Engineering Control Concepts and Their Application to Quantum Control

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Outline

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1. Intro
2. Parameter Estimation for Servo Problems
   - Classical Engineering Concepts
   - Quantum State and Parameter Estimation (Ensemble Systems)
3. Regulator Problems: Feedforward Control
   - Classical Engineering Concepts
   - Quantum Feedforward
4. Filtering
   - Classical Engineering Concepts
   - Quantum Filtering
5. Optimal Feedback
   - Classical Engineering Concepts

Regulator/Servo Problems

- *Regulator* problems: Maintain a desired trajectory (e.g., steady state) in face of input, environmental disturbances or parameter uncertainty
- *Servo* problems: Reach a new steady state / setpoint, possibly at a specified time.
- Both can be achieved while minimizing some resource cost

Time-varying/-invariant Problems

- $\frac{dx_t}{dt} = Ax_t + Bu_t$; assume observation law $z_t = Cx_t$
- Time-invariant: use transfer functions

$$G(s) = C(sI - A)^{-1}B$$
$$z(s) = G(s)u(s)$$
$$z(t) = \mathcal{L}^{-1}\left[C(sI - A)^{-1}Bu(s)\right]$$

- $G(s)$ is *transfer function matrix*: maps any input vector $u(s) = \mathcal{L}[u(t)]$ in Laplace frequency domain to output (state) vector $x(s)$.
- Assess stability in terms of poles of transfer function matrix elements
- For linear systems, irrespective of how modes (eigenvectors) of A are unstable, feedback control can stabilize (if system is observable and controllable). Closed loop fundamental matrix $A^{cl} = A - BK(t)$; $A^{cl,ss} = A - BK(\infty)$ ($K(t)$ is feedback controller gain) then has N eigenvalues on left complex half plane.
- Time-varying: cannot use transfer function theory.

Linear/Bilinear/Nonlinear Problems

- Time-varying, nonlinear systems common in servo problems
- Linearization most common for regulator problems
- Control theory for servo problems discussed extensively in other optimal control talks
- This talk will focus on estimation theory for servo/regulator problems and control theory for regulator problems

Deterministic/Stochastic Problems

Examples of dynamical models arising in engineering control:

- Linear deterministic (ode) control system, parameter uncertainty: $\frac{dy_t}{dt} = Ay_t + Bu_t,\ A_{ij} \sim \mathcal{N}(a_{ij}, \sigma_{ij}^2),\ \ B_{ij} \sim \mathcal{N}(b_{ij}, \delta_{ij}^2)$
- Bilinear deterministic (ode) control system, parameter uncertainty: $\frac{dx_t}{dt} = (A + Bu_t)x_t,\ A_{ij} \sim \mathcal{N}(a_{ij}, \sigma_{ij}^2),\ \ B_{ij} \sim \mathcal{N}(b_{ij}, \delta_{ij}^2)$
- Linear Markovian diffusion process control (Ornstein-Uhlenbeck process): $dy_t = Ay_t dt + Bu_t dt + Dd\omega_t$
- Geometric Brownian motion (multiplicative noise): $dy_t = ay_t dt + bu_t y_t dt + cy_t d\omega_t$

Outline

Goals of parameter estimation

- State estimation: adaptive feedback (open loop) control of multiple output processes
- Dynamical parameter estimation: robust, model predictive optimal control
- Assessment of worst case control performance

Asymptotically Efficient Estimators

- *Likelihood function*: $L(\hat{\theta}|x)$ is joint density of observations expressed as function of unknown parameter vector $\hat{\theta}$. Maximum likelihood estimator $\hat{\theta}_{ML} = \arg\max L(\hat{\theta}|x)$ is best frequentist estimator, achieves CRB
- Fisher information: $I(\theta) = -\mathrm{E}\left[\frac{\partial^2 \ln L(\theta|z)}{\partial\theta\partial\theta'}\right]$; $[I(\theta_0)]^{-1}$ is called the *Cramer-Rao lower bound (CRB)* for consistent estimators.
- *Asymptotically efficient.* A sequence of consistent estimators $\hat{\theta}^m$ is asymptotically efficient if $\sqrt{m}\,[\hat{\theta}^m - \theta_0] \xrightarrow{d} \mathcal{N}[0, mI^{-1}(\theta_0)]$
- Equivalence with and limitations of least squares estimation: equivalent for Gaussian noise. Least squares computationally efficient but can be inaccurate for non-Gaussian noise

Observability of time-variant linear systems

- Consider the time-variant linear system $\frac{dx}{dt} = A(t)x(t)$ in the absence of control, with formal solution $x(t) = U(t)x_0$
- Consider a *linear observer* $z(t) = C(t)x(t) = C(t)U(t)x_0$, where $C(t)$ is $m \times N$
- The aim is to solve for x_0 by making m observations $z(t)$ at each time t
- To obtain a sufficient condition for this solution to exist, left-multiply the observation equation by $U^T(t)C^T(t)$ and integrate over all time:

$$\int_0^T U^T(t)C^T(t)C(t)U(t)\ dt\ x_0 = \int_0^T U^T(t)C^T(t)z(t)\ dt$$

- Let $H(T) = \int_0^T U^T(t)C^T(t)C(t)U(t)\ dt$; note it is an $N \times N$ Gramian matrix. Now solve for x_0:

$$x_0 = H^{-1}(T) \int_0^T U^T(t)C^T(t)z(t)\ dt$$

- H is called the *observability Gramian matrix*.
- Relation to discrete-time recursive least squares estimation (regression) of single state

Dynamical parameter estimation (system identification)

- Method: (linear) regression/step testing for n-th order (vector) autoregressive processes (discrete-time version of n-th order linear ode/sde):

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots + b_1 u_{k-1} + b_2 u_{k-2} + \cdots + \epsilon_k$$

- For step inputs u_k, unique solution for a, b's exists (via left pseudoinverse), if matrix $C^T C$ is full rank, where C is $m \times N$ matrix of regression factors (columns are $[y_{k_1-1}, \cdots, y_{k_m-1}]^T$, etc); requires measurements can be made on timescale where discrete-time approximation valid
- State estimates y_k ("observations") and associated noise ϵ_k can be obtained from MLE/LQ (filters for time-series observations)
- Most useful for time-series observations/estimates of state
- System parameters are extracted using state estimates or direct observations of state
- For independent observations, inversion of dynamical trajectory using evolution times t_k as regression factors, or a dynamical equation that is nonlinear in parameters, results in ill-posed problem

Quantum state estimation, static

Tomography or *Method of Moments* estimation:

$$\mathrm{Tr}(\rho(\hat{\theta})O_i) = d_i,\ 1 \leq i \leq N^2 - 1, \qquad (1)$$

where d_i denotes the sample mean of the observable quantity corresponding to measurement of O_i. Introducing the notation $C_{ij} = \frac{\partial \mathrm{Tr}(\rho(\theta)O_i)}{\partial \theta_j}$, we may solve for the estimated parameter vector as $\hat{\theta} = C^{-1}\mathbf{d}$.

- Not asymptotically efficient: does not reach Cramer-Rao bound for non-Gaussian pdfs. Use MLE instead w likelihood function

$$L\ (\theta \mid X) = \prod_{k=1}^{m} \mathrm{Tr}(\rho(\theta)F_{i_k}), \qquad (2)$$

- Can use more moments (e.g., variances of observation data) to improve estimates

Optimal State estimation, static

- MLE-based quantum state estimation: two diagonal elements of ρ
- Predictions of asymptotic theory are precise in finite samples

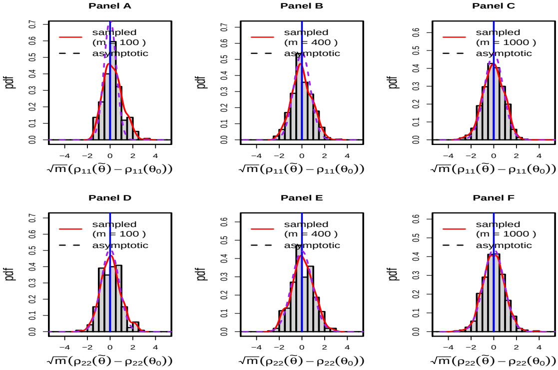

Figure: Finite sample distributions of $\sqrt{m}(\rho_{ii}(\hat{\theta}) - \rho_{ii}(\theta_0))$, for mixed three-level state, MUB bases. Panels A-C, ρ_{11}. (A) $m = 100$; (B) $m = 400$; (C) $m = 1000$. Panels D-F, ρ_{22}. (D) $m = 100$; (E) $m = 400$; (F) $m = 1000$. In each panel, finite sample distributions (1000 simulations) are shown alongside the corresponding asymptotic distribution.

Observability: state estimation, dynamic

- Now let $\mathrm{Tr}(\rho'(\hat{\theta})U^{\dagger}(t)O_i'U(t)) = z_i(t),\ 1 \leq i \leq m$, and $C^T(t) = [\nu\left(O_1(t)\right), \cdots, \nu\left(O_m(t)\right)]$

$$\int_0^T C^T(t)C(t)dt\ \nu(\rho(\hat{\theta})) = \int_0^T C^T(t)z(t)\ dt$$

$$\hat{\theta} = \left[\int_0^T C^T(t)C(t)\ dt\right]^{-1} \int_0^T C^T(t)z(t)\ dt, \tag{3}$$

- Note inversion (MME) does not enforce positive-semidefiniteness constraints
- New aspect of quantum measurement: noncommutative observations; each associated with a *POVM* (resolution of identity)

$$\langle I(\hat{\theta})\rangle = \frac{1}{V_0}\int_{B_{N^2-1}} I(\hat{\theta}, \rho(\theta_0))d\rho(\theta_0), \tag{4}$$

- Assume projective measurements in POVM

$$F_{r(N-1)+i} = V^{(r)}\tilde{F}_i(V^{(r)})^\dagger,$$
$$\tilde{F}_i = |i\rangle\langle i| = diag(0,...,1,...0),$$
$$1 \le i \le N-1, \quad 1 \le r \le N+1.$$

Optimal Observability: state estimation, dynamic

- Bilinear observability: $O_i' = O_i - \frac{\mathrm{Tr}\, O_i}{N} I_N$;

$$\bigoplus_{i=1}^{n} \mathrm{span}\{[\cdots[iH_{j_3},[iH_{j_2},[iH_{j_1},iO_i']]]\cdots]\} = su(N)$$

$H_j \in \{H_0, \mu\}$

- Does not leverage properties of quantum measurement

Definition

A quantum system $\{H_0, \mu\}$ with observable operators $\{O_i\}$ is said to be *optimally observable* if there exists a sequence of controls $\{\varepsilon_k(\cdot)\}$ and associated measurement times t_k such that $\sup_{\{O_{i,\varepsilon_k(\cdot)}(t_k)\}} \langle ||I(\rho_0(\hat{\theta}))||\rangle/m = \sup_{\{F_i\}} \langle ||I(\rho_0(\hat{\theta}))||\rangle/m$, where m denotes the number of measurements, $||\cdot||$ denotes the matrix norm, F_i are the elements of any informationally complete POVM set, and the averaging is in the sense of equation (4).

Optimal observability: state estimation, dynamic

- Formulate optimal measurement basis control problem:

$$\begin{aligned} F(U_k(T)) &= ||V^{(k)} - U_k(T)||^2 \\ &= \mathrm{Tr}[(V^{(k)} - U_k(T))^\dagger (V^{(k)} - U_k(T))] \\ &= 2N - 2\Re\mathrm{Tr}((V^{(k)})^\dagger U_k(T)), \end{aligned} \tag{5}$$

$$V_{pq}^{(r)} = \begin{cases} \delta_{pq}, & r = 0 \\ \frac{1}{\sqrt{N}} \exp[\frac{2\pi i}{N}(rp^2 + pq)], & 1 \leq r \leq N. \end{cases} \tag{6}$$

Density matrix controllability is sufficient (any unitary in coset $\frac{SU(N)}{T^{N-1}}$ informationally equivalent).

Optimal Observability: state estimation, dynamic

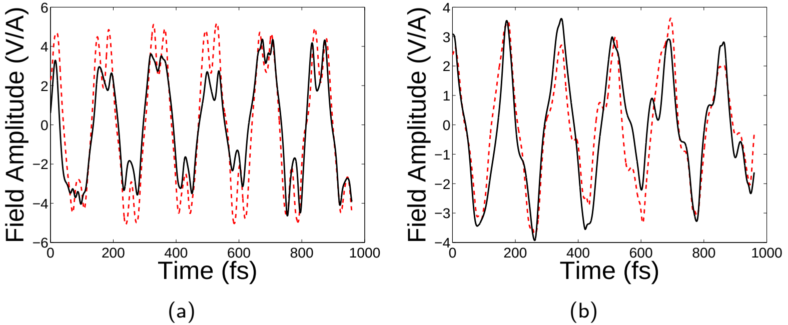

Figure: Optimal fields and power spectra (black, solid) for driving a 3-level system to MUB measurement bases. Noisy fields and spectra are superimposed (red, dashed). a) Field for basis $V^{(1)}$; b) Field for basis $V^{(2)}$. The Frobenius distance between controlled and target bases is < 0.03 for both optimal fields in the absence of noise.

State estimation, dynamic

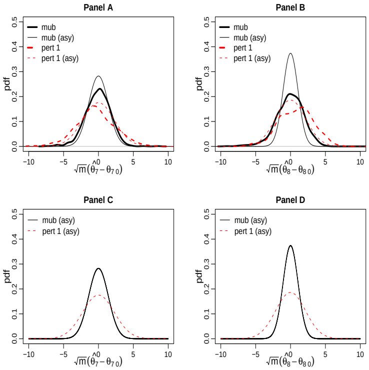

Figure: Distributions of $\sqrt{m}(\hat{\theta}_i - \theta_{i,0})$ for a three-level system, sample size 1000: comparison of perfect and noisy controls. (A) $\sqrt{m}(\hat{\theta}_7 - \theta_{7,0})$; (B) $\sqrt{m}(\hat{\theta}_8 - \theta_{8,0})$. Panels C and D: Asymptotic distributions obtained for A and B, respectively, are isolated for clarity.

Hamiltonian parameter estimation

- Issues with measurement of ultrafast dynamics - regression based on ode difficult; nonlinear inversion or MLE required (avoided by time-series filtering)
- Use i) the observations or ii) the sample means of observations as measurement data
- i) maximize likelihood of parameters given noisy observations:

$$L(\vec{h}|x) = \prod_{k=1}^{m} \mathrm{Tr}(\rho_t(\vec{h}) F_{i_k})$$

ii) solve for parameters by MME (nonlinear) inversion

Hamiltonian parameter estimation

- ii) can be formulated with sample means of "observations" being point estimates of the state ρ_t parameters; nonlinear MME problem following ML estimation of states. Moments: $\hat{\theta}_i(\vec{h}, t) = c_i(t)$
- Advantages of each formulation: ML w i) is asymptotically efficient from perspective of classical probability theory, not optimal within quantum probability theory (but LQ is not, due to non-Gaussian measurement noise and nonlinearity); ii) is not asymptotically efficient classically but can leverage theory of state estimation

Future directions (what should be done): parameter estimation

- Exploit features of noncommutative measurements in Hamiltonian parameter estimation? Choose control inputs $\varepsilon(t)$ - asymptotically efficient within quantum inference - based on mutually unbiased measurement bases
- Use ML based on $L(\vec{h}|x)$ in stage 1, choose optimal measurement times and control inputs in stage 2 refinement based on $\vec{h}$ estimate, reestimate using new $L(\vec{h}|x)$
- How to exploit analytical objective priors (new to quantum) originating due to known geometry of state space in Bayesian estimation? These correspond to the volume (Bures) measure on the space of density matrices
- Exploit results from *ab initio* Hamiltonian parameter calculations to formulate priors for Hamiltonian estimation and apply Bayesian estimation
- Applications to robust model-predictive optimal control

Selected References

- Linear observability: Kalman
- Bilinear observability: D. DAlessandro 2003,2004,2006
- Hamiltonian parameter estimation: Young/Kosut/Whaley (MLE); H. Rabitz et al., others
- Asymptotically optimal measurements (quantum FI): Wootters, Caves, Barndorff-Nielsen
- Optimal observability: Chakrabarti/Ghosh 2011

Outline

Neighboring optimal control

- Engineering concept: noise (environmental, input, measurement) or parameter uncertainty can compromise optimal control strategies; correct during evolution by local linearization and real-time feedback or feedforward (computationally less expensive)
- Linearize nonlinear system around the reference trajectory $(x_r(t), u_r(t))$ (assume it is known from solution to servo problem):

$$A(t) = \frac{\partial F}{\partial x}[x_r(t), u_r(t), t],$$

$$B(t) = \frac{\partial F}{\partial u}[x_r(t), u_r(t), t]$$

  They can be used in integration of the corresponding Riccati equations
- Define deviation variables $\Delta x(t) = x_r(t) - x(t), \; \Delta u(t) = u_r(t) - u(t)$

Local controllability (cont'd)

- *Local controllability*: whether there exists a control perturbation $\Delta u(t)$ that can achieve any arbitrary small perturbation from a nominal (reference) trajectory
- A sufficient condition for local controllability is that the $N \times N$ *controllability Gramian matrix*

$$G(T) = \int_0^T U(T,t')B(t')B^T(t')U^T(T,t')\ dt'$$

  is nonsingular
- This follows because the control perturbation $\Delta u(t)$ necessary to induce a change $dx(T)$ is

$$\Delta u(t) = B^T(t)U^T(T,t)G^{-1}(u,T)dx(T). \tag{7}$$

- Cost functional for perturbative feedback:

$$F(\Delta x(T)) + \frac{1}{2}\int_0^T \Delta x^T(t)Q\Delta x(t) + \Delta u^T(t)R\Delta u(t)\ dt.$$

- Use of this Bolza functional will result in state-dependent feedback form of *neighboring optimal feedback* control $\Delta u(x(t),t)$ that can compensate for noise/uncertainty above; stabilization (pole placement) can be achieved

Feedforward Control: classical input noise

- Motivation - minimize need for feedback
- Direct synthesis ("perfect control") for time-invariant systems: Algebraically solve for feedforward controller transfer function $G_f(s)$ (input-output map $G_f(s) = u(s)/d(s)$, where $d(s)$ is Laplace transform of noisy input)

$$G_f = -\frac{G_d}{G_m G_v G_p}$$

  where G_d is the transfer function for the disturbance input $(\frac{x(s)}{d(s)})$, G_m is the transfer function of the measurement device, G_v is that of the actuator and G_p is that of the process dynamics
- Assume source of uncertainty is noise in input parameters, but environmental noise, measurement noise and dynamical parameter uncertainties are minimal
- Direct synthesis does not work for time-varying systems (e.g., bilinear quantum systems)
- Sensor/actuator time delays can render perfect control impossible

Neighboring Optimal Quantum Feedforward Control: classical input noise

- Consider effects of classical control field input noise on optimal control fidelity
- Bilinear time-varying systems: since direct synthesis not possible, apply neighboring optimal control, using Lagrange functional (rather than Bolza, since state-dependent feedback not required, stabilization is not goal, and quality of control fidelity is improved)
- First assess *local controllability*
- Optimal corrective control perturbation (minimizes fluence of $\delta\varepsilon(t)$):

$$\delta\varepsilon(t) = \nu^T[\mu(t)]\left[\int_0^T \nu[\mu(t)]\nu^T[\mu(t)]dt\right]^{-1}\nu[dA(T)] \qquad (8)$$

where $dA(T) = -U^\dagger(T)\delta U(0)$

Neighboring Optimal Feedforward Control: classical input noise

Local controllability Gramian should be full for local controllability (must be expressed in appropriate minimal parameterization of tangent space for predicted feedforward update to remain on quantum state manifold):

$$S: \; su(N) \to su(N)$$

$$S(\varepsilon, [t_i, t_{i+1}]) = \int_{t_i}^{t_{i+1}} \left[\nu\left(U^\dagger(t,t_i)\mu U(t,t_i)\right)\right]\left[\nu\left(U^\dagger(t,t_i)\mu U(t,t_i)\right)\right]^T \; dt$$

- Expressions on $S_{\mathcal{H}}, \mathbb{CP}^{N-1}, SU(N), SU(N)/T^{N-1}$ etc have been developed
- Feedforward steps:

1. Compute the optimal controls $\varepsilon(t)$ offline
2. On each time interval $[t_i, t_{i+1}]$, compute the Gramian above (offline, prior to the experiment).
3. Measure input disturbances $\delta\varepsilon(t)$ over the interval $[t_{i-1}, t_i - \Delta t]$. Δt denotes the sensor+actuator time delay in the feedforward loop.

Neighboring Optimal Feedforward Control: classical input noise

- Apply the linear map $\delta\varepsilon(t) \to \delta\tilde{U}(t_i, t_{i-1}),\; t \in [t_{i-1}, t_i]$ (where $\tilde{U}(t_i)$ is used to denote the forecast due to the presence of unmeasured disturbances/model misspecification) to obtain a prediction for the effect of measured input disturbances on the dynamical propagator at any given time t_i.
- Apply neighboring optimal feedforward law

$$\delta\varepsilon(t) = \left[\nu\left(U^\dagger(t, t_i)\mu U(t, t_i)\right)\right]^T S^{-1}(\varepsilon, [t_i, t_{i+1}])\nu\left(dA(t_{i+1}, t_i)\right) \tag{9}$$

  on $[t_i, t_{i+1}]$, where the choice $dA(t_{i+1}, t_i) = -\tilde{U}^\dagger(t_i, t_{i-1})\delta\tilde{U}(t_i, t_{i-1})$ is made.
- The functions of time $[U^\dagger(t, t_i)\mu U(t, t_i)]_{jk}$ are linearly combined to create $\delta\varepsilon(t)$: avoid LCM time delays by preloading into pulse shapers
- Or, implement using (electro-optic) pulse shapers with nanosecond response times

Feedforward Control: quantum input noise

- Coherent Feedforward: design a passive controller (that both measures input disturbances and cancels undesired noise)
- Applied to cavity optomechanical systems
- Operator equations of motion are linearized by neglecting quadratic terms producing a time-invariant control system
- Feedforward design goal: cancel effects of phase fluctuations in input field on the outputs by design of the transfer function. Achieved through passive device (with transfer function $G_f(s)$ that rotates input quadratures through s-dependent angle)
- Obtain transfer function representation $y(s) = G(s)u(s)$
- Feedforward design goal: cancel effects of phase fluctuations in input field on the outputs by design of the transfer function.
- Caves formulated block diagrams for linear time-invariant systems from quantum optics; James considered series products as well as diagrams for nonlinear quantum optical control systems

Future Challenges (what should be done)

- Optimal feedforward for open quantum systems (ensemble)
- Application to robust controls
- Analysis of approximation accuracy for physical systems based on Magnus expansion
- Local controllability Gramian determines search effort for adaptive feedback control optimization - connection to quantum control landscapes and singular controls; observable and pure state control systems are more likely to have well-conditioned Gramians than mixed state or gate control systems

Selected References

- Coherent quantum feedforward (time-invariant; quantum noise cancellation): Caves, Kimble
- Feedforward/feedback block diagrams in quantum optics: James, Gough, Nurdin
- Singular controls in quantum control landscapes: R-B. Wu, J. Dominy, H. Rabitz, Schirmer, Tannor
- Optimal coherent feedforward control (classical input noise): Chakrabarti

Outline

Filtering: optimal state estimation of dynamical systems

- **Goals**: for systems with slower dynamics and short time delays, regulate optimal trajectory against dynamical noise or parameter uncertainty; cannot be applied to systems with ultrafast dynamics.
- Since the state covariance of a stochastic dynamical system increases with time of evolution, open loop "optimal" control based on state estimate forecast $\hat{x}(t)$ is prone to error
- *Filtering* methods update the state estimate and its covariance matrix optimally based on additional measurements made during evolution; based on combination of i) state estimate / covariance matrix updates in presence of measurements, but absence of evolution; ii) state estimate / covariance matrix updates in presence of evolution, but absence of measurements
- Filters convert information set (*filtration*) $I(t_0, t) = (z(t_0, t)), u(t_0, t))$ into derived information set $I_D(t_0, t) = (\hat{x}(t_0, t), \Sigma(t_0, t))$. We will restrict attention to Markovian diffusion processes; example: multivariate Ornstein-Uhlenbeck process $dx(t) = Ax(t)\ dt + D\ d\omega(t)$

Kalman Filtering: Linear Gaussian systems

- Filters can be based on different estimators for the state and its covariance; the simplest is the least squares filter
- Kalman developed optimal least squares filter for linear dynamical systems
- Recall, for linear systems in the Heisenberg picture,

$$\int_0^T C^T(t)C(t)dt\ \hat{x}(0) = \int_0^T C^T(t)z(t)\ dt$$

$$\hat{x}(0) = \left[\int_0^T C^T(t)C(t)\ dt\right]^{-1} \int_0^T C^T(t)z(t)\ dt, \tag{10}$$

  where $x = x_0$, i.e., the state was considered stationary and there were only new measurements/observation laws at each time
- Now include dynamical evolution of system; first equation above must be converted to differential equation; combine differential equations due to state evolution and measurements

Kalman filter equations

- Formulate dynamics of measurement in continuous time: $dz(t) = C(t)x(t)dt + Ed\omega(t)$
- Then the *Kalman filter equations* for optimal updating of the state estimate and its error during dynamical evolution of a linear system are (recall C specifies drift in observation law):

$$d\hat{x}(t) = A\hat{x}(t)dt + \Sigma(t)C^T(t)[dz(t) - C(t)\hat{x}(t)dt]; \;\; \hat{x}(0) = \hat{x}_0$$

$$\frac{d\Sigma(t)}{dt} = A\Sigma(t) + \Sigma(t)A^T + DD^T - \Sigma(t)C^T(t)C(t)\Sigma(t); \;\; \Sigma(0) = \Sigma_0$$

  latter is called a Riccati equation; together equivalent to a "stochastic Fokker-Planck equation".
- Kalman filter minimizes state estimate covariance by optimally mixing old and new measurements
- Features shared by all (frequentist) filters for Markovian sp's: a) sde for state estimate update composed of a drift term due to evolution alone and a martingale difference due to new information entering at time t; b) covariance update is a nonlinear ode

Quantum measurement

- $\mathcal{H} \otimes \mathcal{F}$ (latter is Fock space of coupled field) representation of measurement
- $\hat{\rho}_0$, ϕ initial states on $\mathcal{H}, \mathcal{F}$, respectively
- Consider commuting set of projection operators F_z on $\mathcal{F}$, associated with a given measurement basis
- $\mathrm{Tr}[U_t(\hat{\rho}_0 \otimes \phi)U_t^\dagger(I \otimes F_z)] = \mathrm{Tr}[(\hat{\rho}_0 \otimes \phi)U_t^\dagger(I \otimes F_z)U_t]$ specifies the probability distribution $p(z_t)$ of scalar measurement outcomes in that measurement basis at time t_i, following from measurement law $z(t_i) = C(t_i)\hat{x} + w(t_i)$ ($\hat{y}$ is the initial state; $w(t)$ is realization of measurement noise random variable corresponding to outcome z)

Continuous quantum measurement

- Formulate measurements in continuous time by analogy to classical continuous measurement: need to formulate a sde for outcomes F_z of measurements of field
- Stochastic observation law provides posterior pdf of any observable quantity; let output operator be denoted by $Y_t = U_t^\dagger (I \otimes W) U_t$ where W is a given observable on $\mathcal{F}$; compare $z(t)$ above; $dY(t)$ operator sde analogous to scalar $dz(t)$ will follow from sde for U_t, which will also introduce noise
- Homodyne measurement: measures Wiener diffusion process on $\mathcal{F}$: then field observable $W_t = A_t + A_t^\dagger$

Stochastic Schrödinger Equations

- Now obtain the measurement sdes by specifying stochastic dynamics
- Equation of motion for U_t is stochastic differential equation on $\mathcal{H} \otimes \mathcal{F}$
- Markovian SSE:

$$dU_t = \left[\left(-iH - \frac{1}{2} L^\dagger L \right) dt + L dA_t^\dagger - L^\dagger dA_t \right] U_t$$

- The first term is the drift; Lindblad operators on $\mathcal{H}$ specify system-environment coupling of ρ, ϕ
- Second term is diffusion, includes non-Hermitian "quantum noise" increment random variables $dA, dA^\dagger$
- Compare Geometric Brownian motion (linear sde with multiplicative noise)

Stochastic Schrödinger Equations

- Observation stochastic differential equation (observe W on $\mathcal{F}$):

$$Y_t = U_t^\dagger [(I \otimes W)] U_t$$
$$dY_t = (L + L^\dagger)\ dt + dW_t$$

- The first term provides the deterministic component to the observation law (analogous to $C\ dt$ in classical law $dz(t) = Cx(t)dt + Dd\omega(t)$; Cx comes from inner product with ρ_t)

Quantum Filtering: Stochastic Master Equation

- Now add new measurements to free dynamics: obtain stochastic master equation for $\hat{\rho}_t$ (equivalently, any complete set of observable operators X_i):

$$d\hat{\rho}_t = \mathcal{L}(\hat{\rho}_t)\ dt + \sigma(\hat{\rho}_t)\left[dY_t - \langle \hat{\rho}_t, L + L^\dagger \rangle\ dt\right]$$

- Accompanying deterministic (master) equation for covariance matrix of these operators is always nonlinear in state; obtained from $\sigma(\hat{\rho}_t)$ (compare $\Sigma(t)C^T(t)$ in classical filter)

Outline

Stochastic optimal control objectives

- For stochastic dynamics, can no longer aim to drive the system to a precise final state
- Goal: to control moments of a cost functional (cost-to-go): e.g. its expectation:

$$\min_{u(t)} \mathrm{E}\left[F(x(T)) + \frac{1}{2}\int_t^T x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\ dt\right]$$

- This quadratic cost (Bolza functional) can be rewritten in terms quadratic in the mean $\hat{x}(t)$ and the covariance matrix $\Sigma(t)$ of state estimates
- Optimal control must always be expressed in feedback form $\bar{u}(\hat{x}(t), t)$
- Dynamical constraint for LQG controller is (sde)

$$d\hat{x}(t) = A\hat{x}(t)dt + Bu(t)dt + K_e(t)(dz(t) - C(t)\hat{x}(t)dt);$$

control problem is $\min_{u(t)} J_{CE}$ subject to this constraint

- The feedback controller Riccati equation is (propagated backward in time from $S(T)$):

$$\frac{dS(t)}{dt} = -A^T S - SA - Q + S(t)BK_c(t)$$

Quantum Feedback: Hamilton-Jacobi-Bellman Equation

- Operator-valued (new to quantum) optimal control cost functional:

$$J(u(\cdot)) = \int_0^T U_t^\dagger(u(\cdot))C(u(t))U_t(u(\cdot))\ dt + U_T^\dagger(u(\cdot))DU_T(u(\cdot))$$

  where $C(u)$ denotes any operator-valued function of u (e.g., $X(u)$) and $U_t(u)$ denotes the controlled evolution operator. C_t is an (operator-valued) diffusion process. This is expressed in Heisenberg picture.

- "Cost-to-go" on $[t, T]$: $J(\rho, u, t)$ obtained by replacing integral over $[0, T]$ with integral over $[t, T]$; note no state in above equation
- Expectation/moments of associated scalar quantities must be computed in Schrödinger picture by inner product with $\hat{\rho}_t$, the posterior expected density operator conditioned on measurements on $[0, t]$:

$$E_0^t[J(\rho, u, t)] = \mathbb{E}\left[\int_t^T \langle \hat{\rho}_t(u), (C(u))\rangle\ dt + \langle \hat{\rho}_T(u), (D)\rangle\right]$$

Future Challenges: Stochastic Control

- Neighboring optimal LQG q control
- Controller tuning through choice of weighting matrices in operator $C(u)$
- Dynamical parameter estimation: jointly estimate states, parameters by either i) AKF - can be used within robust, model-predictive control formulation; ii) formulation of likelihood function $L(\vec{h}|x_1, \cdots, x_m) = \prod_{i=1}^{m} p(x_{i+1}|x_i, \vec{h})$, using Kalman state estimates for $x_2, \cdots, x_m$ and the conditional probabilities are obtained from the SHE
- Analytical solutions to HJB equation in other canonical cases beyond LQG - possibly including control of covariance matrix; curse of dimensionality
- How to choose optimal measurements for time-series state estimation

References

- Quantum filtering theory: Belavkin, James, H. Mabuchi et al.
- Optimal quantum feedback (HJB equations): Belavkin
- Measurement-based quantum feedback: Wiseman, Milburn, H. Mabuchi et al.