11W5033 Problem Session

Willem Haemers Feb 4, 2011.

Seidel Energy of a graph.

Seidel matrix $S_{ij} = \begin{cases} -1 & \text{if } i \sim j \\ 0 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$

eigenvalues $\sigma_1 \ldots \sigma_n$.

energy $\sum_{i=1}^{n} |\sigma_i| = \mathcal{E}_n$

Invariant under switching and taking complements.

Known: $2\sqrt{\binom{n}{2}} < \mathcal{E}_n \leq n\sqrt{n-1}$

with equality on the right iff the matrix is a symmetric conference matrix.

Complete graph $\mathcal{E}_n = 2(n-1)$

Question is $\mathcal{E}_n \geq 2(n-1)$ for all graphs on n vertices?

True for $n \leq 9$.

R. Craigen

Given any (rectangular) (1,-1)-matrix A, find the minimum value of N such that, for all n>N, **every** H(n) contains A as a submatrix. (Or give an example demonstrating that no such N exists in general.

H(n) = Hadamard matrix of order n

Felix Lazebnik

THE HYPERPLANE ARRANGEMENT GRAPH.

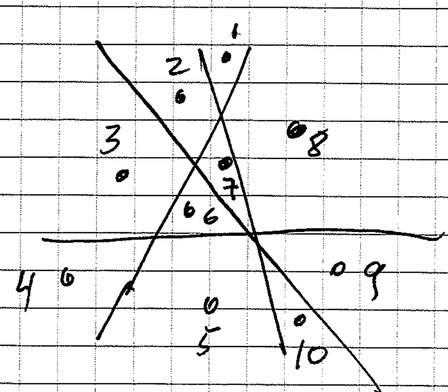
LET $H_1, \dots, H_K$ BE HYPERPLANES IN $\mathbb{R}^n$.

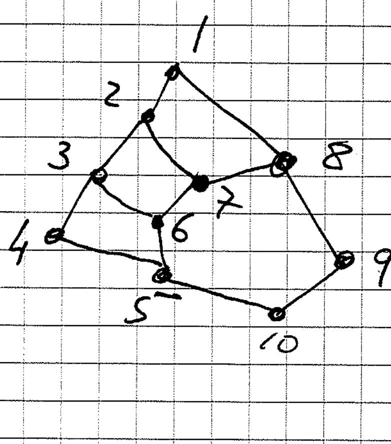
$$V = \left\{ \text{CONNECTED COMPONENTS OF } \mathbb{R}^n \setminus \bigcup_{i=1}^{K} H_i \right\}$$

WE JOIN TWO VERTICES IF THEY SHARE $(n-1)$-dim BOUNDARY.

LET $G = (V, E)$.

CLASSIFY G / STUDY G.





$n = 2$.

S. Friedland, 3 Feb, 2011

Analogs of the van der Waerden and Tverberg conjectures

$S_0(m,\mathbb{R})$ – the set of real symmetric of order m with zero diagonal

1. Definition of haffman of $B=[b_{ij}] \in S_0(2n,\mathbb{R})$

For a graph $G=(V,E)$ with no self-loops denote by $A(G)$ its adjacency matrix. Denote by $\mathcal{M}_k(G)$ – all k matches in G: k-match: $\{(u_1,u_2),\dots,(u_{2k-1},u_{2k})\} \subset E$, $\#\{u_1,\dots,u_{2k}\}=2k$. For $|V|=2n$ $\mathcal{M}_n(G)$ – the set of perfect matchings. Let K_m – be the complete graph on m vertices. Then

$$\mathrm{haf}(B) = \sum_{\{(i_1,i_2),\dots,(i_{2n-1},i_{2n})\} \in \mathcal{M}_n(K_{2n})} b_{i_1 i_2} b_{i_3 i_4} \dots b_{i_{2n-1} i_{2n}}$$

Denote for any $C=[c_{ij}] \in S_0(m,\mathbb{R})$ $\mathrm{haf}_k(C)$ – the sum of all hafnians of principle $2k\times 2k$ submatrices of C.

2. Ψ_{2n}: the set of all $(2n)\times(2n)$ doubly stochastic matrices, which are symmetric, with zero diagonal, and which are convex combinations of symmetric permutation matrices with zero diagonal. Edmonds 1965, $\Psi_{2n} \subseteq S_0(2n,\mathbb{R}_+)$ satisfying (1) $\sum_{j=1}^{2n} b_{ij}=1$, $i=1,\dots,n$, $b_{ij}\ge 0$, $b_{ii}=0$, $b_{ij}=b_{ji}$

(2) $\sum_{(i,j)\in S} b_{ij} \le |S|-1$ for any $S \subseteq \{1,\dots,2n\}$, $|S|$ odd, and $3\le |S| \le 2n-3$

Conjecture $\min_{B\in\Psi_{2n}} \mathrm{haf}_k(B) = \mathrm{haf}_k\left(\frac{1}{2n-1}A(K_{2n})\right)$

for $k=n,n-1,\dots,2$. Equality holds iff $B=\frac{1}{2n-1}A(K_{2n})$

$k=n$ is the analog of van der Waerden conjecture.

$k<n$ – Tverberg's conjecture.

True for $\underline{n=2}$.

S. Akbari

Let F be a field and char $F \neq 2$.

Form a graph $\Gamma_n(F)$ as follows:

$V(\Gamma_n(F)) = GL_n(F)$

Two matrice $A, B \in GL_n(F)$ are adjacent iff

$A + B$ is singular.

Thm. $\omega(\Gamma_n(F)) < \infty$.

Problem (Akbari-Jamaali-Fakhari)

Is the vertex chromatic number of $\Gamma_n(F)$ is finite?

$\chi(\Gamma_2(\mathbb{Q})) = ?$ $\quad \chi(\Gamma_2(\mathbb{R})) = ?$

S. Akbari

Definition. Let D be a 2-(v,k,λ) design with incidence matrix N.

$$N_{v\times b} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ v \end{array} \begin{array}{c} B_1 \; B_2 \; \cdots \; B_b \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \\ & & \end{array} \right] \end{array}$$

A zero-sum flow for D is a nowhere-zero real vector in the null space of N.

A zero-sum k-flow is a zero-sum flow with entries in the set $\{\pm 1, \pm 2, \ldots, \pm(k-1)\}$.

Thm. The incidence matrix of every non-symmetric 2-(v,k,λ) design admits an integral zero-sum flow.

Conjecture 1. (Akbari- Khosrovshahi- Mofidi)
The incidence matrix of every non-symmetric design admit a zero-sum 4-flow.

Conjecture 2. (Akbari- Khosrovshahi- Mofidi)
The incidence matrix of every $STS(v)$, $v>7$, admits a zero-sum 3-flow.

Seth Meyer

The t-term rank of a matrix

Let $A \in \{0,1\}^{n\times n}$ and recall that a "transversal of A" is a selection of '1's in A such that no two come from the same row or column We denote the term rank of A (the size of the largest transversal in A) by $\rho(A)$ or $\rho_1(A)$. We can extend this definition to say that a "t-transversal" is a selection of '1's in A such that no two come from the same column and no more than t come from any row. We then define the t-term rank of A as the size of the largest t-transversal in A and we write $\rho_t(A)$ for this number. Many results about $\rho(A)$ carry over to results about $\rho_t(A)$, see [1] for more information.

Given non-negative non-increasing integral vectors $R = (r_1, \dots, r_n)$ and $S = (s_1, \dots, s_n)$ we can define the class of matrices

$$\mathcal{A}(R,S) = \{A \in \{0,1\}^{n\times n} : \Sigma_{k=1}^{n} a_{i,k} = r_i, \Sigma_{l=1}^{n} a_{l,k} = s_j, \forall i,j\}.$$

We write $\overline{\rho}_t(R,S) = \overline{\rho}_t$ to be the maximum t-term rank of a matrix in $\mathcal{A}(R,S)$. For the usual term rank ($t = 1$), Haber proved that there exists $A \in \mathcal{A}(R,S)$ with a transversal of length $\overline{\rho}_1$ on the upper left backwards diagonal, that is, in the $\overline{\rho}_1$ rows and columns with the largest number of '1' entries (R and S are non-increasing by assumption).

We have shown that for a given t, there is an $A \in \mathcal{A}(R,S)$ with $\rho_k(A) = \overline{\rho}_k$ for all $1 \le k \le t$ (it simultaneously realizes the first maximum t term ranks). Furthermore, the t-transversal occurs in the rectangle formed by the top $\overline{\rho}_1$ rows and the leftmost $\overline{\rho}_t$ columns. In addition, the number of '1's in each row of this t-transversal is non-increasing.

QUESTION: Can we extend Haber's result to say anything further about the positions of the '1's in the t-transversal?

References

[1] R.A. Brualdi, K.P. Kiernan, S.A. Meyer, and M.W. Schroeder, On the t-Term Rank of a Matrix, Submitted.

Mike Schroeder

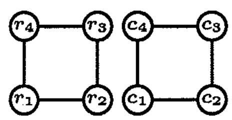
Row and Column Orthogonal $(0,1)$ Matrices

Let A be a $(0,1)$-matrix. Label the rows $\mathcal{R} = \{R_1, \ldots, R_m\}$ and the columns $\mathcal{C} = \{C_1, \ldots, C_n\}$. Define the **row orthogonal realization of** A to be a graph $G_r(A)$ whose vertices are $\mathcal{R}$ and $R_i \sim R_j$ if the rows for R_i and R_j are orthogonal. This is essentially the set coloring problem; assign to each R_i a set of colors from $\mathcal{C}$ and render all permissable edges which maintains a proper set coloring.

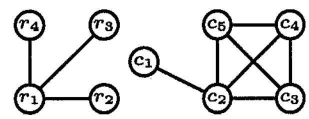
Similarly, define $G_c(A)$ as the **column orthogonal realization of** A. This in a sense is the dual graph of the incidence matrix, having vertices $\mathcal{C}$ with assigned colors from $\mathcal{R}$.

Graphs G and H are **jointly realizable** if there exists a $(0,1)$-matrix A for which $G_r(A) \cong G$ and $G_c(A) \cong H$, and say (G,H) has a joint-orthogonal realization (JOR). If $G \cong H$ we say G is **self-realizable** if such a matrix can be found whose rows and columns realize G, i.e. G has a self-JOR. If a matrix A can be found which is symmetric, we say G is **symmetric realizable**, or G has a SJOR. Below is an example of a SJOR for C_4 and a JOR for $(S_4, K_4 \cup \{e\})$.

	c_1	c_2	c_3	c_4
r_1	1		1	
r_2		1		1
r_3	1		1	
r_4		1		1



	c_1	c_2	c_3	c_4	c_5
r_1		1			
r_2	1		1		
r_3	1			1	
r_4	1				1



We have some criteria for the existence of SJORs.

- A graph G has an SJOR implies that $\theta\left(\overline{G}\right) \leq n$, where θ gives the clique cover number of a graph.
- For bipartite graphs with parts W and B, we say W is *pairwise neighborhood incomplete* for each $x, x' \in W$, $N(x) \cup N(x') \neq B$. We similarly can discuss B being pairwise neighborhood-incomplete. If a bipartite graph G has partition which are pairwise neighborhood-incomplete, then G has an SJOR.

There are infinite families for which there are SJORs, like complete graphs, complete multi-partite graphs, bipartite graphs with a pendent vertex (which will includes trees), bipartite graphs with diameter $d \geq 9$, cycles C_n with $n \geq 3$ and $n \neq 6$ ($n = 6$ is impossible), and Q_n (the Hamming graph) when $n \geq 5$ or $n = 2$ ($n = 3$ is impossible).

We have many questions – but two burning questions would be

1. Does Q_4 have a SJOR? there isn't an SJOR?
2. Does there exist a graph G and matrix A for which A is a self-JOR of G, but ~~cannot be made symmetric after row and column permutation~~?

References

[1] Row and Column Orthogonal (0,1)-Matrices, A. Berliner, R. Brualdi, L. Deaett, K. P. Kiernan, M. W. Schroeder, *Linear Algebra and its Applications*, Volume 429, Issues 11-12, 1 December 2008, Pages 2732-2745

FERENC SZOLLOSI

Problem 1 (General). *Characterize those $m \times m$ unimodular matrices which can be submatrices of an $n \times n$ unit Hadamard matrix.*

The following problem is motivated by the need of a complete characterization of unit Hadamard matrices of order 6 [3].

Problem 2 (Particular). *Give a complete algebraic characterization to those matrices*

$$E(a,b,c,d) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}$$

featuring unimodular complex numbers a, b, c and d which can appear as a 3×3 submatrix in a 6×6 unit Hadamard matrix.