Control and Numerics: Continuous versus discrete approaches

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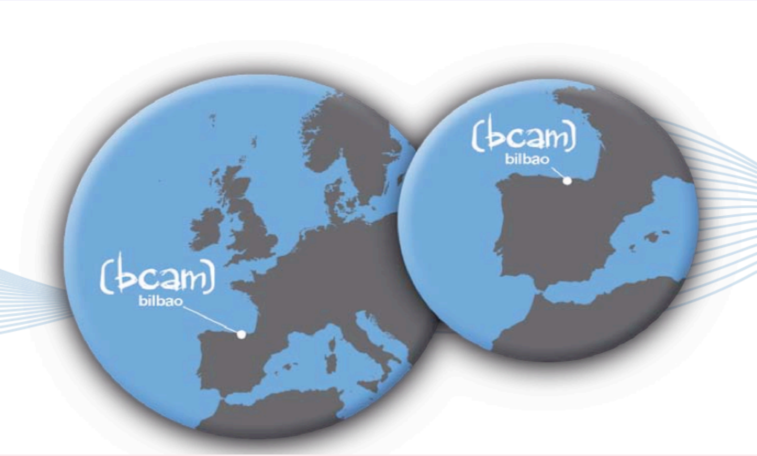
Outline

Outline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Motivation

Control problems for PDE are interesting for at least two reasons:

- They emerge in most real applications. PDE as the models of Continuum and Quantum Mechanics. Furthermore, in real world, there is always something to be optimized, controlled, optimally shaped, etc.
- Answering to these control problems often requires a deep understanding of the underlying dynamics and a better master of the standard PDE models.

Surprisingly enough, this has led to an important ensemble of new tools and results and some fascinating problems are still widely open.
Furthermore, these kind of techniques are of application in some other fields, such as inverse problems theory, optimal shape design and parameter identification issues.

These problems are also challenging and important from the viewpoint of numerical analysis and

Often, classical intuition based on finite-dimensionality and smoothness fails...

Warning!

Warning:

Probably, this lecture will be only of interest for you if:

- You have a model for your problem.
- This model is a PDE (or closely related).
- It is preferably of conservative nature. Highly dissipative dynamics may lead to other phenomena.
- There is a control you want to compute numerically...

Continuous versus discrete

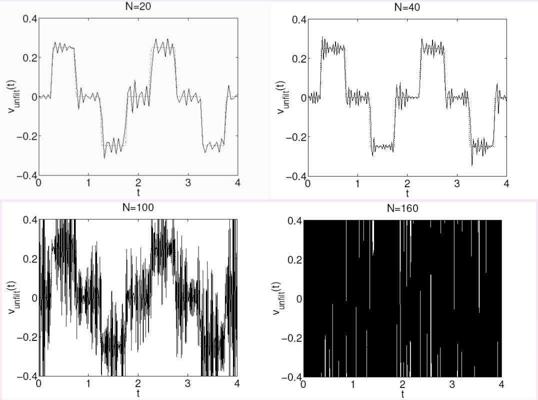
Two approaches:

- **Continuous:** PDE first + Control $\longrightarrow$ implement the resulting algorithm numerically.
- **Discrete:** Replace the PDE and the control problem by a discrete version $\longrightarrow$ Apply discrete control tools

Do these processes lead to the same result?

$$\text{CONTROL} + \text{NUMERICS} \\ = \\ \text{NUMERICS} + \text{CONTROL?}$$

NO!!!!!!



E. Z., SIAM Review, 47 (2) (2005), 197-243.

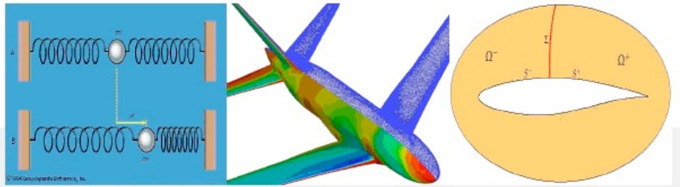
- Important progresses has been done in recent years by many authors.
- In this lecture I report mainly on the joint developed in collaboration with Sylvain Ervedoza, Sorin Micu, Liviu Ignat, Aurora Marica, Martin Gugat, ... among others concerning the wave equation, as a prototype of infinite dimensional purely conservative dynamical system.
- But the list of contributors to the field is much longer: R. Glowinski, J. L. Lions, M. Asch, G. Lebeau, G. Leugering, J. M. Coron, O. Glass, M. Tucsnak, M. Negreanu, C. Castro, Ch. Schwab, N. Cindea, J.A. Infante, F. Macià,....

The instabilities above are due to high frequency numerical spurious solutions.

Numerical analysis ensures that all solutions of the PDE can be approximated by the numerical ones but it does not guarantee that other virtual numerical realities emerge.

And in fact they often do and can produce damage.

These issues are also very relevant and have been investigated systematically in the context of aeronautics optimal shape design.



The two approaches

Discrete: Discretization + Control

- Advantages: Discrete clouds of values. No fine regularity issues. Automatic differentiation. Black box optimization,...
- Drawbacks:
  - "Invisible" geometrical aspects.
  - Scheme dependent.

Continuous: Continuous control theory + discretization.

- Advantages: "Simpler" formal computations. Solver independent. Sensitive to fine regularity issues.
- Drawbacks:
  - Needs a significant amount of PDE and Functional Analysis theory.
  - Subtle for complex problems.
  - More sophisticated algorithms

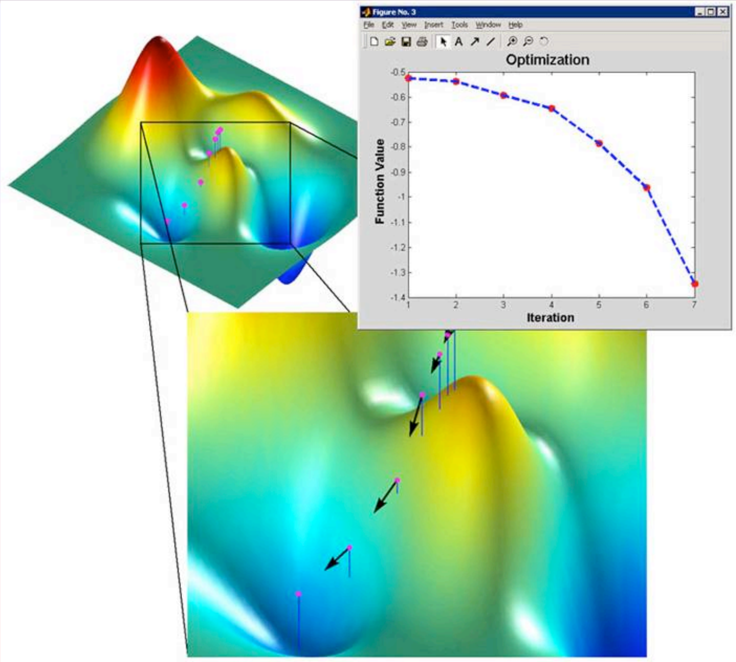
In the end, in a way or another, a discrete optimization problem needs to be solved.

Steepest descent:

$$u_{k+1} = u_k - \rho \nabla J(u_k).$$

Discrete version of continuous gradient systems

$$u'(\tau) = -\nabla J(u(\tau)).$$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Wave control

Most of this talk will be devoted to discuss recent development on the control of wave-like equations. But most of the results presented here can be applied for the Schrödinger equation that, to a large extent, can be thought as being a wave model with infinite velocity of propagation.

THE 1-D CONTROL PROBLEM

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control on one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control.
The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1.$$

THE 1-D OBSERVATION PROBLEM

The control problem above is equivalent to the following one, on the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & 0 < x < 1,\, 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,0) = \varphi^0(x),\, \varphi_t(x,0) = \varphi^1(x), & 0 < x < 1. \end{cases}$$

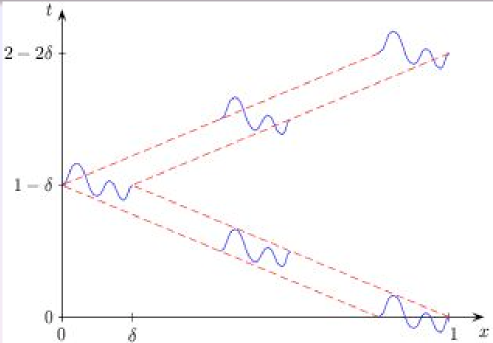
The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|\varphi_x(x,t)|^2 + |\varphi_t(x,t)|^2 \right] dx = E(0), \ \ \forall 0 \leq t \leq T.$$

The question is then reduced to analyze whether the folllowing inequality is true. This is the so called observability inequality:

$$E(0) \leq C(T) \int_0^T |\varphi_x(1,t)|^2 \, dt.$$

The answer to this question is easy to gues: The observability inequality holds if and only if $T \geq 2$.



$$E(0) \leq C(T) \int_0^T |u_x(1,t)|^2 \, dt.$$

Wave localized at $t = 0$ near the extreme $x = 1$ that propagates with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.

This observability inequality is easy to prove by several means.

- Use D'Alambert's formula

$$\varphi = f(x+t) + g(x-t)$$

 indicating that information propagates along rays with velocity one, and bounces on the boundary points.

- Use the Fourier representation of solutions in which it is clearly seen that solutions are periodic with time-period 2.
- Multipliers: Multiply the equation by $x\varphi_x$, φ_t and φ and integrate by parts....

CONSTRUCTION OF THE CONTROL:

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \varphi_x(1,t),$$

where u is the solution of the adjoint system corresponding to initial data $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(\varphi^0, \varphi^1) = \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1}\times H_0^1},$$

in the space $H_0^1(0,1) \times L^2(0,1)$.

Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that $\varphi_x(1,t) \in L^2(0,T)$ (hidden regularity). Moreover,

COERCIVITY OF J = OBSERVABILITY INEQUALITY.

CONCLUSION

- The 1-d wave equation is controllable from one end, in time 2, twice the length of the interval.
- Similar results are true in several space dimensions. The region in which the observation/control applies needs to be large enough to capture all rays of Geometric Optics. According to the terminology coined in the paper by Bardos - Lebeau - Rauch, this is the so-called Geometric Control Condition (GCC).
- When the GCC is not satisfied one may control projections of solutions into eigenfunctions clusters with a cost that increases exponentially as the frequency function increases.

Tools to prove observability inequalities

- Explicit D'Alembert's formula:

$$\varphi(x,t) = f(x+t) + g(x-t);$$

- Fourier series:

Ingham's Theorem. (1936) *Let $\{\mu_k\}_{k\in \mathbf{Z}}$ be a sequence of real numbers such that*

$$\mu_{k+1} - \mu_k \geq \gamma > 0, \ \forall k \in \mathbf{Z}.$$

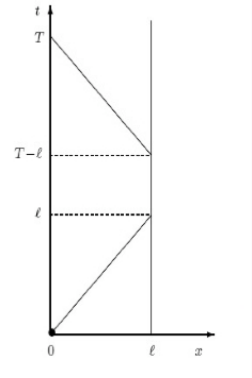
Then, for any $T > 2\pi/\gamma$ there exists $C(T,\gamma) > 0$ such that

$$\frac{1}{C(T,\gamma)} \sum_{k\in \mathbf{Z}} \mid a_k \mid^2 \leq \int_0^T \left| \sum_{k\in \mathbf{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq C(T,\gamma) \sum_{k\in \mathbf{Z}} \mid a_k \mid^2$$

for all sequences of complex numbers $\{a_k\} \in \ell^2$

- Sidewise energy propagation:

$$[\varphi_{tt} - \varphi_{xx} = 0] \equiv [\varphi_{xx} - \varphi_{tt} = 0.]$$



Pointwise controllers and observers

Take $x_0 \in (0,1)$. How much energy we can recover from measurements done on x_0?

$$\varphi(x_0, t) = \sum_{k \in \mathbf{Z}} a_k e^{ik\pi t} \sin(k\pi x_0).$$

Furthermore, if $T > 2$,

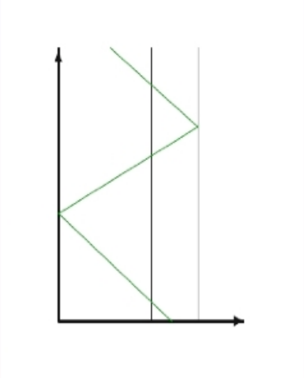
$$\int_0^T \left| \sum a_k e^{ik\pi t} \sin(k\pi x_0) \right|^2 dt \sim \sum \sin^2(k\pi x_0) |a_k|^2.$$

Obviously, two cases:

- When x_0 is irrational: $\sin^2(k\pi x_0) \neq 0$ for all k and the quantity under consideration is a norm, i.e. it provides information on all the Fourier components of the solutions.
- The case: $x_0 \in \mathbf{Q}$, some of the weights $\sin^2(k\pi x_0)$ vanish an the quadratic term is not a norm.

But, even if, $\sin^2(k\pi x_0) \neq 0$ for all k, the norm under consideration is not the L^2-one we expect!!!!

Can we explain this in terms of rays, and the propagation of waves (and antiwaves)?



If x_0 is rational we can build a finite number of rays and anti-rays that always intersect in x_0 for the time interval $(0, 2)$ of periodicity of solutions.

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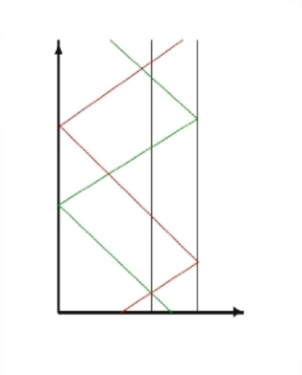

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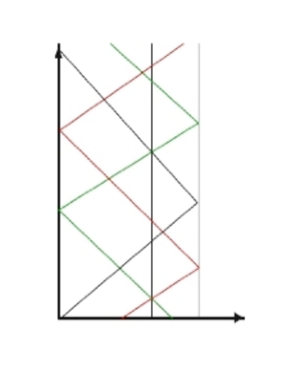

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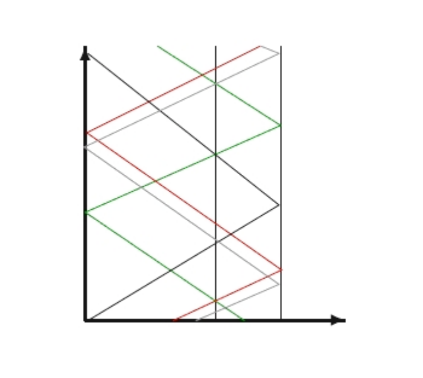

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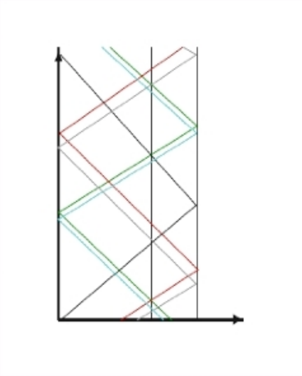

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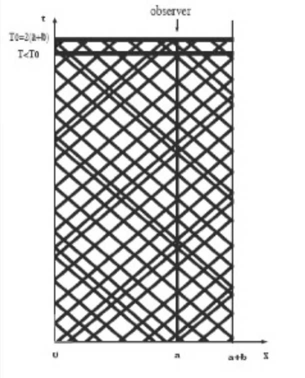


If x_0 is rational we can build a finite number of rays and anti-rays that always intersect in x_0 for the time interval $(0, 2)$ of periodicity of solutions.

The case x_0 irrational.

Can we expect that

$$\left|\sin(k\pi x_0)\right| \geq \alpha > 0, \quad \forall k?$$

This is impossible!!!!
Indeed, this would mean that

$$\left|k\pi x_0 - m\pi\right| \geq \beta$$

for all $k, m \in \mathbf{Z}$. And this is obviously false.
For suitable irrational numbers x_0 we can get

$$\left|k\pi x_0 - m\pi\right| \geq \beta/k.$$

And this is the best we can get.
In this case we get an observation inequality but with a loss of one derivative.
For some other irrational numbers (Liouville ones, for instance) the degeneracy may be arbitrary fast.

Conclusion

Making measurements in the interior of the domain is a much less robust process than doing it on the boundary (actually, one boundary measurement=two measurements in the interior since the boundary condition adds one).

In some cases we fail to capture all the Fourier components and, even if we are able to do it, this does not happen in the energy space, but there is a loss of at least one derivative.

$$E_{x_0} \leq C \int_0^T |\varphi(x_0, t)|^2 dt.$$

Similar phenomena occur for lumped control problems, an in the control of the linearized bilinear control system for the Scrödinger equation with laser beams.

This shows that the rigorous analysis of these problems at the PDE level can be subtle.

The continuous approach: Gradient algorithms

The control was characterized as being the minimizer over $H_0^1(0,1) \times L^2(0,1)$ of

$$J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1} \times H_0^1} .$$

We produce an algorithm in which:

- We replace J by some numerical approximation J_h with an order h^θ.
- We apply a gradient iteration algorithm to J_h.

The following holds

Theorem

(S. Ervedoza & E. Z., 2011)
In $K \sim C|\log(h)|$ iterations, the controls v_h^K obtained after applying K iterations of the gradient algorithm to J_h fulfill:

$$||v - v_h^K|| \leq C|\log(h)|^{\max(\theta,1)} h^\theta .$$

- This continuous approach is easy to be implemented.
- Note however that it does not yield any result on the control of the approximated finite-dimensional dynamics but only the convergence towards the continuous control.
- Similar results have been proved by S. Micu, M. Tucsnak and N. Cindea in the context of D. Russell's iteration for computing controls out of feedback stabilization results. Our result shows that it is a fact that systematically can be proved once the functional setting for the control and the numerical approximation is clear enough.

The continuous approach: Exact penalization

It is also natural analyze the issue of controllability from the point of view of optimal control. In some sense, the exact control property has to be a limit of some optimal control problem when the penalty parameter is large enough....

Theorem

(M. Gugat & E. Z., 2011)
There exists a constant γ (that can be computed explicitly out of the observability constant) such that when minimizing

$$\frac{1}{2}||v||^2_{L^2(0,T)} + \gamma||(y(T), y_t(T))||_{L^2(0,1)\times H^{-1}(0,1)}$$

over the class of solutions of

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

The discrete approach: Numerical phantoms

FROM FINITE TO INFINITE DIMENSIONS IN PURELY CONSERVATIVE SYSTEMS.....

Set $h = 1/(N+1) > 0$ and consider the mesh

$$x_0 = 0 < x_1 < ... < x_j = jh < x_N = 1 - h < x_{N+1} = 1,$$

which divides $[0,1]$ into $N+1$ subintervals
$I_j = [x_j, x_{j+1}]$, $j = 0, ..., N$.
Finite difference semi-discrete approximation of the wave equation:

$$\begin{cases} u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\, j = 1, \ldots, N \\ u_j(t) = 0, & j = 0,\, N+1,\, 0 < t < T \\ u_j(0) = u_j^0,\, u_j'(0) = u_j^1, & j = 1, \ldots, N. \end{cases}$$

The energy of the semi-discrete system (obviuosly a discrete version of the continuous one)

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[\mid u_j' \mid^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right].$$

It is constant in time.

Is the following observability inequality true?

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt$$

$$\left(-\frac{u_N(t)}{h} = \frac{u_{N+1} - u_N(t)}{h} \sim u_x(1, t). \right)$$

YES! It is true for all $h > 0$ and for all time T.

BUT, FOR ALL $T > 0$ (!!!!!)

$$C_h(T) \to \infty, \quad h \to 0.$$

CONCLUSION

The classical convergence (consistency+stability) does not guarantee continuous dependence for the observation problem with respect to the discretization parameter.

WHY?

Convergent numerical schemes do reproduce all continuous waves but, when doing that, they create a lot of spurious (non-realistic, purely numerical) high frequency solutions. This spurious solutions distroy the observation properties and are an obstacle for the controls to converge as the mesh-size gets finer and finer.

SPECTRAL ANALYSIS

Eigenvalue problem

$$-\frac{1}{h^2}\left[w_{j+1}+w_{j-1}-2w_j\right]=\lambda w_j,\ j=1,\dots,N$$
$$w_0=w_{N+1}=0.$$

The eigenvalues $0<\lambda_1(h)<\lambda_2(h)<\cdots<\lambda_N(h)$ are

$$\lambda_k^h=\frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$$

and the eigenvectors

$$w_k^h=\left(w_{k,1},\dots,w_{k,N}\right)^T:w_{k,j}=\sin(k\pi jh),\ k,j=1,\dots,N.$$

It follows that

$$\lambda_k^h\to\lambda_k=k^2\pi^2,\ \text{as }h\to 0$$

and the eigenvectors coincide with those of the wave equation.

Then, the solutions of the semi-discrete system may be written in Fourier series as follows:

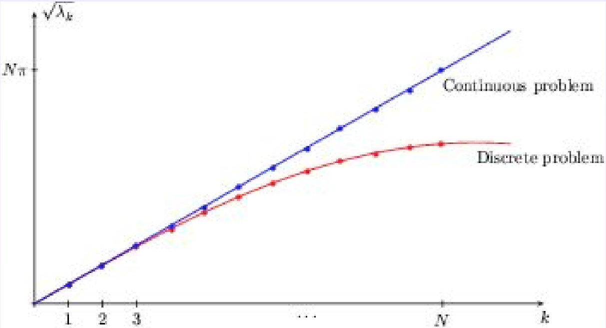
$$\vec{u} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h}\, t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h}\, t\right) \right) \vec{w}_k^h.$$

Compare with the Fourier representation of solutions of the continuous wave equation:

$$u = \sum_{k=1}^{\infty} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x)$$

The only relevant difference is that the time-frequencies do not quite coincide, but they converge as $h \to 0$.

DISPERSION DIAGRAM: LACK OF GAP.



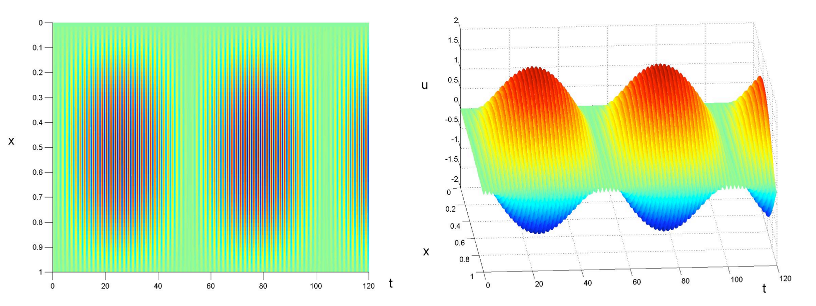
Graph of the square roots of the eigenvalues both in the continuous and in the discrete case. The gap is clearly independent of k in the continuous case while it is of the order of h for large k in the discrete one.

SPURIOUS NUMERICAL SOLUTION

$$\vec{u} = \exp\left(i\sqrt{\lambda_N(h)}\, t\right) \vec{w}_N - \exp\left(i\sqrt{\lambda_{N-1}(h)}\, t\right) \vec{w}_{N-1}.$$

Spurious semi-discrete wave combining the last two eigenfrequencies with very little gap:

$$\sqrt{\lambda_N(h)} - \sqrt{\lambda_{N-1}(h)} \sim h.$$



$h = 1/61$, ($N = 60$), $0 \leq t \leq 120$. The solution exhibits a time-periodicity property with period τ of the order of $\tau \sim 50$

GAP

=

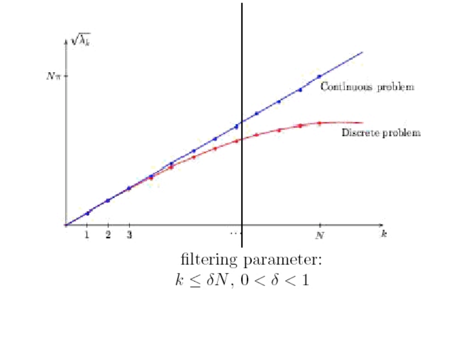
GROUP VELOCITY

=

VELOCITY OF PROPAGATION OF HIGH FREQUENCY WAVE PACKETS.

Filtering

WHAT IS THE REMEDY?



To filter the high frequencies, i.e. keep only the components of the solution corresponding to indexes: $k \leq \delta/h$ with $0 < \delta < 1$.
Filtering reestablishes the gap condition, then waves propagate with a speed which is uniform with respect to h and the observability inequality becomes uniform too.

CONCLUSION

Given any $T > 2$, choose $0 < \delta < 1$ such that $T > 2/\cos(\pi\delta/2)$.

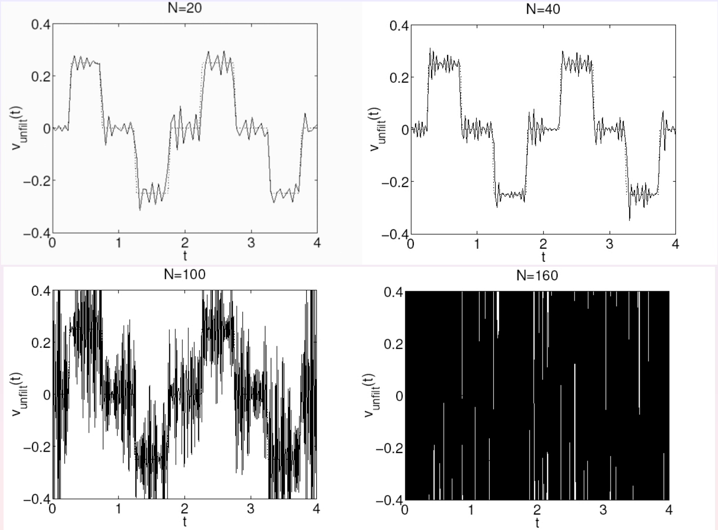
The choice of $0 < \delta < 1$ is obviously possible since $2/T < 1$.

Then, we can control UNIFORMLY ON h the solution PARTIALLY:

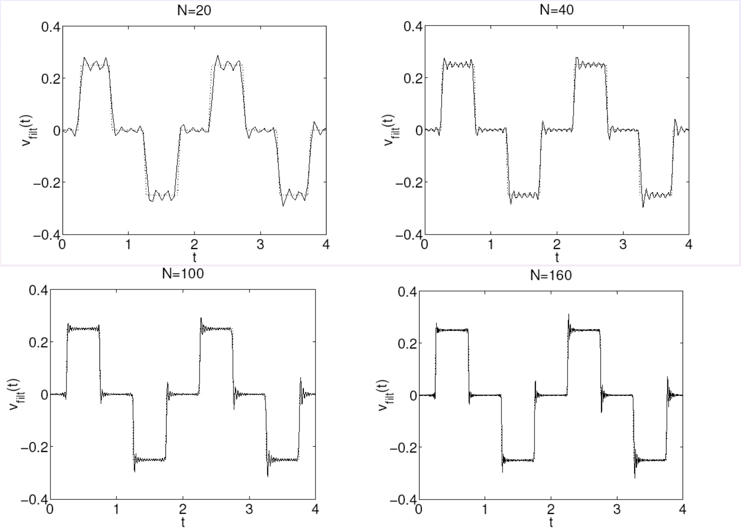
$$\pi_\delta(y(T), y_t(T)) = 0$$

and

the numerical controls $\quad v_h \to v, \qquad$ the control of the wave equation.



Without filtering, the control diverges as $h \to 0$.



With appropriate filtering the control converges as $h \to 0$.

TWO-GRID ALGORITHM

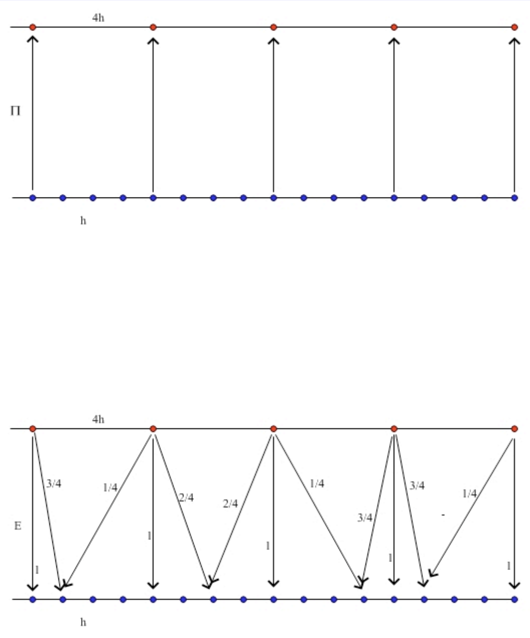
High frequencies producing lack of gap and spurious numerical solutions correspond to large eigenvalues

$$\sqrt{\lambda_N^h} \sim 2/h.$$

When refining the mesh

$$h \to h/2, \qquad \sqrt{\lambda_{2N}^{h/2}} \sim 4/h.$$

All solutions on the coarse mesh when projected to the fine one are no longer pathological.
TWO GRIDS $\sim$ FILTERING WITH PARAMETER $\delta = 1/2$.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Further results:

- Black box to transfer control results from time-continuous conservative semigroups into time-discrete ones. This leads to a huge class of control results for fully discrete schemes (S. Ervedoza, Ch. Zheng & EZ, JFA, 2008).
- Controls can be ensured to be smooth when the data to be controlled are smooth minimizing the following variant of the functional J:

$$J_\eta(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T \eta(t) |\varphi_x(1, t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >$$

  with $\eta(0) = \eta(T) = 0$.
- Convergence rates for filtered controls. In the present finite-difference context, for initial data to be controlled with one more derivative $(y^0, y^1) \in H_0^1(0,1) \times L^2(0,1)$ the convergence rate of controls in $L^2(0, T)$ is $h^{2/3}$.

- More sophisticated numerical algorithms based on Discontinuous Galerkin and higher order Finite Element Methods have been developed by A. Marica.

Some references

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