Curve modeling in shape spaces

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Outline

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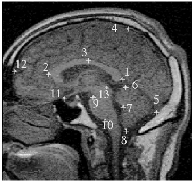
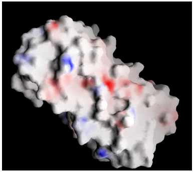
SHAPE

- Geometrical properties that are invariant under certain registration transformations. Some examples...





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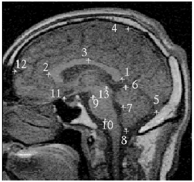
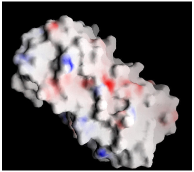
- Euclidean Shape: point sets which invariant under translation, rotation and scale





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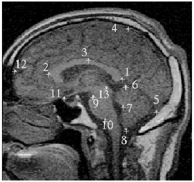
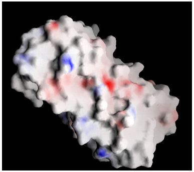
- Size-and-shape: point sets which invariant under translation and rotation





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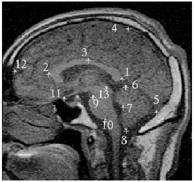
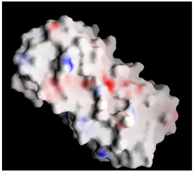
- Closed outline shapes: curves which are invariant under diffeomorphic transformations of arc-length.





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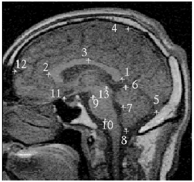
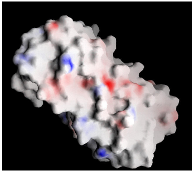
- Note: Quotient spaces often appropriate.





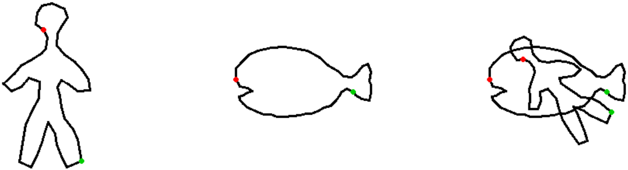
Landmark shapes

- EXAMPLE: Object: k points in m dimensions $X \in \mathbb{R}^{km}$
- Transformation group: Translation, rotation and scale.
- Here there are $k = 50$ points in $m = 2$ dimensions.

- Kendall's (1984) shape space: $\Sigma_m^k = S^{(k-1)m-1}/SO(m)$.
- Quotient space: Pre-shape sphere with rotation removed.

Practicalities: Procrustes matching

We wish to register the man (X_1) on to the fish (X_2), using translation, rotation and scale.
X_1, X_2 are $k \times m$ matrices, which are centered ($1_k^T X_j = 0$)



X_1 $\quad$ X_2 $\quad$ X_1^P, X_2

Explicit solution: SVD. Procrustes shape distance $d([X_1^P], [X_2])$.

Outline

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- A PRE-SHAPE has had location and scale removed BUT NOT rotation. It lives on a sphere.

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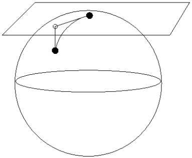
- Given two pre-shapes μ, z the minimal geodesic between their shapes corresponds to

$$\Gamma(s) = \mu \cos s + z \sin s, \quad 0 \leq s \leq \pi/2$$

where z has been Procrustes rotated to μ.



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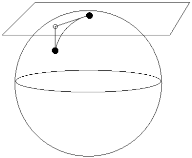


- Note that this is the horizontal lift of the minimal geodesic in shape space.

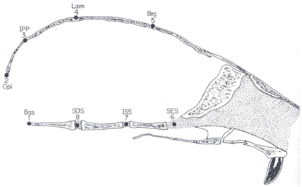
EXAMPLE

GEODESIC PATH: Fish $->$ Fishman $->$ Man

$$\Gamma(s) = \mu \cos s + z \sin s, \quad 0 \leq s \leq \pi/2.$$

EXAMPLE: RAT SKULLS

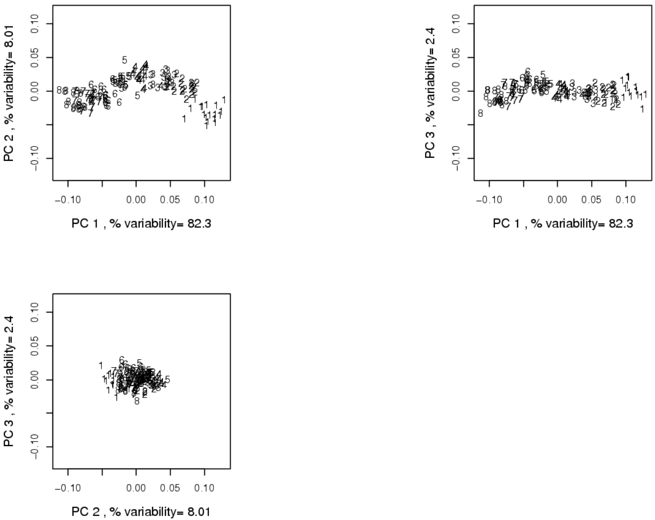
Bookstein's (1991) rat skull data.
The rats were carefully X-rayed at age N days, where $N \in \{7, 14, 21, 30, 40, 60, 90, 150\}$,



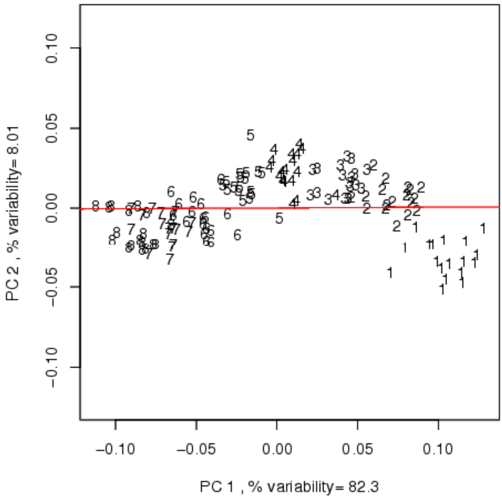
[Image: Bookstein, 1991]

There are $n = 18$ rats with complete sets of $k = 8$ landmarks at each age in $m = 2$ dimensions.

PCA in tangent space



Minimal geodesic



SHAPE CURVE FAMILY

- Joint work with Kim Kenobi and Huiling Le (Nottingham).
- Let us consider an extension by introducing another pre-shape w_2 which is orthogonal to w_1 and μ.
- Also, consider a function $t_1(s)$ which gives the position in the direction w_2 for each s.
- The shape curve lifted to the the pre-shape space is then defined as:

$$\Gamma_{(\mu,w_1,w_2;\,t_1)}(s) = \cos\{t_1(s)\}\,\{(\cos s)\mu + (\sin s)\,w_1\} + \sin\{t_1(s)\}\,w_2,$$

QUADRATIC SHAPE CURVE

$$t_1(s) = a_0 + a_1 s + a_2 s^2$$

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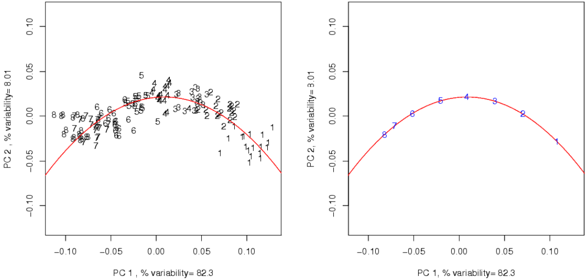


ESTIMATION

Best fitting curve: Minimise

$$F(\underline{a}) = \sum_{i=1}^{g} \sum_{j=1}^{n_i} d^2\{[Z_{ij}], \gamma(\hat{s}_i)\}$$

over the parameters $\underline{a}$, where $\hat{s}_i$ minimises

$$F_{\gamma,i}(s) = \sum_{j=1}^{n_i} d^2\{[Z_{ij}], \gamma(s)\}, \quad i = 1, \ldots, g,$$

and $\gamma()$ is the shape corresponding to pre-shape $\Gamma()$, $[Z]$ is the shape of Z, and $d()$ is a shape distance.

PRACTICALITIES

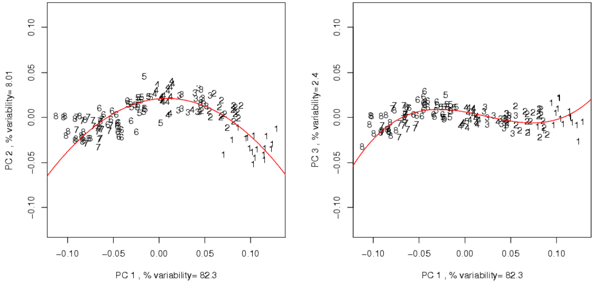
- In practice we often have the shapes of μ, w_1 estimated to be almost identical to the Procrustes mean and the first shape PC.
- The use of the Procrustes mean and shape PCAs gives an excellent approximation in many applications.
- For small s and $t_1(s)$:

$$\Gamma_{(\mu, w_1, w_2; t_1)}(s) \approx \mu + s w_1 + t_1(s) w_2.$$

QUADRATIC-CUBIC SHAPE CURVE

$$t_1(s) = a_0 + a_1 s + a_2 s^2$$
$$t_2(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3$$

$$\Gamma_{(\mu, w_1, w_2, w_3;\, t_1, t_2)}(s) = \cos\{t_2(s)\}\, \Gamma_{(\mu, w_1, w_2;\, t_1)}(s) + \sin\{t_2(s)\}\, w_3,$$



HYPOTHESIS TESTS

In the quadratic-cubic model there are seven free parameters, $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3\}$, which specify the curves.
We set up three hypotheses which express the different relationships.

- H_0: $a_0 = a_1 = \ldots = b_3 = 0$ (Geodesic)
- H_1: At least one of a_0, a_1, a_2 is non-zero and $b_0 = b_1 = b_2 = b_3 = 0$. (Quadratic)
- H_2: At least one of a_0, a_1, a_2 is non-zero and at least one of b_0, b_1, b_2, b_3 is non-zero. (Quadratic-cubic)

LIKELIHOOD RATIO TEST

- Using a complex Watson model

$$f([z]) \propto \exp(\kappa \cos^2 d([z],[\mu])),$$

gives the log-likelihoods under the three models as

$$l_0 = 4766.26, l_1 = 5040.40, l_2 = 5076.82.$$

Thus $-2(l_0 - l_1) = 548.27, -2(l_1 - l_2) = 72.85$.

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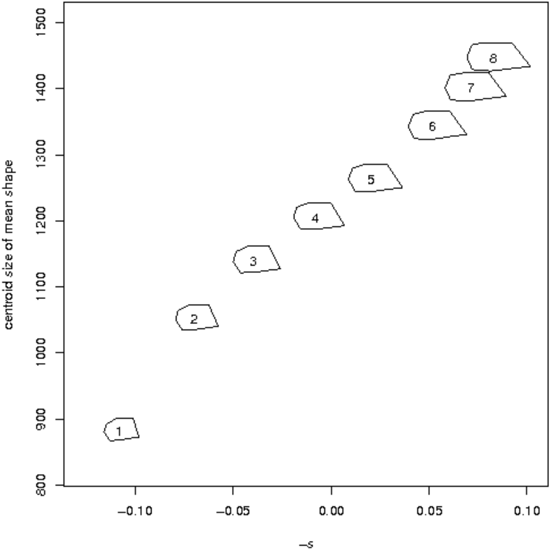
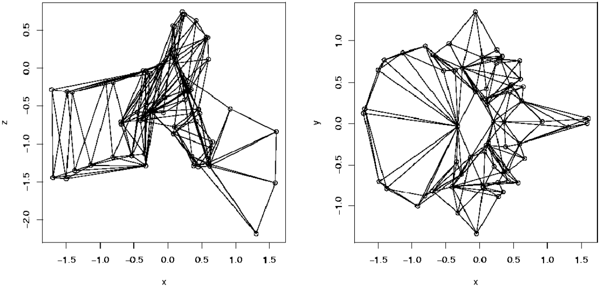
- Comparing these statistics with a χ_3^2 and a χ_4^2 distribution respectively shows that each reduction in the sum of squares is highly significant.
- There is strong evidence against the geodesic and quadratic models in favour of the quadratic-cubic model.

GROWTH ALONG THE QUADRATIC-CUBIC CURVE

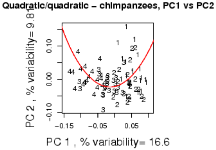


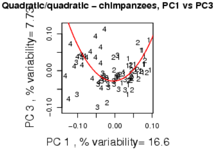
ANOTHER EXAMPLE: HOMINIDS

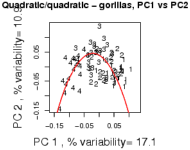


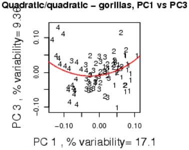
Lumbar vertebra 1-4. Chimpanzee, Gorilla, Human.
$k = 62$ landmarks in $m = 3$ dimensions, $n = 22$ per group.
Data from Paul O'Higgins.

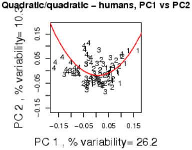
VERTEBRA


Quadratic/quadratic – chimpanzees, PC1 vs PC2
PC 2 , % variability= 9.81
PC 1 , % variability= 16.6
−0.15 −0.05 0.05
−0.10 0.00 0.10


Quadratic/quadratic – chimpanzees, PC1 vs PC3
PC 3 , % variability= 7.73
PC 1 , % variability= 16.6
−0.10 0.00 0.10
−0.10 0.00 0.10


Quadratic/quadratic – gorillas, PC1 vs PC2
PC 2 , % variability= 10.9
PC 1 , % variability= 17.1
−0.15 −0.05 0.05
−0.15 −0.05 0.05


Quadratic/quadratic – gorillas, PC1 vs PC3
PC 3 , % variability= 9.36
PC 1 , % variability= 17.1
−0.10 0.00 0.10
−0.10 0.00 0.10


Quadratic/quadratic – humans, PC1 vs PC2
PC 2 , % variability= 10.3
PC 1 , % variability= 26.2
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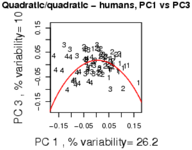
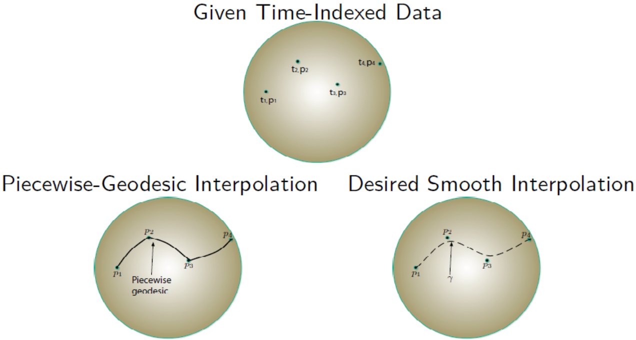
−0.15 −0.05 0.05 0.15


Quadratic/quadratic – humans, PC1 vs PC3
PC 3 , % variability= 10
PC 1 , % variability= 26.2
−0.15 −0.05 0.05 0.15
−0.15 −0.05 0.05 0.15

Outline

SAMSI AOOD program, North Carolina

- Joint work with Jingyong Su, Anuj Srivastava and Eric Klassen (Florida State) and Huiling Le (Nottingham).
- Consider points p_i on manifold M at times t_i, $i = 1, \ldots, n$.



[Credit: Jingyong Su]

Motivating example

Sequence of outlines of a dancer: 100 points located on outline in 2D.

- Translation, rotation and scale invariance.
- Manifold is Kendall's shape space (complex projective space).

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- Manifold is Kendall's shape space (complex projective space).
- Either: interpolate between data points (here shapes in 2D)

- Or: smooth a noisy sequence of shapes

Roughness penalty approach

- Find an optimal path $\hat{\gamma}$ by minimizing

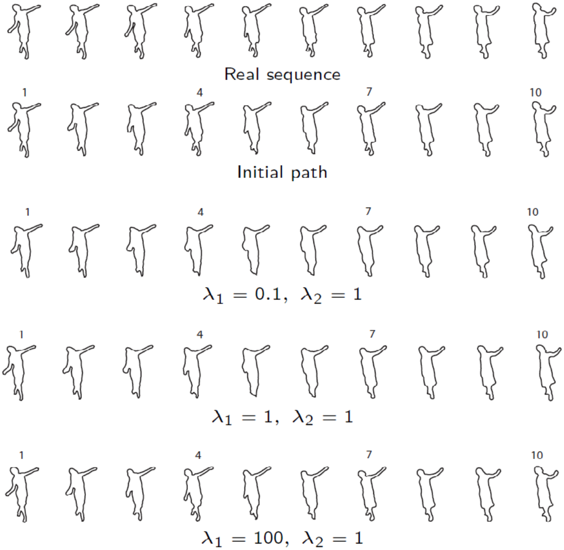
$$\frac{\lambda_1}{2}\sum_{i=1}^{n} d(\gamma(t_i), p_i)^2 + \frac{\lambda_2}{2}\mathcal{R}(\gamma).$$

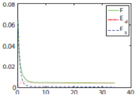
- Example roughness penalty: $\mathcal{R}(\gamma) = \int_0^1 \left\langle \frac{D^2\gamma}{dt^2}, \frac{D^2\gamma}{dt^2} \right\rangle dt$. Objective function: Data term and Smoothing term.

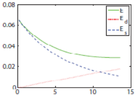
$$S = \lambda_1 E_d + \lambda_2 E_s.$$

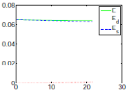
- Using the Palais metric an explicit expression for the gradient is obtained, leading to a practical fitting algorithm.
- Can use cross-validation to choose λ_2/λ_1.

Video dancer



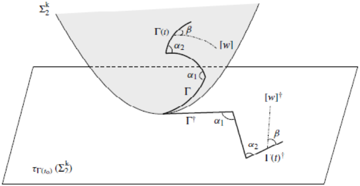






OTHER APPROACHES

- Unrolling and unwrapping splines: Kume et al. (2007)



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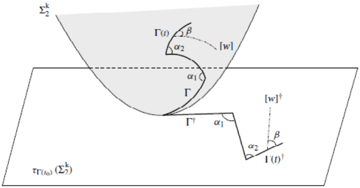

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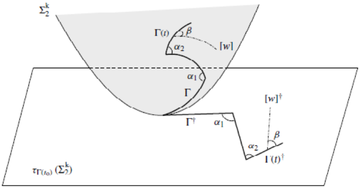

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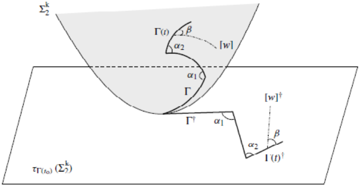

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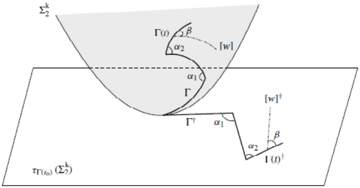

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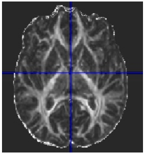
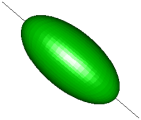
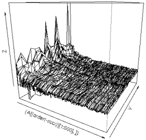
- Tangent space functional curves: Kent et al. (2001)
- Principal nested spheres: Jung et al (2011)
- Local Polynomial Regression: Yuan et al (2011)

Outline

Discussion

- Many other types of non-Euclidean data, high-dimensional, requiring new statistical methodology.

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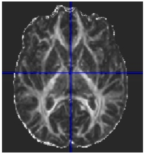 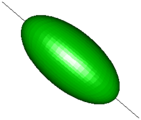 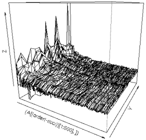

- Statistics/biology/computer science/mathematics/other bridges

Further information

- Kenobi, K., Dryden, I.L. and Le, H. (2010). Shape curves and geodesic modelling *Biometrika*. **97** (3): 567-584.
- J. Su, I. L. Dryden, E. Klassen, H. Le and A. Srivastava (2011). Fitting Smoothing Splines to Time-Indexed, Noisy Points on Nonlinear Manifolds. *Submitted to Journal of Image and Vision Computing*

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Thank you!

Palais Metric

- Samir et al. (2011) used the Palais metric for computing the gradient of the objective function S.
- Let γ be a twice differentiable path in manifold M and let v, w be two smooth vector fields along γ, i.e. $v(t), w(t) \in T_{\gamma(t)}(M)$ for $t \in [0, 1]$. Then, the second-order Palais (1963) metric is:

$$< v(0), w(0) >_{\gamma(0)} + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{dt}(0) \right\rangle_{\gamma(0)} + \int_0^1 \left\langle \frac{D^2 v}{dt^2}, \frac{D^2 w}{dt^2} \right\rangle_{\gamma(t)}$$

- The explicit form of the gradient is useful for in a practical algorithm for fitting the spline.