Are Landscape Traps Lurking to Impede Quantum Control?

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Outline

Quantum Control Landscapes

Consider a quantum system consisting of a Hamiltonian interacting with an external field $\varepsilon(t)$ that obeys the Schrödinger equation,

$$H(t) = H_0 - \mu\varepsilon(t), \quad i\hbar\frac{\partial U(t,0)}{\partial t} = H(t)U(t,0)$$

and the control objectives at some final time T:

- State-to-state transition probability from $|i\rangle$ to $|f\rangle$: $P_{i\to f} = |\langle f|U(T,0)|i\rangle|^2$
- Unitary propagator control to generate target W: $||W - U(T,0)||^2 = 2N - 2\mathrm{Re}(\mathrm{Tr}(W^\dagger U(T,0)))$

The *Quantum Control Landscape* defines the relationship between the objective and the field $\varepsilon(t)$.

When is the landscape trap-free?

For both the $\mathrm{P}_{i\to f}$ and $||W - U||^2$ control objectives, the respective landscapes can be shown to contain no suboptimal trapping extrema provided the following criteria are satisfied:

- The target quantum system is controllable
- The Jacobian $\frac{\delta U(T,0)}{\delta \varepsilon(t)} = -\frac{i}{\hbar} U(T,0) U^\dagger(t,0) \mu U(t,0)$ is full-rank
- No constraints are placed on the controls $\varepsilon(t)$

Possible sources of constraints:

- Fluence cost on $\varepsilon(t)$ in order to restrict pulse energy
- Choice of final time T that is too short
- Insufficient discretized time resolution of $\varepsilon(t)$.
- Inaccuracies in the search algorithm

Are traps lurking around every corner? What is the evidence?

Model quantum systems

Simulations performed on N-level model quantum systems. H_0 has either rigid rotor or anharmonic oscillator energy levels:

$$H_0^{rot} = \sum_{j=0}^{N-1} \gamma\, j\,(j+1)\, |j\rangle\langle j| \tag{1}$$

$$H_0^{osc} = \sum_{j=0}^{N-1} \left[\omega(j+1/2) - \frac{\omega^2}{\mathcal{D}}(j+1/2)^2\right] |j\rangle\langle j| \tag{2}$$

The dipole matrix μ preferentially allows transitions between nearby states

$$\mu_{ij} \sim \frac{1}{i-j} \tag{3}$$

For the $\mathrm{P}_{i\to f}$ objective, $\mu_{if} = 0$

Optimization procedure

Algorithm:

- Select initial control field discretized on $[0, T]$ with p time-points
- Employ gradient algorithm with conservative step-size to climb the landscape, with the field value at each time-point allowed to vary freely
- Stop when objective value decreases at two consecutive iterations (i.e., a trap is reached) OR convergence criterion is satisfied

Target Objectives:

- $\mathrm{P}_{i \to f}$: $|1\rangle \to |5\rangle$, $|1\rangle \to |10\rangle$, and $|1\rangle \to |N\rangle$
- $||W - U||^2$: random unitary W, CNOT and FT quantum gates.

Initial results for $P_{i\to f}$ optimization

Convergence criterion: $P_{i\to f} > 0.999$, p=2048 time-points, T=28.

N	target $\|i\rangle \to \|f\rangle$	# runs	traps?
5	$\|1\rangle \to \|N\rangle$	500	no
10	all	730	no
15	all	820	no
20	all	800	no
30	$\|1\rangle \to \|5\rangle$ and $\|1\rangle \to \|10\rangle$	450	no
	$\|1\rangle \to \|N\rangle$	130	6 at $P_{i\to f}\sim 0.9975$
40	$\|1\rangle \to \|5\rangle$ and $\|1\rangle \to \|10\rangle$	460	no
	$\|1\rangle \to \|N\rangle$	160	8 at $P_{i\to f}\sim 0.9972$
100	$\|1\rangle \to \|5\rangle$ and $\|1\rangle \to \|10\rangle$	70	no
Total	all	4120	14 traps

"Traps" only occur for $|1\rangle \to |N\rangle$ transition with large N. Are these real?

Are the initially observed traps real?

Which conditions show trapping behavior?

N	case	p=2048?	p=4096?	optimized?
30	1	yes	no	no
	2	yes	no	no
	3	yes	no	no
	4	yes	no	no
	5	yes	no	no
	6	yes	yes	no
40	1	yes	yes	no
	2	yes	yes	no
	3	yes	yes	no
	4	yes	yes	no
	5	yes	yes	no
	6	yes	yes	no
	7	yes	yes	no
	8	yes	no	no

No traps upon increasing p and then optimizing with gradient algorithm

Results from K. W. Moore and H. Rabitz, arXiv:quant-ph/1006.3702v3, submitted to *Phys. Rev. A*

Results for $||W - U||^2$

Convergence criterion: $||W - U||^2 < 0.001 \times 4N$. For $N \leq 4$, p=1024; for $N = 8$, p=2048; for $N \geq 16$, p=4096. T=14 or 28.

N	# runs	traps?
2	6200	no
4	10380	no
8	7850	no
16	460	no
32	130	no
Total	25020	no traps!

No searches got trapped with sufficient time resolution.

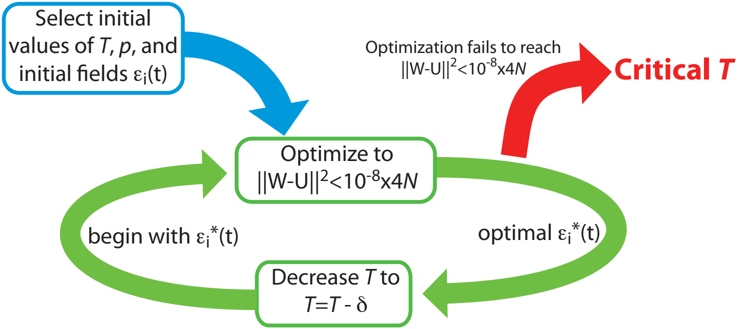
Results from K.W. Moore, R. Chakrabarti, G. Riviello, and H. Rabitz, *Phys. Rev. A* **83** 012326 (2011)

What minimum value of T is needed for $||W - U||^2$?

Coupled spin systems of m qubits,

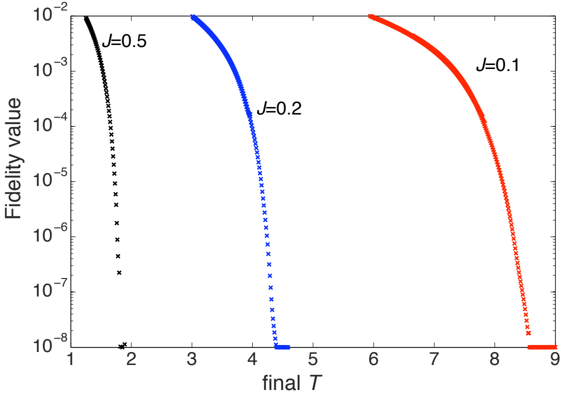
$$H(t) = \sum_{i=1}^{m} \omega_i \sigma_z^i + \sum_{i=1}^{n-1} \sum_{j>i}^{n} \sum_{k=x,y,z} J^{i,j} \sigma_k^i \sigma_k^j + \sum_{i=1}^{m} \sigma_x^i \varepsilon_i(t)$$

Procedure to find the minimum T:



Results for 2-qubit systems

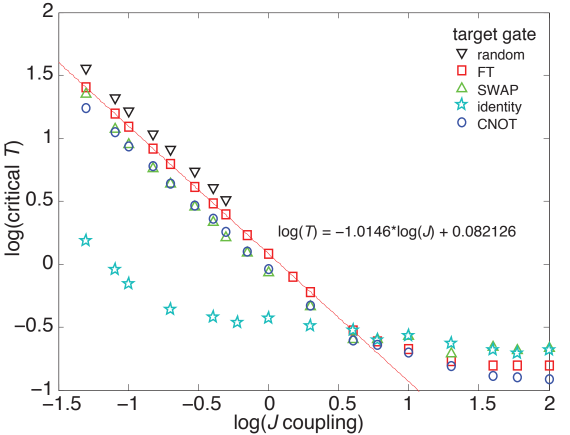
Target W is CNOT gate in $SU(N)$. Choice of coupling constant J determines the attainable value of $||W - U||^2$ at a given value of T.



Stronger coupling makes high fidelity attainable at lower T

Relationship between critical T and coupling J

Critical T is proportional to $1/J$ for weak coupling constants



Each target W has its own relationship between T and J

How many variables are needed to avoid false traps on the $P_{i\to f}$ landscape?

Consider parameterizing the control field $\varepsilon(t)$ by M variables:

$$\varepsilon(t) = A(t) \sum_{m=1}^{M} \cos(\omega_m t + \phi_m) \tag{4}$$

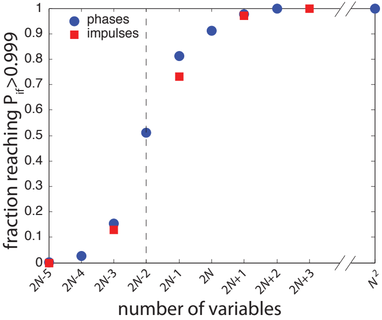
$$\varepsilon(t) = \sum_{m=1}^{\frac{M}{2}} \beta_m \delta(t - t_m) \tag{5}$$

with variables ϕ_m (4) or β_m and t_m (5).

Expect no traps on the landscape if a judicious set of N^2 variables with the freedom to define $U(T, 0)$ are used. Can we get away with fewer variables?

Results with parameterized control fields $\varepsilon(t)$

For phase controls, N ranges from 3 through 8. For impulse controls, N=4, 8. Convergence criterion is $P_{i\to f} > 0.999$.



- The gradient for $P_{i\to f}$ can be written in terms of $2N$-2 independent functions of time.
- 50% of searches reach $P_{i\to f} > 0.999$ with $2N$-2 variables, and no traps are observed when at least $2N+2$ variables are employed.

Summary

- When care is taken to ensure that no constraints are present on the controls, no traps are observed during optimization of $\mathrm{P}_{i\to f}$ and $||W-U||^2$.
- Traps introduced by insufficient time resolution can be lifted by increasing the number of time-points describing the control field $\varepsilon(t)$.
- Choice of T can determine the attainable target fidelity. To avoid traps on the landscape, T must be sufficiently large.
- Judiciously chosen small sets of variables can work, but care is needed if the highest fidelity is desired.