Projection bodies in complex vector spaces

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Projection body

V: real vector space of dimension n.
$\mathcal{K}(V)$: space of compact convex bodies in V.

Definition

Let $\Pi : \mathcal{K}(V) \to \mathcal{K}(V)$ be the operator defined from

$$\begin{aligned} h(\Pi K, u) &= \mathrm{vol}_{n-1}(K|u^\perp), \quad u \in S^{n-1} \\ &= \frac{n}{2} V(K, \ldots, K, [-u, u]), \\ &= \int_{S^{n-1}} h([-u, u], v) dS(K, v) = \int_{S^{n-1}} |\langle u, v\rangle| dS(K, v). \end{aligned}$$

where $h(L, \cdot) : S^{n-1} \to \mathbb{R}$, $L \in \mathcal{K}(V)$ is the support function of L.

ΠK is the projection body of $K \in \mathcal{K}(V)$.

Properties of the projection body

a) Invariant under translations, i.e.

$$\Pi(K+x)=\Pi(K),\quad x\in V, K\in\mathcal{K}(V).$$

b) $GL(V,\mathbb{R})$-contravariant, i.e.

$$\Pi(\phi K)=|\det\phi|\phi^{-t}(\Pi K),\quad \phi\in GL(V,\mathbb{R}), K\in\mathcal{K}(V).$$

c) It is a continuous Minkowski valuation, i.e.

$$\Pi(K\cup L)+\Pi(K\cap L)=\Pi(K)+\Pi(L),\ K,L\in\mathcal{K}(V), K\cup L\in\mathcal{K}(V),$$

where $+$ denotes de Minkowski sum on V.

Characterization of the projection body (Ludwig)

V: real vector space of dimension n.

If the operator $Z : \mathcal{K}(V) \to \mathcal{K}(V)$ is

- translation invariant,
- $SL(V, \mathbb{R})$-contravariant,
- continuous Minkowski valuation,

then $Z = c\Pi$, $c \in \mathbb{R}^+$.

The converse also holds.

Complex projection bodies (A.-Bernig)

W: complex vector space of complex dimension m, $m \geq 3$.

If the operator $Z : \mathcal{K}(W) \to \mathcal{K}(W)$ is

- translation invariant,
- $SL(W, \mathbb{C})$-contravariant,
- continuous Minkowski valuation,

then $Z = \Pi_C$ where $C \subset \mathbb{C}$ is a convex body and

$$h(\Pi_C K, u) = V(K, \dots, K, Cu), \quad u \in S^{2m-1},$$

$Cu = \{cu \,:\, c \in C \subset \mathbb{C}\}$.

The converse also holds for every $C \in \mathcal{K}(\mathbb{C})$.

Complex projection bodies (A.-Bernig)

For $m = 2$,

$$Z : \mathcal{K}(W) \to \mathcal{K}(W)$$

given by

$$h(ZK, u) = \mu(\det(K, u))$$

with μ a continuous, translation invariant, monotone valuation of degree 1 and

$$\det(K, u) = \{\det(k, u) \; : \; k \in K\} \subset \mathbb{C}$$

satisfies all the required properties.

Idea of the proof

$\Leftarrow$) Direct from the properties of mixed volumes and the support function.

$\Rightarrow$)

i) McMullen decomposition.

ii) Z cannot be of degree k, $k \neq 2m-1$.

iii) If the degree of Z is $2m-1$, then $Z = \Pi_C$:

- McMullen description of real-valued valuations of degree $n-1$.
- The involved function is a function of one complex variable.
- It is also subbadditive. Thus, the support function of a convex body in $\mathbb{C}$.

Idea of the proof: i)

McMullen decomposition (1977): Let Val be the space of real-valued, translation invariant, continuous valuations on V and $\mathrm{Val}_k \subset \mathrm{Val}$ the subspace of valuations of degree k. Then,

$$\mathrm{Val} = \bigoplus_{k=0,\dots,n} \mathrm{Val}_k.$$

In our case:

$$h(ZK,\cdot) = \sum_{k=0}^{2m} f_k(K,\cdot),$$

with $f_k(K,\cdot)$ 1-homogeneous and subadditive for k_0, k_1, the minimal and maximal indices with $f_k \neq 0$.

Idea of the proof: ii)

$\mathbf{k = 0}$: the Euler characteristic is the only 0-degree valuation.
$\mathbf{k = 2m}$: the volume is the only $2m$-degree valuation.
$\mathbf{1 \leq k < 2m - 1}$: define

$$\tilde{Z}(K) = \int_{S^1} \int_{S^1} q_1 Z(q_2 K) dq_1 dq_2$$

and use the Klain's injectivity theorem.

Injectivity theorem (Klain 2000): Let $\mu \in \mathrm{Val}_k(V)$ even and $E \subset V$ a k-dimensional subspace. Then, there exists a function $Kl_\mu : \mathrm{Gr}_k(V) \to \mathbb{R}$ which uniquely determines μ and

$$\mu(K) = \mathrm{Kl}_\mu(E)\,\mathrm{vol}(K), K \in \mathcal{K}(E).$$

$\mathbf{k = 2m - 1}$:

Theorem (McMullen 1980): If $\mu \in \mathrm{Val}_{n-1}(V)$, there exists a continuous, 1-homogeneous function $f : V^* \to \mathbb{R}$ with

$$\mu(K) = \int_{S^{n-1}} f(v) dS(K, v) = V(K, \ldots, K, f).$$

Moreover, f is unique up to a linear function.

In our case:

$$h(ZK, u) = V(K, \ldots, K, f_u).$$

Using the $SL(W, \mathbb{C})$-contravariance, we get $f_u \equiv f \circ h^*$, for all $h \in SL(W, \mathbb{C})$ with $h(u) = u$.

Moreover, $f_u(\xi_1 + \xi_2) = f_u(\xi_1)$ if $(\xi_2, u) = 0$.

Idea of the proof: iii)

Thus, $f_u(\xi) = G((\xi, u)) = G(\langle \xi, u\rangle + i\langle \xi, Ju\rangle)$ with $G : \mathbb{C} \to \mathbb{R}$ continuous, 1-homogenous function.

Using that $h(ZK, u)$ is a support function and the Minkowski's existence theorem, we get that G is a convex function.

Classical Brunn-Minkowski inequalities

Let $K, L \in \mathcal{K}(V)$ and $0 \leq \lambda \leq 1$. Then,

- $\mathrm{vol}((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda)\,\mathrm{vol}(K)^{1/n} + \lambda\,\mathrm{vol}(L)^{1/n}$, with equality for $\lambda \in (0,1)$ iff K and L lie in parallel hyperplanes or are homothetics.
- $W_i(K+L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}$, with equality iff K and L are homothetics.
- $V((K+L)[n-i], \mathbf{C})^{1/(n-i)} \geq$ $V(K[n-i], \mathbf{C})^{1/(n-i)} + V(L[n-i], \mathbf{C})^{1/(n-i)}$, where $\mathbf{C} = (K_1, \ldots, K_i)$.

Brunn-Minkowski inequality for Π_C

Let $K, L \in \mathcal{K}(W)$ with non-empty interior. Then

$$\mathrm{vol}(\Pi_C(K+L))^{1/2m(2m-1)} \geq \mathrm{vol}(\Pi_C K)^{1/2m(2m-1)} + \mathrm{vol}(\Pi_C L)^{1/2m(2m-1)},$$

with equality iff K and L are homothetic.

Symmetry property

Let $\mathbf{K} := (K_1, \ldots, K_{2m-1}), \mathbf{L} := (L_1, \ldots, L_{2m-1}) \in \mathcal{K}(W)^{2m-1}$ and $C \subset \mathcal{K}(\mathbb{C})$. Then,

$$V(\Pi_C \mathbf{K}, \mathbf{L}) = V(\Pi_{\overline{C}} \mathbf{L}, \mathbf{K}),$$

with $h(\Pi_C \mathbf{K}, u) = V(K_1, \ldots, K_{2m-1}, Cu)$, $u \in S^{2m-1}$.