The Sine Transform of Isotropic Measures

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joint work with F.E. Schuster

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The Cosine Transform

The cosine transform assigns to each finite (signed) Borel measure μ on S^{n-1} the continuous function

$$\mathcal{C}(\mu)(u) = \int_{S^{n-1}} |u \cdot v| \, d\mu(v), \qquad u \in S^{n-1}.$$

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Zonoids

If μ is an even (positive) Borel measure on S^{n-1}, then $\mathcal{C}\mu(u)$ is the support function of a unique convex body $\mathcal{C}_\mu \subset \mathbb{R}^n$:

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- C_μ is centered and origin-symmetric.
- $\mathcal{C}_\mu$ is a zonoid, i.e. can be approximated by finite Minkowski sums of segments.

Finite dimensional subspaces

Theorem (Bolker 1969; Lewis 1978)

Each n-dimensional subspace F of $L_1(S^{n-1})$ is isometric to the Banach space $(\mathbb{R}^n, \|.\|_F)$ whose dual spaces norm is given by

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- *μ is even,*
- *μ is not concentrated on any great-sphere,*
- *μ is isotropic.*

Finite dimensional subspaces

Example

- $F := \text{span}\{u \mapsto u \cdot x : x \in \mathbb{R}^n\} = \mathcal{H}_1^n$. Then the representing measure is suitably normalized spherical Lebesgue measure

$$\lambda = \frac{1}{\kappa_n}\, d\sigma$$

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- μ has total mass n.
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- The set of all *discrete* isotropic measures is a weakly dense subset of all isotropic measures (due to F. Barthe).

Isotropic Position

The minimal surface area of a convex body $K \subseteq \mathbb{R}^n$ is defined as

$$\partial(K) := \inf\{S(\Phi K) : \Phi \in \mathrm{SL}(n)\},$$

where $S(\cdot)$ is the surface area.

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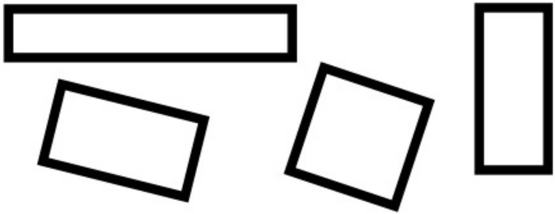

K is in surface isotropic position if $S(K) = \partial(K)$.

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Theorem (Petty 1961)

Let $K \subseteq \mathbb{R}^n$ be a convex body. Then there exists a linear transformation $\Phi \in SL(n)$ such that ΦK is in surface isotropic position. This Φ is unique up to orthogonal transformations.

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Let $K \subseteq \mathbb{R}^n$ be a convex body. K is in surface isotropic position if and only if the surface area measure μ_K is (up to normalization) isotropic.

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Remark:

- Extremizers exist due to compactness.

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Proof.

Makes use of the Urysohn inequality:

$$\left(\frac{\mathrm{Vol}(K)}{\kappa_n}\right)^{1/n} \leq \frac{1}{n\,\kappa_n}\int_{S^{n-1}} h(K,u)\,du$$

Where equality only holds if $h(K,u)$ is constant.

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Makes use of the Urysohn inequality:

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The sine transform assigns to each finite (signed) Borel measure μ on S^{n-1} the continuous function

$$\mathcal{S}(\mu)(u) = \int_{S^{n-1}} (1 - |u \cdot v|^2)^{1/2} \, d\mu(v), \qquad u \in S^{n-1}.$$

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- $\mathcal{S}$ annihilates odd measures.
- $\mathcal{S}\mu = c_n \cdot \mathcal{R}(\mathcal{C}\mu)$, where $\mathcal{R}$ is the Radon transform and c_n a constant only depending on the dimension.

Disc Bodies

If μ is an even (positive) Borel measure on S^{n-1}, then $\mathcal{S}\mu(u)$ is the support function of a unique convex body $\mathcal{S}_\mu \subset \mathbb{R}^n$:

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- $\mathcal{S}_\mu$ is centered and origin-symmetric.
- $\mathcal{S}_\mu$ is a disc body, i.e. can be approximated by finite Minkowski sums of ($n-1$ dimensional) discs. Disc bodies constitute a subclass of zonoids.

Examples

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- If K is a convex body in $\mathbb{R}^n$ then

$$\frac{1}{2(n+1)}\,\mathcal{S}\mu_K(u) = \int_{-\infty}^{\infty} \mathrm{Vol}_{n-2}\left(K \cap (u^\perp + t\,u)\right)\,dt,$$

where $\mathrm{Vol}_{n-2}(L)$ denotes the $n-2$ dimensional surface area of the $n-1$ dimensional body L. This characterization of $\mathcal{S}$ is due to Schneider.

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where λ is suitably normalized spherical Lebesgue measure and ν is any cross measure; Here τ_n and η_n not only stay bounded but also satisfy

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \eta_n = 1.$$

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$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \eta_n = 1.$$

Remark: These inqualities are *asymptotically* optimal.

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Main tool is the Brascamp-Lieb inequality: Let μ be a discrete measure such that $\frac{1}{n-1}\,\mu$ is isotropic, say

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Choose $f_1(x) = \ldots = f_m(x) = \exp(-\|x\|)$.

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Re-interpretation of the previous results:

$$\begin{array}{rcccl} \text{parallelotopes} & \leq & \partial(K)^{-1}\mathrm{Vol}(\Pi K)^{1/n} & \leq & \text{ellipsoids} \\ \text{ellipsoids} & \leq & \partial(K) \quad \mathrm{Vol}(\Pi^* K)^{1/n} & \leq & \text{parallelotopes} \end{array}$$

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The End

Thank you for your attention!