{u} \rangle \geq 0\}.$$

and let

$$U = \{u \in S^{n-1}; \langle u, z - z_0 \rangle > 0\}.$$

Then $u \to y(u) := \frac{u}{\langle u, z - z_0 \rangle}$ is a one-to-one mapping from U onto H, and it is easy to check that

$$G(u) = F\big(y(u)\big) \text{ for every } u \in U.$$

It follows that $G = c$ on U, and since $G(u) + G(-u) = \mathrm{vol}(\mathrm{L})$ for every $u \in S^{n-1}$, $G = \mathrm{vol}(\mathrm{L}) - \mathrm{c}$ on $-U$. Now, $S^{n-1} \cap (z - z_0)^{\perp}$ is contained in the closures of both U and of $-U$ in S^{n-1}. Since G is continuous on S^{n-1}, $G = c = 1 - c = \frac{\mathrm{vol}(\mathrm{L})}{2}$ on S^{n-1}. Thus$L - z$ is centrally symmetric by a classical result (Funk-Falconer).

Additional comments and some open problems

The bodies $C(L)$ and $I(L, z)$ are not in general convex : $C(L)$ is always convex only for $n \leq 3$ (Meyer) and Brehm proved that if Δ_n is a simplex in $\mathbb{R}^n$, $n \geq 4$, $C(\Delta_n)$ is not convex.

Additional comments and some open problems

The bodies $C(L)$ and $I(L,z)$ are not in general convex : $C(L)$ is always convex only for $n \leq 3$ (Meyer) and Brehm proved that if Δ_n is a simplex in $\mathbb{R}^n$, $n \geq 4$, $C(\Delta_n)$ is not convex. However $C(L)$, and $I(L, g(L))$, where $g(L)$ is the centroid of L, are *almost convex*, and even **almost ellipsoids**, in the sense that for some constants $c > d > 0$, independent on n and L, such that for every $u \in S^{n-1}$, one has

$$\frac{d}{\mathrm{vol(L)}^{\frac{3}{2}}}\Big(\int_{L-g(L)}\langle x,u\rangle^2 dx\Big)^{\frac{1}{2}} \leq \frac{1}{\max_t \mathrm{vol}\big(\mathrm{L}\cap(\mathrm{tu}+\mathrm{u}^{\perp})\big)} = \rho_{C(L)}(u)$$

$$\leq \frac{1}{\mathrm{vol(L}\cap\mathrm{u}^{\perp})} = \rho_{I\left(L,g(L)\right)}(u) \leq \frac{c}{\mathrm{vol(L)}^{\frac{3}{2}}}\Big(\int_{L-g(L)}\langle x,u\rangle^2 dx\Big)^{\frac{1}{2}}.$$

Additional comments and some open problems

The bodies $C(L)$ and $I(L,z)$ are not in general convex : $C(L)$ is always convex only for $n \leq 3$ (Meyer) and Brehm proved that if Δ_n is a simplex in $\mathbb{R}^n$, $n \geq 4$, $C(\Delta_n)$ is not convex. However $C(L)$, and $I(L, g(L))$, where $g(L)$ is the centroid of L, are *almost convex*, and even **almost ellipsoids**, in the sense that for some constants $c > d > 0$, independent on n and L, such that for every $u \in S^{n-1}$, one has

$$\frac{d}{\mathrm{vol(L)}^{\frac{3}{2}}}\left(\int_{L-g(L)} \langle x,u\rangle^2 dx\right)^{\frac{1}{2}} \leq \frac{1}{\max_t \mathrm{vol}\left(\mathrm{L} \cap (\mathrm{tu} + \mathrm{u}^{\perp})\right)} = \rho_{C(L)}(u)$$

$$\leq \frac{1}{\mathrm{vol(L \cap u^{\perp})}} = \rho_{I\left(L,g(L)\right)}(u) \leq \frac{c}{\mathrm{vol(L)}^{\frac{3}{2}}}\left(\int_{L-g(L)} \langle x,u\rangle^2 dx\right)^{\frac{1}{2}}.$$

For L centrally symmetric, this was proved by Hensley (Ball for sharp constants), in the general case by Schütt (Fradelizi for sharp constants) (see also Milman-Pajor).

Additional comments and some open problems

By definition one has $\rho_{I(L,g(L))} \leq \rho_{CI(L,g(L))}$.

Additional comments and some open problems

By definition one has $\rho_{I(L,g(L))} \leq \rho_{CI(L,g(L))}$. A natural question is :

Open problem 1. Does there exist a universal constant $C > 0$, independent on the convex body L in $\mathbb{R}^n$ and on $n \geq 1$, such that $\rho_{CI(L,g(L))} \leq C\rho_{I(L,g(L))}$?

Additional comments and some open problems

By definition one has $\rho_{I(L,g(L))} \leq \rho_{CI(L,g(L))}$. A natural question is :

Open problem 1. Does there exist a universal constant $C > 0$, independent on the convex body L in $\mathbb{R}^n$ and on $n \geq 1$, such that $\rho_{CI(L,g(L))} \leq C\rho_{I(L,g(L))}$? This would say that the the radial functions of $C(L)$, $CI(L)$ and $I(L, g(L))$ are all equivalent.

Additional comments and some open problems

By definition one has $\rho_{I(L,g(L))} \leq \rho_{CI(L,g(L))}$. A natural question is :

Open problem 1. Does there exist a universal constant $C > 0$, independent on the convex body L in $\mathbb{R}^n$ and on $n \geq 1$, such that $\rho_{CI(L,g(L))} \leq C\rho_{I(L,g(L))}$? This would say that the the radial functions of $C(L)$, $CI(L)$ and $I(L, g(L))$ are all equivalent.

An equivalent problem is the following : Let K be a convex body in $\mathbb{R}^n$ with Santaló point is at 0. Does there exist an absolute constant $C > 0$, independent on n and K such that

$$\mathrm{vol}\big((\mathrm{P_u K})^{*\mathrm{P_u z}}\big) \geq \mathrm{C}\,\mathrm{vol}\big((\mathrm{P_u K})^{*0}\big) \text{ for every } z \in \mathrm{int}(K) \text{ ?}$$

Additional comments and some open problems

By definition one has $\rho_{I(L,g(L))} \leq \rho_{CI(L,g(L))}$. A natural question is :

Open problem 1. Does there exist a universal constant $C > 0$, independent on the convex body L in $\mathbb{R}^n$ and on $n \geq 1$, such that $\rho_{CI(L,g(L))} \leq C\rho_{I(L,g(L))}$? This would say that the the radial functions of $C(L)$, $CI(L)$ and $I(L, g(L))$ are all equivalent.

An equivalent problem is the following : Let K be a convex body in $\mathbb{R}^n$ with Santaló point is at 0. Does there exist an absolute constant $C > 0$, independent on n and K such that

$$\mathrm{vol}\big((\mathrm{P_u K})^{*\mathrm{P_u z}}\big) \geq \mathrm{C}\,\mathrm{vol}\big((\mathrm{P_u K})^{*0}\big) \text{ for every } z \in \mathrm{int}(K) \text{ ?}$$

Or, given a convex $M \subset u^\perp$, with Santaló point $s(M)$, and a convex body K in $\mathbb{R}^n$, with Santaló point $s(K)$, such that $P_u K = M$, does

$$\mathrm{vol}(\mathrm{M}^{*\mathrm{s(M)}}) \geq \mathrm{Cvol}(\mathrm{M}^{*\mathrm{P_u s(K)}})$$

for some universal constant $C > 0$?

If one could prove that in this situation, for some universal constant $c > 0$, the following is true

$$P_u s(K) - s(M) \in \frac{c}{n}\big(M - s(M)\big)$$

Additional comments and some open problems

If one could prove that in this situation, for some universal constant $c > 0$, the following is true

$$P_u s(K) - s(M) \in \frac{c}{n}\big(M - s(M)\big)$$

then an affirmative answer could be given, using the following lemma :

Additional comments and some open problems

If one could prove that in this situation, for some universal constant $c > 0$, the following is true

$$P_u s(K) - s(M) \in \frac{c}{n}\big(M - s(M)\big)$$

then an affirmative answer could be given, using the following lemma :

Lemma

Let V be a convex body in $\mathbb{R}^n$ and $x, y \in \operatorname{int}(V)$. Then

$$(1 - ||x - y||_{V-y})^n \operatorname{vol}(V^{*x}) \leq \operatorname{vol}(V^{*y}) \leq \frac{\operatorname{vol}(V^{*x})}{(1 - ||y - x||_{V-x})^n}$$

Additional comments and some open problems

It is known (see Milman-Pajor) that for some affine mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, $M := AL$ is *isotropic*, that is satisfies $\mathrm{vol(M)} = 1$ and

$$\left(\int_{M-g(M)} \langle x, u\rangle^2 dx \right)^{\frac{1}{2}} = c_M \text{ for all } u \in S^{n-1}.$$

where $\mathbf{c_M}$ is *the isotropy constant* of M. Problem 1 is equivalent to

Additional comments and some open problems

It is known (see Milman-Pajor) that for some affine mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, $M := AL$ is *isotropic*, that is satisfies $\mathrm{vol}(\mathrm{M}) = 1$ and

$$\left(\int_{M-g(M)} \langle x, u \rangle^2 dx \right)^{\frac{1}{2}} = c_M \text{ for all } u \in S^{n-1}.$$

where $\mathbf{c_M}$ is *the isotropy constant* of M. Problem 1 is equivalent to

Open problem 2. Let M be an isotropic convex body. Is $CI(M)$ equivalent to the Euclidean ball, independently on M and n ?

Additional comments and some open problems

It is known (see Milman-Pajor) that for some affine mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, $M := AL$ is *isotropic*, that is satisfies $\mathrm{vol}(\mathrm{M}) = 1$ and

$$\left(\int_{M-g(M)} \langle x, u \rangle^2 dx \right)^{\frac{1}{2}} = c_M \text{ for all } u \in S^{n-1}.$$

where $\mathbf{c_M}$ is *the isotropy constant* of M. Problem 1 is equivalent to

Open problem 2. Let M be an isotropic convex body. Is $CI(M)$ equivalent to the Euclidean ball, independently on M and n ?

Of course, problems 1 and 2 are non-trivial only if L or M are not centrally symmetric. The particular case of the simplex is open :

Additional comments and some open problems

It is known (see Milman-Pajor) that for some affine mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, $M := AL$ is *isotropic*, that is satisfies $\mathrm{vol(M)} = 1$ and

$$\left(\int_{M-g(M)} \langle x, u \rangle^2 dx \right)^{\frac{1}{2}} = c_M \text{ for all } u \in S^{n-1}.$$

where $\mathbf{c_M}$ is *the isotropy constant* of M. Problem 1 is equivalent to

Open problem 2. Let M be an isotropic convex body. Is $CI(M)$ equivalent to the Euclidean ball, independently on M and n ?

Of course, problems 1 and 2 are non-trivial only if L or M are not centrally symmetric. The particular case of the simplex is open :

Open problem 3. Let Δ_n be a simplex in $\mathbb{R}^n$ with $g(\Delta_n) = 0$. Is there a constant c such that for every $n \geq 2$ and every $u \in S^{n-1}$

$$\mathrm{vol}(\Delta_n \cap u^\perp) \leq c\,\mathrm{vol}\Big(\big(P_u(\Delta_n^{*g})\big)^{*s}\Big) = c\,\mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s}\Big) \text{ ?}$$

Additional comments and some open problems

Observe that when Δ_n is a regular simplex inscribed in the Euclidean ball, since $(\Delta_n)^* = -n\Delta_n$, one has

$$(\Delta_n \cap u^\perp)^{*0} = P_u\big((\Delta_n)^{*0}\big) = P_u(-n\Delta_n)$$

and thus

Additional comments and some open problems

Observe that when Δ_n is a regular simplex inscribed in the Euclidean ball, since $(\Delta_n)^* = -n\Delta_n$, one has

$$(\Delta_n \cap u^\perp)^{*0} = P_u\big((\Delta_n)^{*0}\big) = P_u(-n\Delta_n)$$

and thus

$$\mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s}\Big) = \frac{1}{n^{n-1}}\mathrm{vol}\big((P_u\Delta_n)^{*s}\big)\,.$$

Let $e_1, \ldots, e_{n+1}$, $|e_i| = 1$, be the vertices of Δ_n so that $0 = e_1 + \cdots + e_{n+1}$ and for $1 \leq i \neq j \leq n+1$, $\langle e_i, e_j\rangle = -\frac{1}{n}$.

Additional comments and some open problems

Fact. *Let* $A \subset \{1, \ldots, n+1\}$ *satisfy* $1 \le k := \mathrm{card}(A) \le n$. *Define*

$$u_A = \frac{\sum_{i \in A} e_i}{|\sum_{i \in A} e_i|} = \sqrt{\frac{n}{k(n+1-k)}} \sum_{i \in A} e_i \in S^{n-1}.$$

Then 0 *is the centroid of* $\Delta_n \cap u_A^{\perp}$.

Additional comments and some open problems

Fact. *Let $A \subset \{1,\dots,n+1\}$ satisfy $1 \le k := \mathrm{card}(A) \le n$. Define*

$$u_A = \frac{\sum_{i\in A} e_i}{|\sum_{i\in A} e_i|} = \sqrt{\frac{n}{k(n+1-k)}}\sum_{i\in A} e_i \in S^{n-1}.$$

Then 0 is the centroid of $\Delta_n \cap u_A^{\perp}$.

We get thus:

Proposition For every $A \subset \{1,\dots,n+1\}$, with $1 \le \mathrm{card}(A) \le n$, one has : $||u_A||_{CI(\Delta_n,0)} = ||u_A||_{I(\Delta_n,0)}$.

Additional comments and some open problems

Additional comments and some open problems

When $u^\perp \cap \Delta_n$ is a simplex, one can also conclude :

Proposition Let $u \in S^{n-1}$, and if $u = \sum_{i=1}^{n+1} u_i e_i \in S^{n-1}$ with $\sum_{i=1}^{n+1} u_i = 0$ and $u_1, \dots, u_n \geq 0 > u_{n+1}$, then $u^\perp \cap \Delta_n$ is a simplex and

$$\rho_{I(\Delta_n,0)(u)} = \mathrm{vol}(\Delta_n \cap u^\perp) = \frac{1}{(n-1)!} \frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n}{2}-1}} \frac{1}{\prod_{i=1}^{n}(u_i + \sum_{j=1}^{n} u_j)}$$

and

$$\rho_{CI(\Delta_n,0)(u)} = \mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s}\Big) = \frac{1}{(n-1)!} \frac{n^{\frac{n}{2}+1}}{(n+1)^{\frac{n+1}{2}}} \frac{1}{\sum_{i=1}^{n} u_i}.$$

When $u^\perp \cap \Delta_n$ is a simplex, one can also conclude :

Proposition Let $u \in S^{n-1}$, and if $u = \sum_{i=1}^{n+1} u_i e_i \in S^{n-1}$ with $\sum_{i=1}^{n+1} u_i = 0$ and $u_1, \dots, u_n \geq 0 > u_{n+1}$, then $u^\perp \cap \Delta_n$ is a simplex and

$$\rho_{I(\Delta_n,0)(u)} = \mathrm{vol}(\Delta_n \cap u^\perp) = \frac{1}{(n-1)!} \frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n}{2}-1}} \frac{1}{\prod_{i=1}^n (u_i + \sum_{j=1}^n u_j)}$$

and

$$\rho_{CI(\Delta_n,0)(u)} = \mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s}\Big) = \frac{1}{(n-1)!} \frac{n^{\frac{n}{2}+1}}{(n+1)^{\frac{n+1}{2}}} \frac{1}{\sum_{i=1}^n u_i}.$$

Thus $CI(\Delta_n, 0)$ has $2n+2$ small faces around $u = \pm e_i$, $1 \leq i \leq n+1$. It is easy to check that for such directions $u \in S^{n-1}$ one has

$$1 \leq \frac{\mathrm{vol}(\Delta_n \cap u^\perp)}{\mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s}\Big)} \leq \frac{e}{2}.$$

Additional comments and some open problems

But these u's around which the two norms are equal or almost equal are not enough to be able to conclude.

But these u's around which the two norms are equal or almost equal are not enough to be able to conclude.

THANK YOU FOR YOUR ATTENTION

But these u's around which the two norms are equal or almost equal are not enough to be able to conclude.

THANK YOU FOR YOUR ATTENTION

THE END