Natural maps, differentiable rigidity, Ricci and scalar curvature

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joint with L. Bessières, G. Courtois and S. Gallot

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Banff, April 18th, 2011

Outline

Differentiable Rigidity

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- Farrell-Jones : Y, X negatively curved, $n \geq 5$, f homotopy equivalence $\Rightarrow f \sim$ homeomorphisme.

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For $\epsilon > 0$ small, need to take large cover $\tilde{X}$.

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Theorem (B-. Courtois, Gallot)

(X, g_0) closed hyperbolic, Y closed, dominates X, $n \geq 3$. For any metric g on Y,

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Under the same hypothesis,

$$\mathrm{minvol}(Y) = \mathrm{minvol}(X) \Longrightarrow Y \text{ diffeo. to } X$$

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Perelman's works $\rightsquigarrow$ true if $n = 3$.

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- Bishop, $\mathrm{Ricci}(g) \geqslant -(n-1) \Rightarrow h(g) \leqslant n-1 = h(g_0)$.

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Replace f by a "better" map, take a minimizing sequence and study the limit.

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$\forall c > h(g)$, there exists $F_c : Y \to X$, homotopic to f $(n \geqslant 3)$, such that

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- If $\operatorname{Ricci}(g) \geqslant -(n-1)g$, $\operatorname{vol}(X, g_0) \leqslant \operatorname{vol}(Y, g)$.

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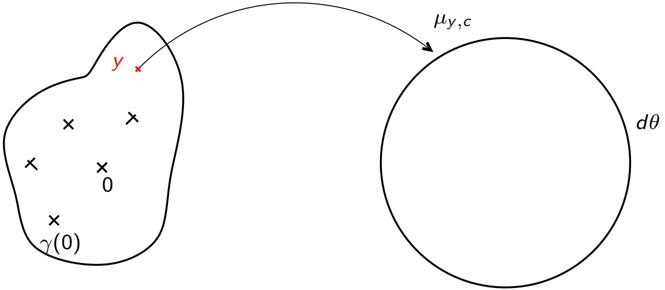

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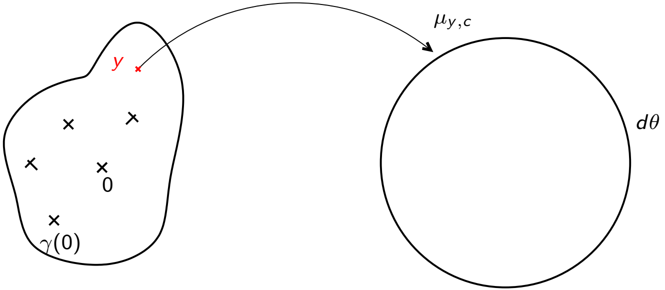

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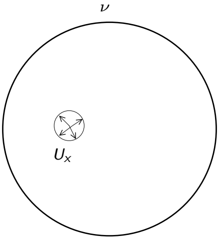

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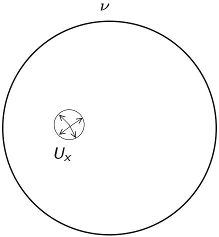

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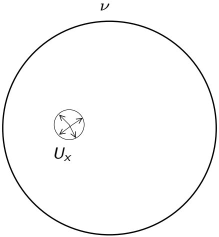

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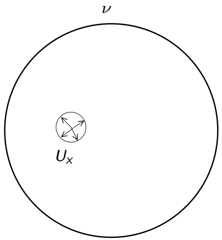

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Rigidity $\rightsquigarrow$ if $|\,\mathrm{Jac}(F(y))| = \left(\frac{h(g)}{h(g_0)}\right)^n$, then $D_y F =$ homothety.

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(Y_∞, d_∞) complete length space.

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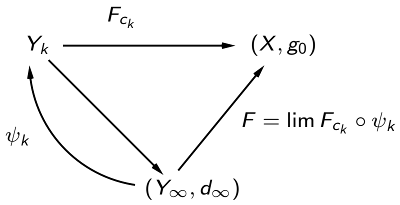

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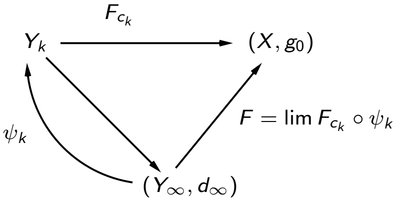

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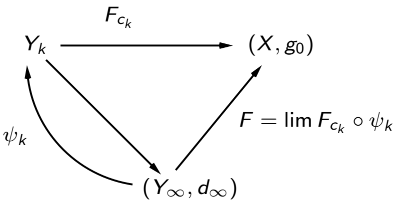

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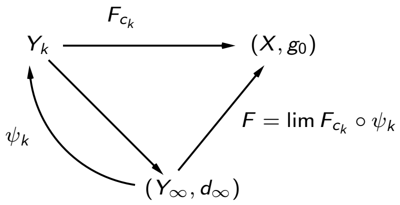

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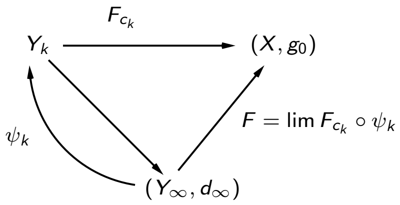

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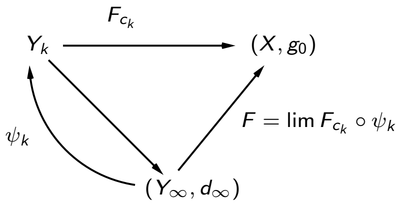

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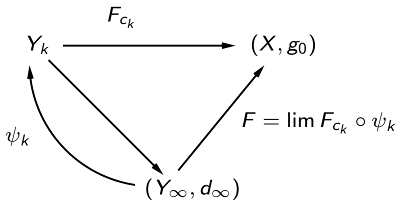

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- F_k almost volume preserving $\Rightarrow$ F is volume preserving.

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This yields the contradiction !

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Outline

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2. Use the Ricci flow and give another proof of Cheeger-Colding.
3. Build a natural map adapted to the scalar curvature question.
4. What about integrals of curvature ?