Solving Ill-posed Algebraic Problems

--- *A Geometric Perspective*

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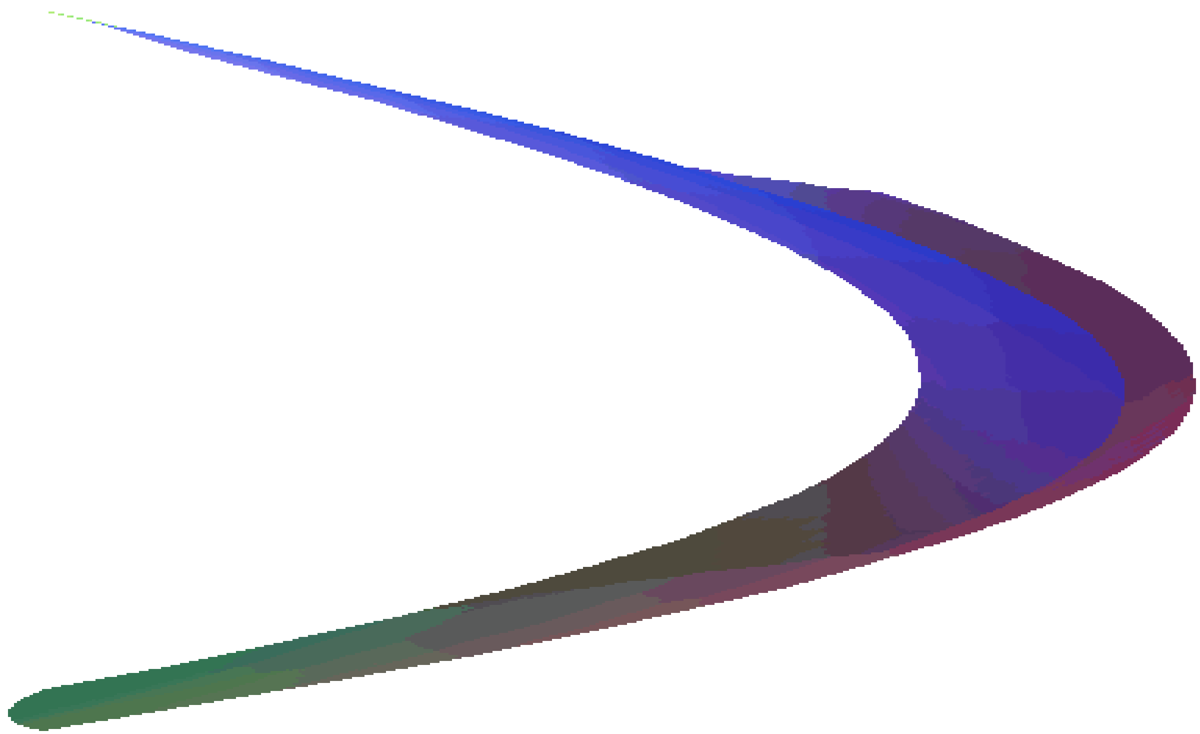


BIRS, March 1, 2010 (supported in part by NSF under Grant DMS-0715137)

Example: Polynomial root/factorization problem:

Exact coefficients

2372413541474339676910695241133745439996376
-21727618192764014977087878553429208549790220
83017972998760481224804578100165918125988254
-175233447692680232287736669617034667590560789
228740383018936986749432151287201460989730173
-194824889329268365617381244488160676107856145
110500081573983216042103084234600451650439725
-41455438401474709440879035174998852213892159
9890516368573661313659709437834514939863439
-1359954781944210276988875203332838814941903
82074319378143992298461706302713313023249

Exact roots

1.072753787571903102973345215911852872073...
0.422344648788787166815198898160900915499...
0.422344648788787166815198898160900915499...
2.603418941910394555618569229522806448999...
2.603418941910394555618569229522806448999 ...
2.603418941910394555618569229522806448999 ...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...

Inexact coefficients

2372413541474339676910695241133745439996376
-21727618192764014977087878553429208549790220
83017972998760481224804578100165918125988254
-175233447692680232287736669617034667590560780 9
228740383018936986749432151287201460989730170 3
-194824889329268365617381244488160676107856140 5
110500081573983216042103084234600451650439720 5
-41455438401474709440879035174998852213892159
9890516368573661313659709437834514939863439
-1359954781944210276988875203332838814941903
82074319378143992298461706302713313023249

"attainable" roots

1.072753787571903102973345215911852872073...
0.422344648788787166815198898160900915499...
0.422344648788787166815198898160900915499...
2.603418941910394555618569229522806448999...
2.603418941910394555618569229522806448999 ...
2.603418941910394555618569229522806448999 ...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...

Coeff. in hardware precision

2372413541474339676910695241133745439996376
-21727618192764014977087878553429208549790220
83017972998760481224804578100165918125988254
-175233447692680232287736669617034667590560789
228740383018936986749432151287201460989730173
-194824889329268365617381244488160676107856145
110500081573983216042103084234600451650439725
-41455438401474709440879035174998852213892159
9890516368573661313659709437834514939863439
-1359954781944210276988875203332838814941903
82074319378143992298461706302713313023249

“attainable” roots

1.072753787571903102973345215911852872073...
0.422344648788787166815198898160900915499...
0.422344648788787166815198898160900915499...
2.603418941910394555618569229522806448999...
2.603418941910394555618569229522806448999 ...
2.603418941910394555618569229522806448999 ...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...
1.710524183747873288503605282346269140403...

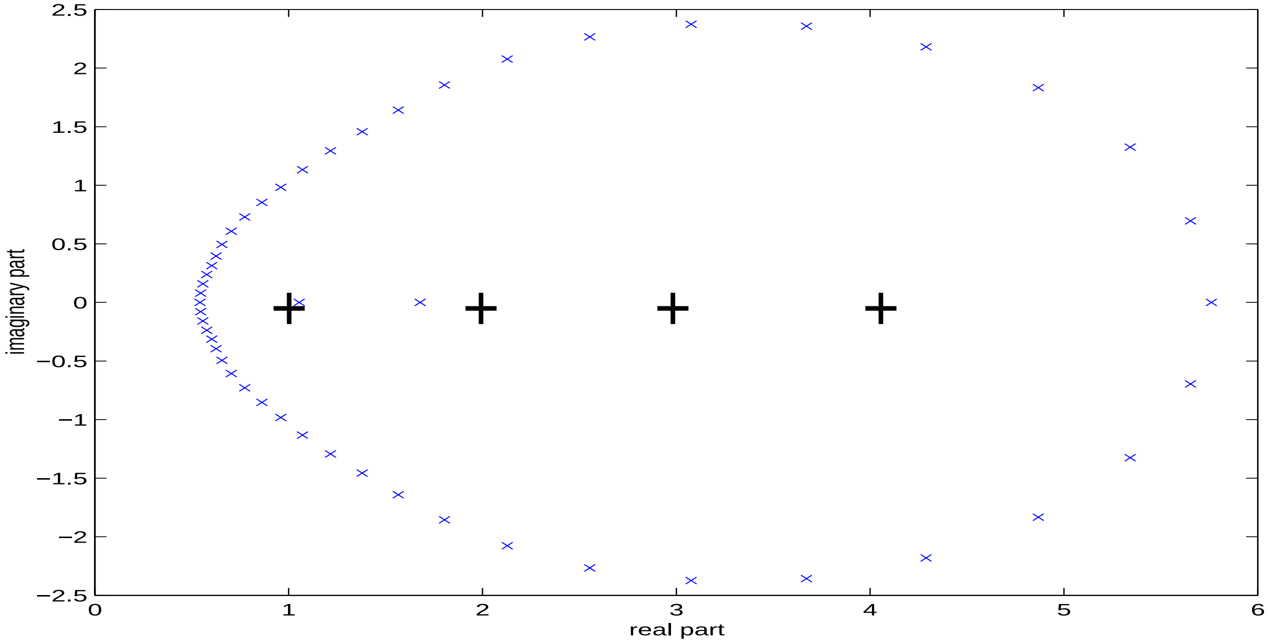
The highest multiplicity is only 4!

For polynomial

$$(x-1)^{20}(x-2)^{15}(x-3)^{10}(x-4)^{5}=0$$

with coefficients in hardware precision:

The computed roots:



E:\ApproximateGCD.mw

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Greatest Common divisor (GCD) of *exact polynomials*

```
> f;
```

$$\frac{513}{217}x^3 z - \frac{127}{311}x^2 z^2 - \frac{1026}{217}x^2 y z + \frac{254}{311}x y z^2 + \frac{1539}{217}x z^2 - \frac{381}{311}z^3$$

```
> g;
```

$$\frac{213}{131}x^2 y z - \frac{59}{77}x^4 - \frac{426}{131}z y^2 x + \frac{118}{77}x^3 y + \frac{639}{131}y z^2 - \frac{177}{77}z x^2$$

```
> gcd(f,g);
```

$$x^2 - 2 y x + 3 z$$

When coefficients become *inexact:*

```
> F := evalf(f);
```

$$F := 2.3640553\, x^3 z - 0.40836013\, x^2 z^2 - 4.7281106\, x^2 y z + 0.81672026\, x y z^2 + 7.0921659\, x z^2 - 1.2250804\, z^3$$

```
> G := evalf(g);
```

$$G := 1.6259542\, x^2 y z - 0.76623377\, x^4 - 3.2519084\, z y^2 x + 1.5324675\, x^3 y + 4.8778626\, y z^2 - 2.2987013\, z x^2$$

```
> gcd(F,G);
```

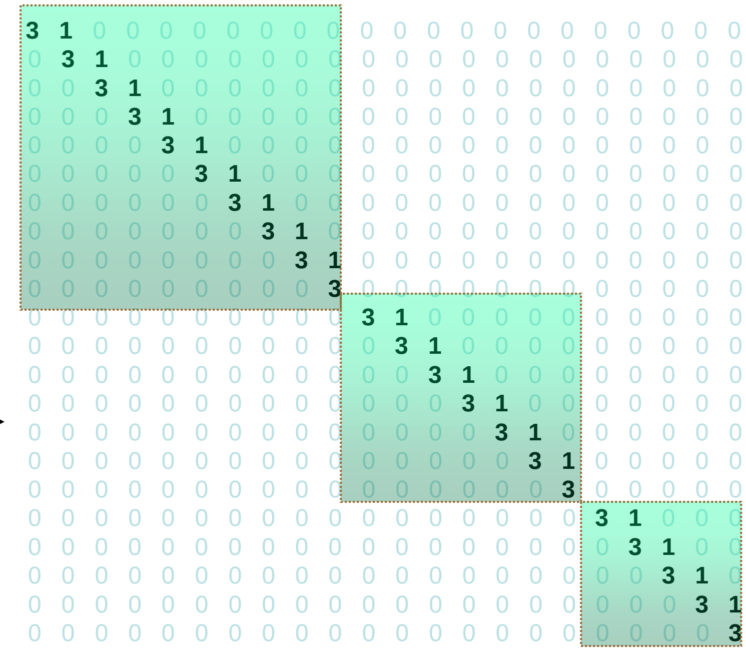
$$1.$$

```
>
```

Jordan Canonical Form (JCF)

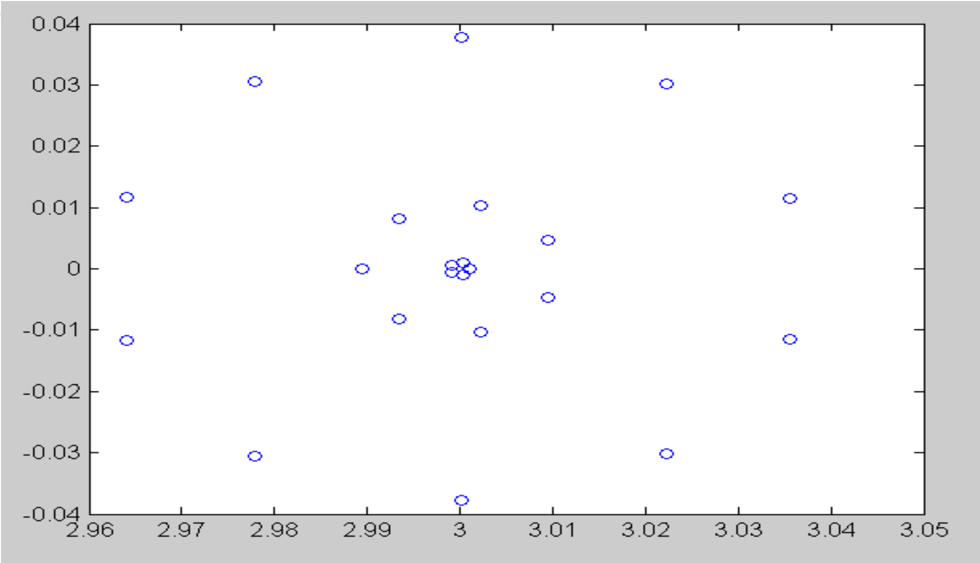
0	10	3	0	-1	-1	-4	0	0	-5	-5	0	1	0	0	-1	0	-5	-1	0	-3	-1
0	5	9	-1	3	-2	-1	1	1	-2	-2	1	-1	1	1	2	-1	-1	1	0	-1	1
3	1	7	2	-2	-11	1	0	6	-4	-3	6	0	5	-1	0	-3	-2	-1	0	0	0
-1	1	5	2	3	1	-1	0	0	0	0	0	-1	0	1	2	0	0	1	0	-1	1
-4	-2	-9	-2	6	19	-2	0	-8	8	6	-8	1	-7	1	-2	4	4	2	0	0	-1
0	-1	1	-1	1	2	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	1	9	-2	4	-3	3	3	1	-2	-2	1	0	1	2	1	-1	-1	1	0	-1	0
1	0	1	0	0	-2	0	3	4	0	0	3	0	2	0	0	-1	0	0	0	0	0
1	-4	-2	0	1	4	1	0	3	5	4	0	-2	0	0	1	0	3	1	0	1	1
-1	1	-2	1	-1	3	-1	-1	-3	3	0	-3	0	-2	-1	0	1	0	0	0	0	0
5	2	6	2	-3	-16	1	0	12	-5	-1	12	0	9	-1	0	-5	-3	-2	0	0	0
-1	4	0	1	-2	-4	-1	0	0	-5	-4	3	4	0	-1	-2	0	-3	-1	0	-1	-2
1	0	1	0	0	-2	0	0	2	0	0	2	3	2	0	0	-1	0	0	0	0	0
0	-1	4	-3	3	-1	1	1	0	0	0	0	-2	3	3	1	0	0	0	0	0	1
0	2	12	-1	2	-7	0	0	2	-4	-3	2	-3	2	4	6	-1	-2	0	0	-1	3
-4	-1	-5	-2	2	12	-1	0	-7	4	3	-7	0	-6	1	3	4	2	1	0	0	0
0	11	8	1	-2	-12	-3	0	6	-9	-8	6	1	5	0	-1	0	-7	-2	0	-3	-1
-2	0	7	-2	5	1	-1	1	-2	0	0	-2	0	-1	1	1	0	4	3	-1	-1	0
3	2	6	2	-2	-7	1	0	2	-5	-4	2	-2	2	0	3	-1	-3	1	2	0	2
5	-12	-10	2	-3	1	5	-1	0	6	6	0	0	0	-2	-1	0	6	0	3	5	0
4	-9	0	1	0	1	4	-1	0	4	4	0	-4	0	0	4	0	4	0	1	6	4
2	0	2	0	0	-3	0	0	3	0	0	3	0	3	0	0	-2	0	0	0	[illegible]	[illegible]

JCF



$$\lambda(A)=\{\quad 3 \quad\}$$

$$\lambda(A+10^{-15}E)$$



Matrix rank problem

```
> A := Matrix([[11/7, 18/7, 15/7, 10/7], [50/21, 64/21, 37/21, 41/21], [19/7, 26/7, 17/7, 16/7], [38/21,
  52/21, 34/21, 32/21], [38/21, 52/21, 34/21, 32/21]]);
```

$$A := \begin{bmatrix} \frac{11}{7} & \frac{18}{7} & \frac{15}{7} & \frac{10}{7} \\ \frac{50}{21} & \frac{64}{21} & \frac{37}{21} & \frac{41}{21} \\ \frac{19}{7} & \frac{26}{7} & \frac{17}{7} & \frac{16}{7} \\ \frac{38}{21} & \frac{52}{21} & \frac{34}{21} & \frac{32}{21} \\ \frac{38}{21} & \frac{52}{21} & \frac{34}{21} & \frac{32}{21} \end{bmatrix}$$

```
> Rank(A), NullSpace(A);
```

$$2, \left\{ \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{4} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ \frac{-7}{4} \\ 1 \\ 0 \end{bmatrix} \right\}$$

```
> B := evalf(A,8);
```

$$B := \begin{bmatrix} 1.5714286 & 2.5714286 & 2.1428571 & 1.4285714 \\ 2.3809524 & 3.0476190 & 1.7619048 & 1.9523810 \\ 2.7142857 & 3.7142857 & 2.4285714 & 2.2857143 \\ 1.8095238 & 2.4761905 & 1.6190476 & 1.5238095 \\ 1.8095238 & 2.4761905 & 1.6190476 & 1.5238095 \end{bmatrix}$$

```
> Rank(B), NullSpace(B);
```

$$4, \{\}$$

Factoring a multivariate polynomial:

```
> f := 6*x^3*y+3*x^2*y+x^2+4*x*y^2+2*y^2+2/3*y;
```

$$f := 6\,x^3\,y + 3\,x^2\,y + x^2 + 4\,x\,y^2 + 2\,y^2 + \frac{2}{3}\,y$$

```
> factor(f);
```

$$\frac{1}{3}\,(3\,x^2 + 2\,y)\,(6\,x\,y + 3\,y + 1)$$

A factorable polynomial → approximation → irreducible

```
> g := evalf(f);
```

$$g := 6.\,x^3\,y + 3.\,x^2\,y + x^2 + 4.\,x\,y^2 + 2.\,y^2 + 0.6666666667\,y$$

```
> factor(g);
```

$$6.000000000\,x^3\,y + 3.000000001\,x^2\,y + 1.000000000\,x^2 + 4.000000001\,x\,y^2 + 2.000000000\,y^2 + 0.6666666666\,y$$

Distorted Cyclic Four system in floating point form:

$$0.7071067810\, z_1 + 0.5773502693\, z_2 + z_3 + z_4 = 0$$

$$2.449489743\, z_1 z_2 z_3 + 3.464101616\, z_2 z_3 z_4 + 4.242640686\, z_3 z_4 z_1 + 2.449489743\, z_4 z_1 z_2 = 0$$

$$0.4082482906\, z_1 z_2 z_3 + 0.5773502693\, z_2 z_3 z_4 + 0.7071067810\, z_3 z_4 z_1 + 0.4082482906\, z_4 z_1 z_2 = 0$$

$$0.4082482906\, z_1 z_2 z_3 z_4 - 1. = 0$$

1-dimensional solution set —approximation→ Isolated solutions

$\{z_1 = -1.414213562 + 0.00001424974386\, I,\ z_4 = -0.9999999999 - 0.00001007609052\, I,\ z_3 = 0.9999999999 + 0.00001007609052\, I,\ z_2 = 1.732050807 - 0.00001745230071\, I\}$

$\{z_1 = 0.00001424974386 - 1.414213562\, I,\ z_3 = 0.00001007609052 + 0.9999999999\, I,\ z_2 = -0.00001745230071 + 1.732050807\, I,\ z_4 = -0.00001007609052 - 0.9999999999\, I\}$

$\{z_4 = 0.00001007609052 - 0.9999999999\, I,\ z_1 = -0.00001424974386 - 1.414213562\, I,\ z_3 = -0.00001007609052 + 0.9999999999\, I,\ z_2 = 0.00001745230071 + 1.732050807\, I\}$

$\{z_4 = 0.9999999999 - 0.00001007609052\, I,\ z_1 = 1.414213562 + 0.00001424974386\, I,\ z_3 = -0.9999999999 + 0.00001007609052\, I,\ z_2 = -1.732050807 - 0.00001745230071\, I\}$

$\{z_1 = 1.414213562 - 0.00001424974386\, I,\ z_2 = -1.732050807 + 0.00001745230071\, I,\ z_4 = 0.9999999999 + 0.00001007609052\, I,\ z_3 = -0.9999999999 - 0.00001007609052\, I\}$

$\{z_2 = 0.00001745230071 - 1.732050807\, I,\ z_4 = 0.00001007609052 + 0.9999999999\, I,\ z_3 = -0.00001007609052 - 0.9999999999\, I,\ z_1 = -0.00001424974386 + 1.414213562\, I\}$

$\{z_3 = 0.00001007609052 - 0.9999999999\, I,\ z_4 = -0.00001007609052 + 0.9999999999\, I,\ z_2 = -0.00001745230071 - 1.732050807\, I,\ z_1 = 0.00001424974386 + 1.414213562\, I\}$

$\{z_2 = 1.732050807 + 0.00001745230071\, I,\ z_4 = -0.9999999999 + 0.00001007609052\, I,\ z_1 = -1.414213562 - 0.00001424974386\, I,\ z_3 = 0.9999999999 - 0.00001007609052\, I\}$

$\{z_4 = 1.,\ z_3 = 1.,\ z_2 = -1.732085712,\ z_1 = -1.414185063\}$

$\{z_4 = 1.,\ z_3 = 1.,\ z_2 = -1.732015903,\ z_1 = -1.414242062\}$

$\{z_1 = 1.414185063,\ z_2 = 1.732085712,\ z_4 = -1.,\ z_3 = -1.\}$

$\{z_2 = 1.732015903,\ z_1 = 1.414242062,\ z_4 = -1.,\ z_3 = -1.\}$

$\{z_2 = -1.732085712\, I,\ z_1 = -1.414185063\, I,\ z_4 = 1.\, I,\ z_3 = 1.\, I\}$

$\{z_2 = -1.732015903\, I,\ z_1 = -1.414242062\, I,\ z_4 = 1.\, I,\ z_3 = 1.\, I\}$

$\{z_2 = 1.732085712\, I,\ z_1 = 1.414185063\, I,\ z_4 = -1.\, I,\ z_3 = -1.\, I\}$

$\{z_2 = 1.732015903\, I,\ z_1 = 1.414242062\, I,\ z_4 = -1.\, I,\ z_3 = -1.\, I\}$

A well-posed problem: (Hadamard, 1923)

the solution satisfies

- existence
- uniqueness
- continuity w.r.t data

An ill-posed problem is *infinitely* sensitive to perturbation

tiny perturbation ➔ huge error

Ill-posed problems are common in applications

- image restoration
- IVP for stiction damped oscillator
- some optimal control problems
- air-sea heat fluxes estimation

... ...

- deconvolution
- inverse heat conduction
- electromagnetic inverse scatering
- the Cauchy prob. for Laplace eq.

Ill-posed problems are common in algebraic computing

- Multiple roots of polynomials

- Polynomial GCD

- Factorization of multivariate polynomials

- The Jordan Canonical Form

- Multiplicity structure/zeros of polynomial systems

- Matrix rank/kernel

- Uncontrollability and unobservability mode/subspace (control theory)

- Gröbner basis

...

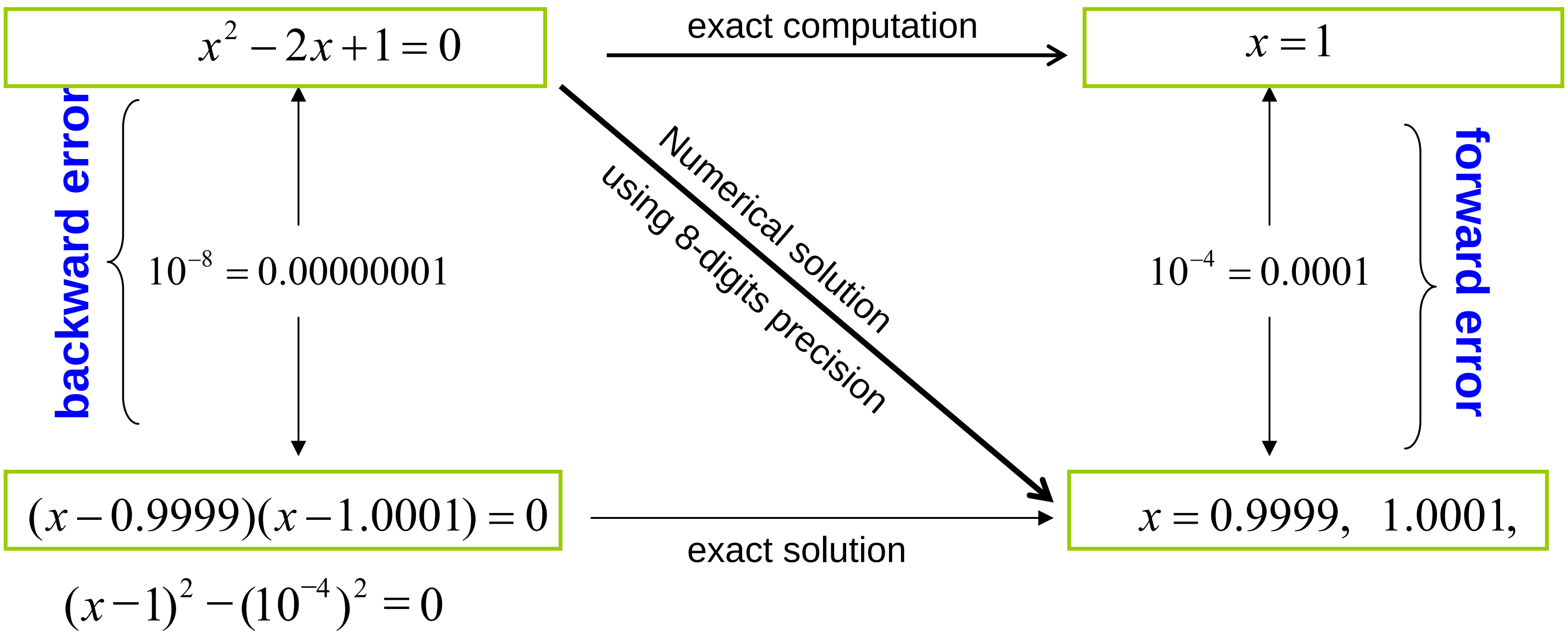
A frontier in scientific computing

Though frequently needed in application, the adequate handling of such ill-posed … problems is hardly ever touched upon in numerical analysis textbooks.

--- Arnold Neumaier, SIAM Review

What is a "numerical solution"?

To solve $x^2 - 2x + 1 = 0$ with 8 digits precision:

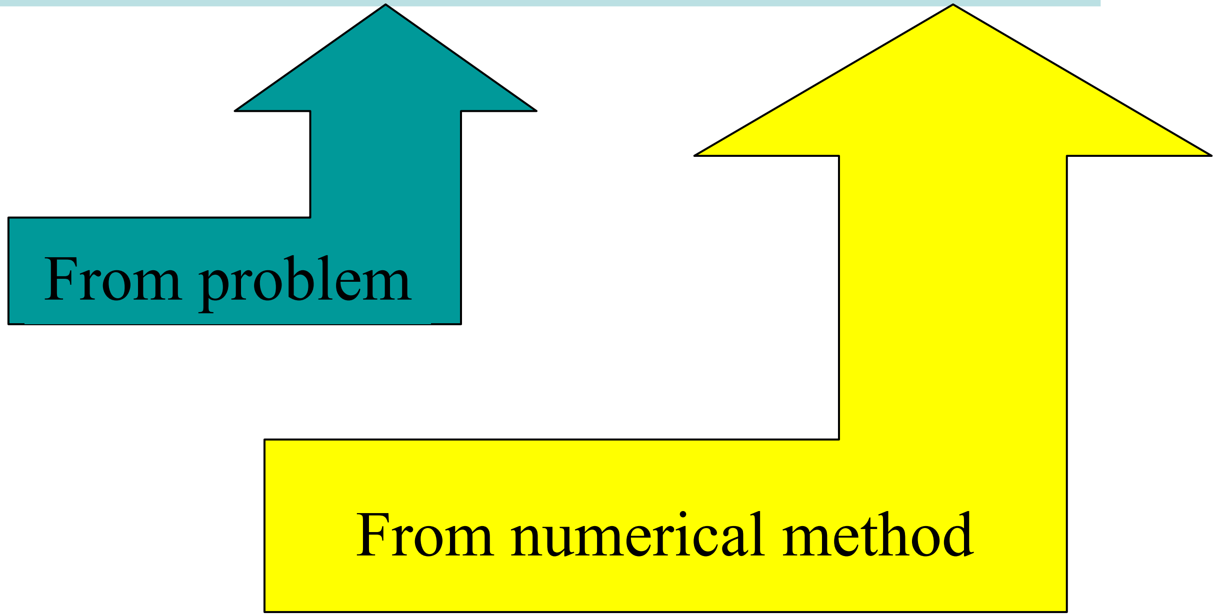


backward error: 0.00000001 -- method is good

forward error: 0.0001 -- problem is bad

The condition number

[Forward error] $\leq$ [Condition number] [Backward error]



A large condition number

<=> The problem is sensitive or, ill-conditioned

An ∞ condition number <=> The problem is ill-posed

Are ill-posed problems solvable in numerical computation?

A numerical algorithm seeks the exact solution of a nearby problem

Ill-posed problems are infinitely sensitive to data perturbation

Conclusion: Ill-posed problems are intractable in numerical computation

On difficulties of computing JCF:

C. Moler and **C. Van Loan**, SIAM Review, 2003: *… [T]he JCF cannot be computed using floating point arithmetic. A single rounding error may cause some multiple eigenvalue to become distinct or vise versa, altering the entire structure …*

S. Barnett and **R. Cameron**, Introduction to Mathematical Control Theory, 1985: *It should be noted that although the Jordan form is of fundamental theoretical importance it is of little use in practical computation, being generally very difficult to compute.*

J. Demmel, Applied Numerical Linear Algebra, 1997: *The Jordan form tells everything we want to know about a matrix … But it is bad to compute the Jordan form for two numerical reasons: First reason: It is discontinuous… Second reason: it can not be computed stably in general.*

G.W. Stewart, Matrix Algorithms vol II, 1998: *[T]he (Jordan) form is virtually uncomputable. Perturbations in the matrix send the eigenvalues flying… [A]ttempts to compute the Jordan canonical form of a matrix have not been very successful…*

R.A. Horm & C.R. Johnson, Matrix Analysis, 1990: *There is no hope of computing such an object in a stable way, os the Jordan canonical form is little used in numerical applications*

Are ill-posed problems solvable in numerical computation?

A numerical algorithm seeks the exact solution of a nearby problem

Ill-posed problems are infinitely sensitive to data perturbation

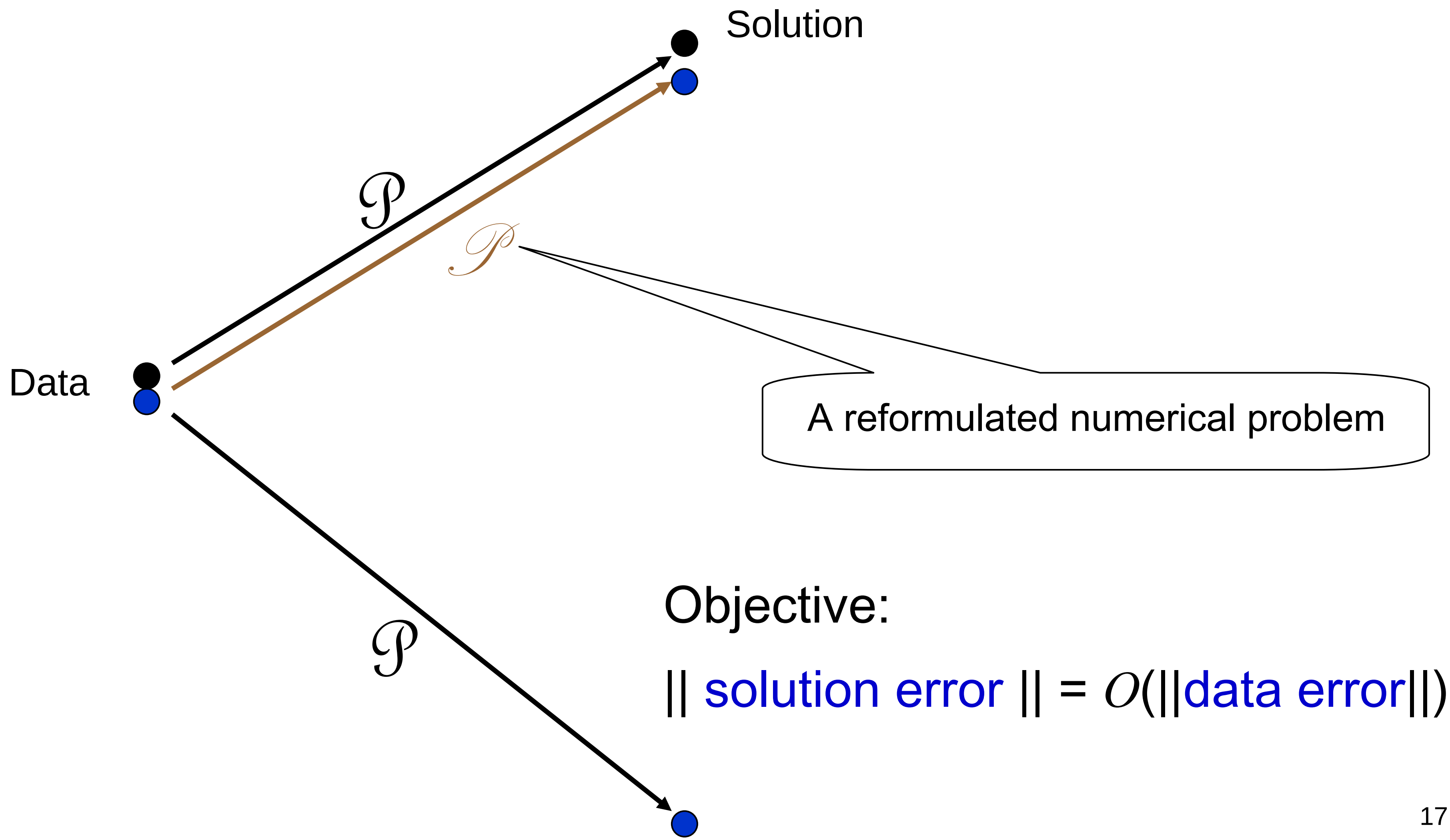
Conclusion: Ill-posed problems are intractable in numerical computation

What to do: Fix the problem (i.e. regularization)

Challenge in solving ill-posed problems:

Can we recover the lost solution
when the problem is inexact?

$\mathcal{P} : \mathcal{D}ata \rightarrow \mathcal{S}olution$



Objective:

|| solution error || = $\mathcal{O}$(||data error||)

What's coming up:

- The geometry: Why problems are ill-posed, why they are solvable

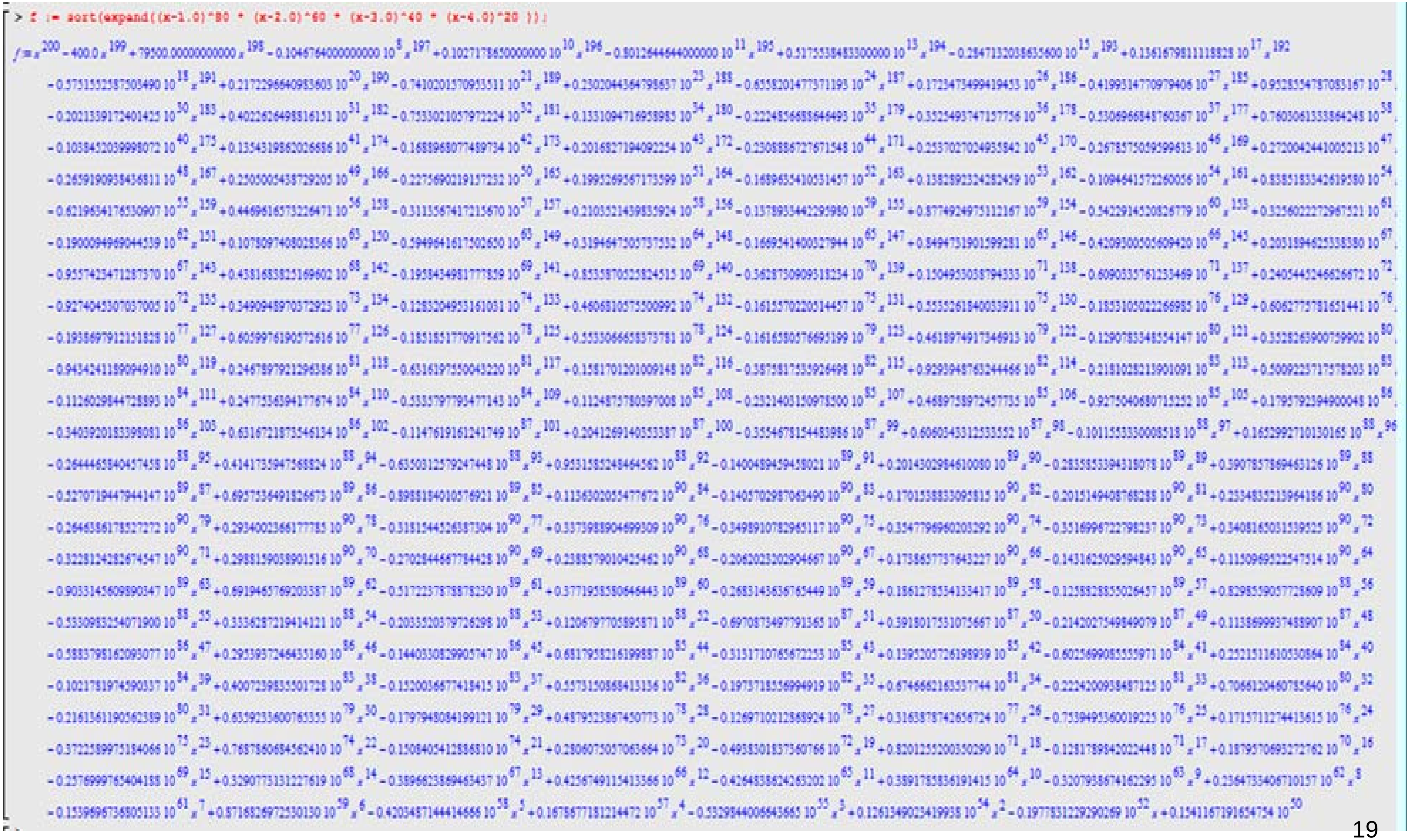
- The regularization principle: How to reformulate a numerical problem

- The well-posedness theorem: For the reformulated numerical problem
  if the data is sufficiently accurate, then the solution satisfies
  -- existence
  -- uniqueness
  -- Lipschitz continuity w.r.t. data
  -- |solution error| = O(|data error|)

- The two-staged strategy: Solve the regularized numerical problem
  via two optimizations

Sample result: For polynomial $(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$

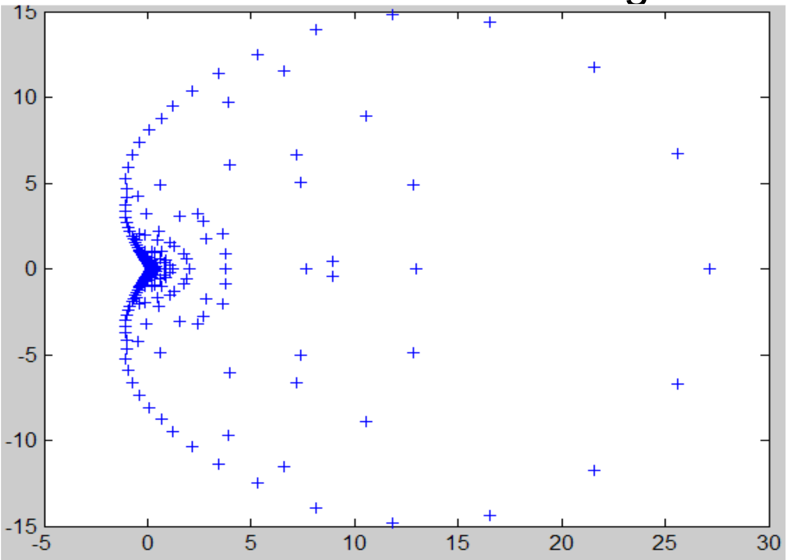
with (inexact) coefficients in hardware precision



Sample result: For polynomial $(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$

with (inexact) coefficients in hardware precision

Conventional root-finding



Numerical factorization:

```
>> [F,res,fcnd]  =  uvFactor(f,1e-10,1);

THE CONDITION NUMBER:                           914.329
THE BACKWARD ERROR:                      5.71e-015
THE ESTIMATED FORWARD ROOT ERROR:        1.04e-011

  FACTORS

   ( x -      3.999999999999990 )^20
   ( x -      3.000000000000008 )^40
   ( x -      1.999999999999998 )^60
   ( x -      1.000000000000000 )^80
```

Z. Zeng, '03, '04, '05, '09

[It is] the most efficient and reliable algorithm for [numerical gcd]

Hans J. Stetter, *Numerical Polynomial Algebra*

[The algorithm] accurately calculates polynomial roots of high multiplicity without using multiprecision arithmetic (as usually required) even if the coefficients are inexact. This is the first work to do that, and is a remarkable achievement .

J.M McNamee, *Numerical Methods for Roots of Polynomials, Part I*

Case study: Polynomial factorization (simplified)

$(x - t)^3 = -t^3 + (3t^2)\,x + (-3t)\,x^2 + x^3$

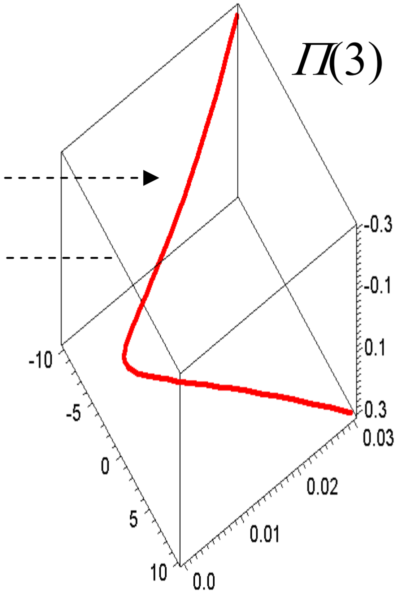
$$F(t) = \begin{bmatrix} -t^3 \\ 3t^2 \\ -3t \end{bmatrix}$$

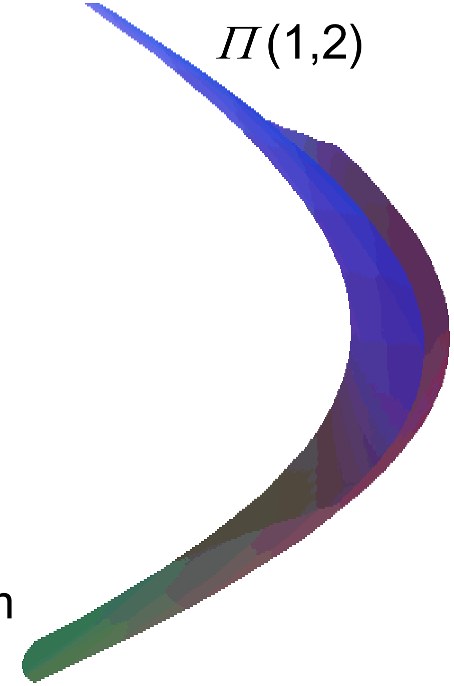
t

F

F^{-1}

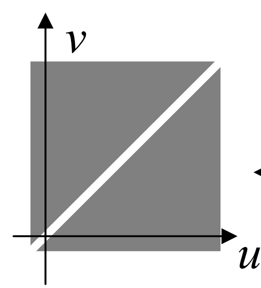
diffeomorphism





$(x - u)^1 (x - v)^2 = -uv^2 + (v^2 + 2uv)\,x + (-2v - u)\,x^2 + x^3$

$$G(u,v) = \begin{bmatrix} -uv^2 \\ v^2 + 2uv \\ -2v - u \end{bmatrix}$$



G

G^{-1}

diffeomorphism

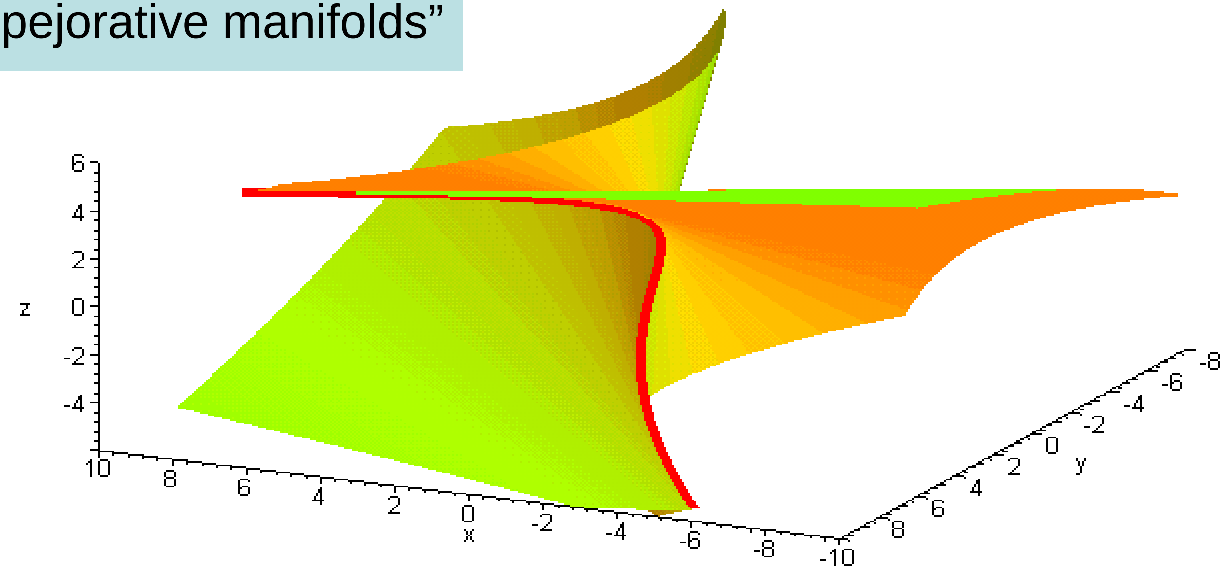
Polynomails form (factorization) manifolds

Are ill-posed problems really sensitive?

Kahan: It is a misconception.

W. Kahan's observation (1972)

- Problems form a "pejorative manifolds"



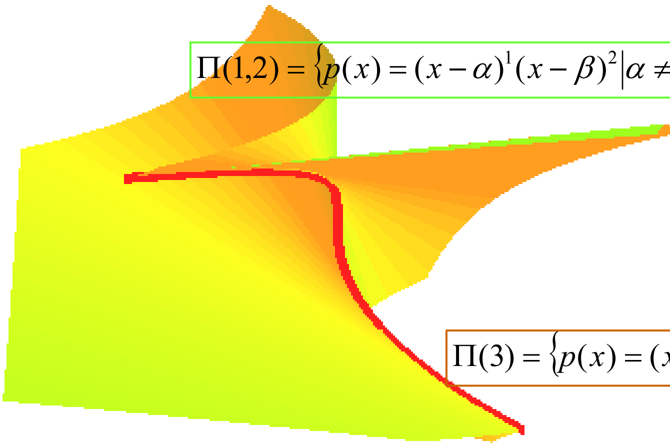
Plot of pejorative manifolds of degree 3 polynomials with multiple roots

- Ill-posedness: a tiny perturbation pushes the problem out of the manifold

- A problem is <u>not</u> sensitive at all if it stays on the manifold.

Stratification of factorization manifolds of degree 3 polynomials

$\Pi(1,1,1) = \{ p(x) = (x-\alpha)^1 (x-\beta)^1 (x-\gamma)^1 \mid \alpha \neq \beta \neq \gamma \}$



$$\overline{\Pi(3)} \subset \overline{\Pi(1,2)} \subset \overline{\Pi(1,1,1)} = C^3$$

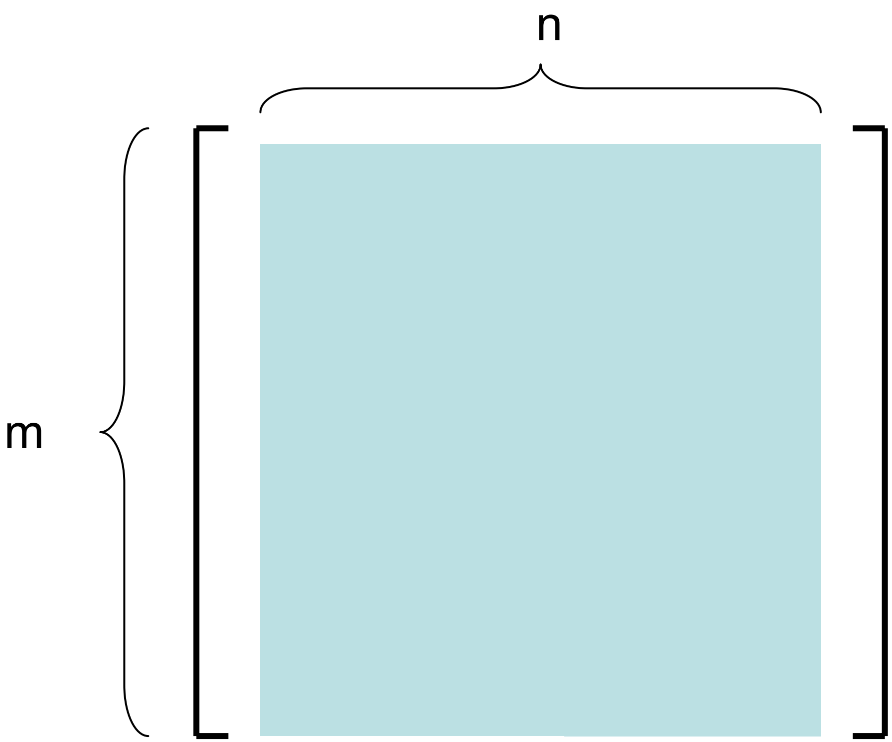
Codimensions: 2 1 0

Factorization manifold stratification of degree 4 polynomials:

$$\overline{\Pi(4)} \subset \overline{\Pi(2,2)}, \quad \overline{\Pi(4)} \subset \overline{\Pi(1,3)}, \quad \overline{\Pi(2,2)} \subset \overline{\Pi(1,1,2)}, \quad \overline{\Pi(1,3)} \subset \overline{\Pi(1,1,2)} \subset \overline{\Pi(1,1,1,1)} = C^4$$

Codimensions: 3 2 1 0

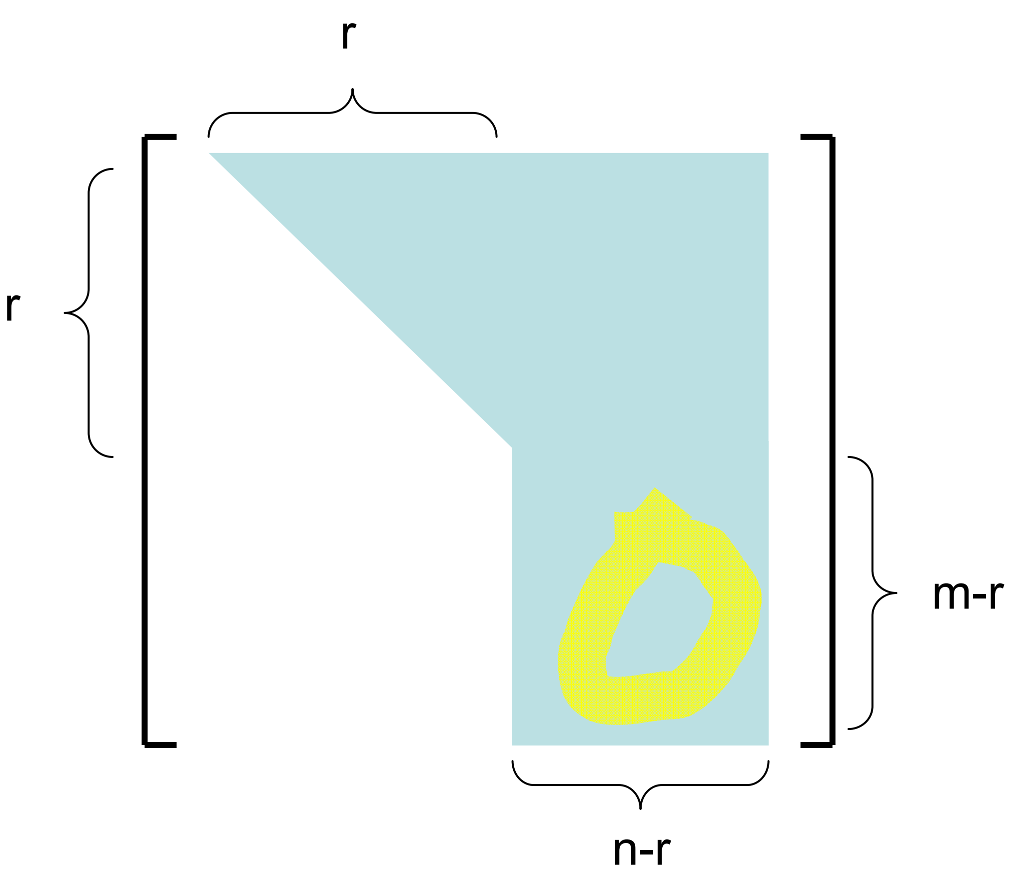
Manifold mxn matrices or rank r:



$$M_r^{m\times n}$$

= { all mxn matrices of rank r }

Manifold mxn matrices or rank r:



$$M_r^{m\times n}$$

$$= \{ \text{all } mxn \text{ matrices of rank } r \}$$

$$\text{codim}(M_r^{m\times n}) = (m-r)(n-r)$$

$$\overline{M_0^{m\times n}} \subset \overline{M_1^{m\times n}} \subset \overline{M_2^{m\times n}} \subset \cdots \subset \overline{M_n^{m\times n}}$$

Polynomial GCD manifold $P_k^{m,n} = \{(p,q) \mid \deg(GCD(p,q)) = k\}$

$$\left.\begin{aligned} p(x) &= p_0 + p_1 x + p_2 x^2 + \cdots + p_m x^m \\ q(x) &= q_0 + q_1 x + q_2 x^2 + \cdots + q_n x^n \end{aligned}\right\} \in \; C^{(m+1)+(n+1)}$$

deg(GCD(p,q)) = k

$$p(x) = (u_0 + u_1 x + \cdots + u_k x^k)(v_0 + v_1 x + \cdots + v_{m-k} x^{m-k})$$

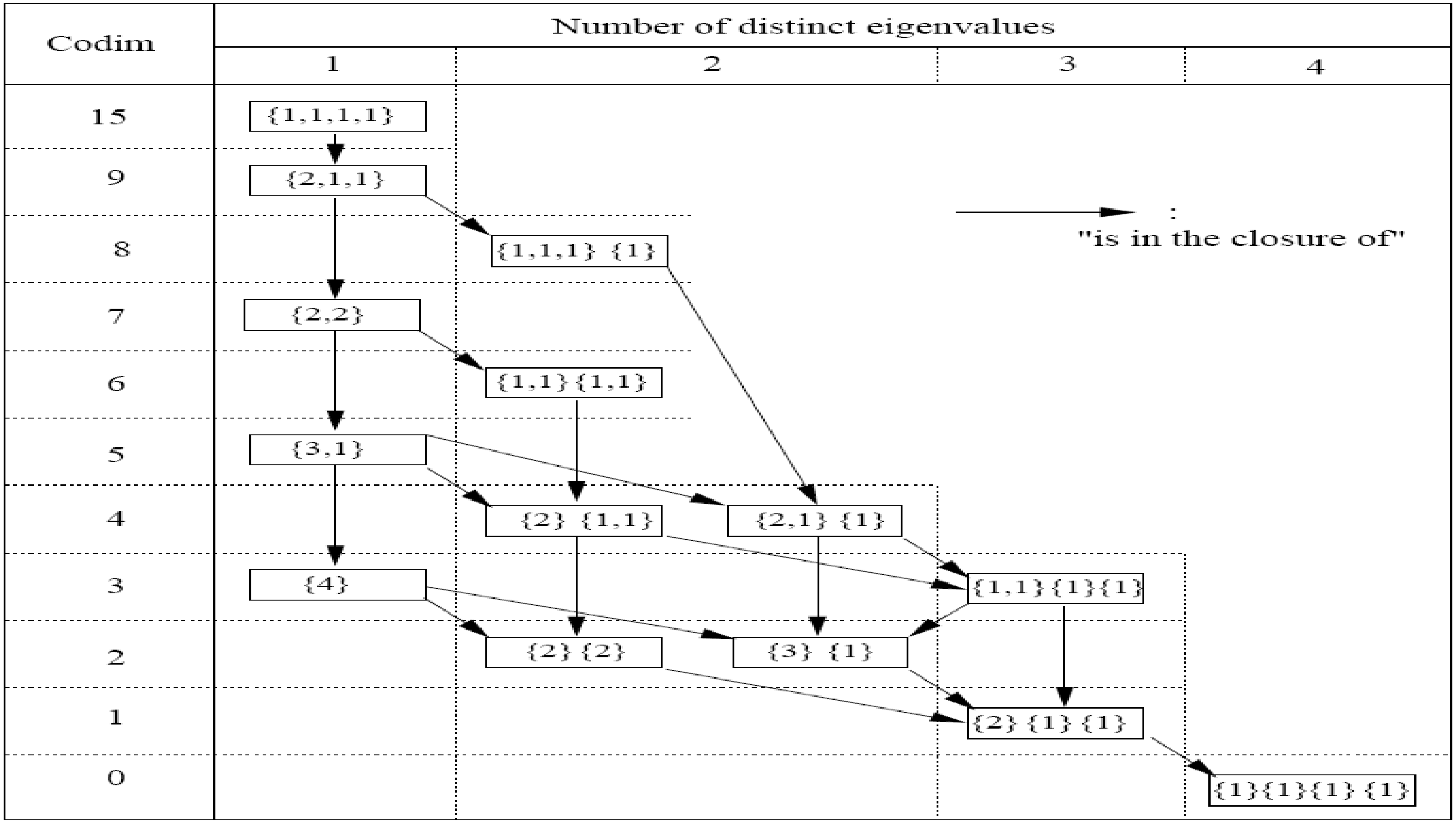
$$q(x) = (u_0 + u_1 x + \cdots + u_k x^k)(w_0 + w_1 x + \cdots + w_{n-k} x^{n-k})$$

$$\gamma_0 u_0 + \gamma_1 u_1 + \cdots + \gamma_k u_k = 1$$

$$\begin{aligned} \mathrm{codim}\left(P_k^{m\times n}\right) &= (m+1)+(n+1)-\left[(k+1)+(m-k+1)+(n-k+1)-1\right] \\ &= k \end{aligned}$$

$$\overline{P_n^{m\times n}} \subset \overline{P_{n-1}^{m\times n}} \subset \cdots \subset \overline{P_1^{m\times n}} \subset \overline{P_0^{m\times n}}$$

Manifolds of 4x4 matrices defined by Jordan structures

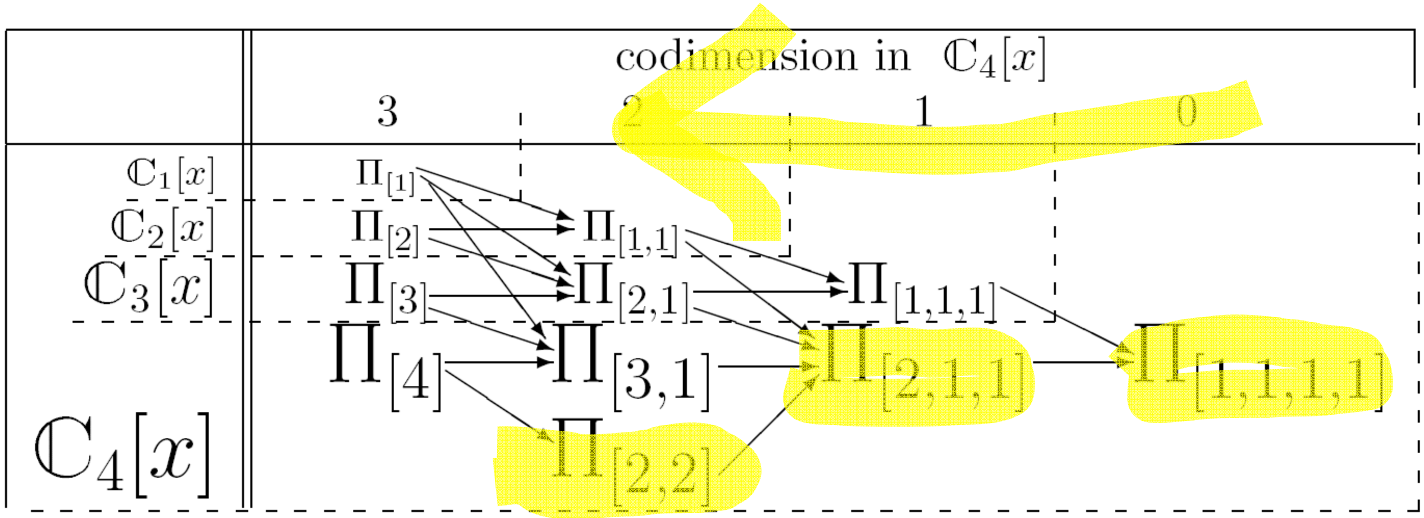


e.g. {2,1} {1} is the structure of 2 eigenvalues in 3 Jordan blocks of sizes 2, 1 and 1

Factorization manifolds and their stratification (Zeng, 2009)

$$\Pi_{[k_1 k_2 \cdots k_n]} = \left\{ a_0 (a_1 x + b_1)^{k_1} (a_2 x + b_2)^{k_2} \cdots (a_n x + b_n)^{k_n} \;\middle|\; a_i, b_i \in C, a_i b_j \neq a_j b_i, \forall i \neq j \right\}$$

$$\subset \quad C_m[x] = \left\{ c_0 + c_1 x + \cdots + c_m x^m \;\middle|\; c_i \in C \right\}$$

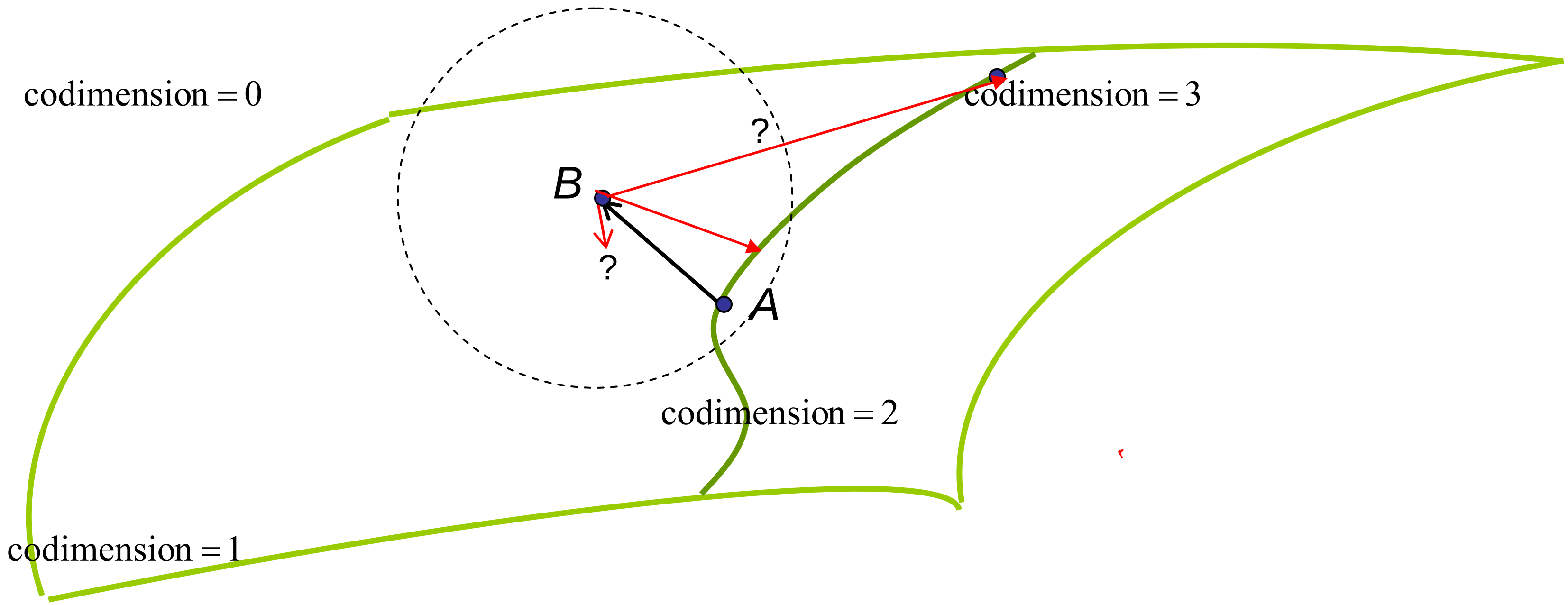


$p(x) \in \Pi_{[2,2]} \quad \rightleftarrows \quad \mathrm{dist}(p, \Pi_{[2,2]}) = \mathrm{dist}(p, \Pi_{[2,1,1]}) = \mathrm{dist}(p, \Pi_{[1,1,1,1]}) = 0$

Theorem: $p(x) \in \Pi_{[k_1 \ldots k_n]}$ if and only if

$$\mathrm{codim}(\Pi_{[k_1 \ldots k_n]}) = \max \{ \mathrm{codim}(\Pi) \mid \mathrm{dist}(p, \Pi) = 0 \}$$

Illustration of ill-posedness manifolds

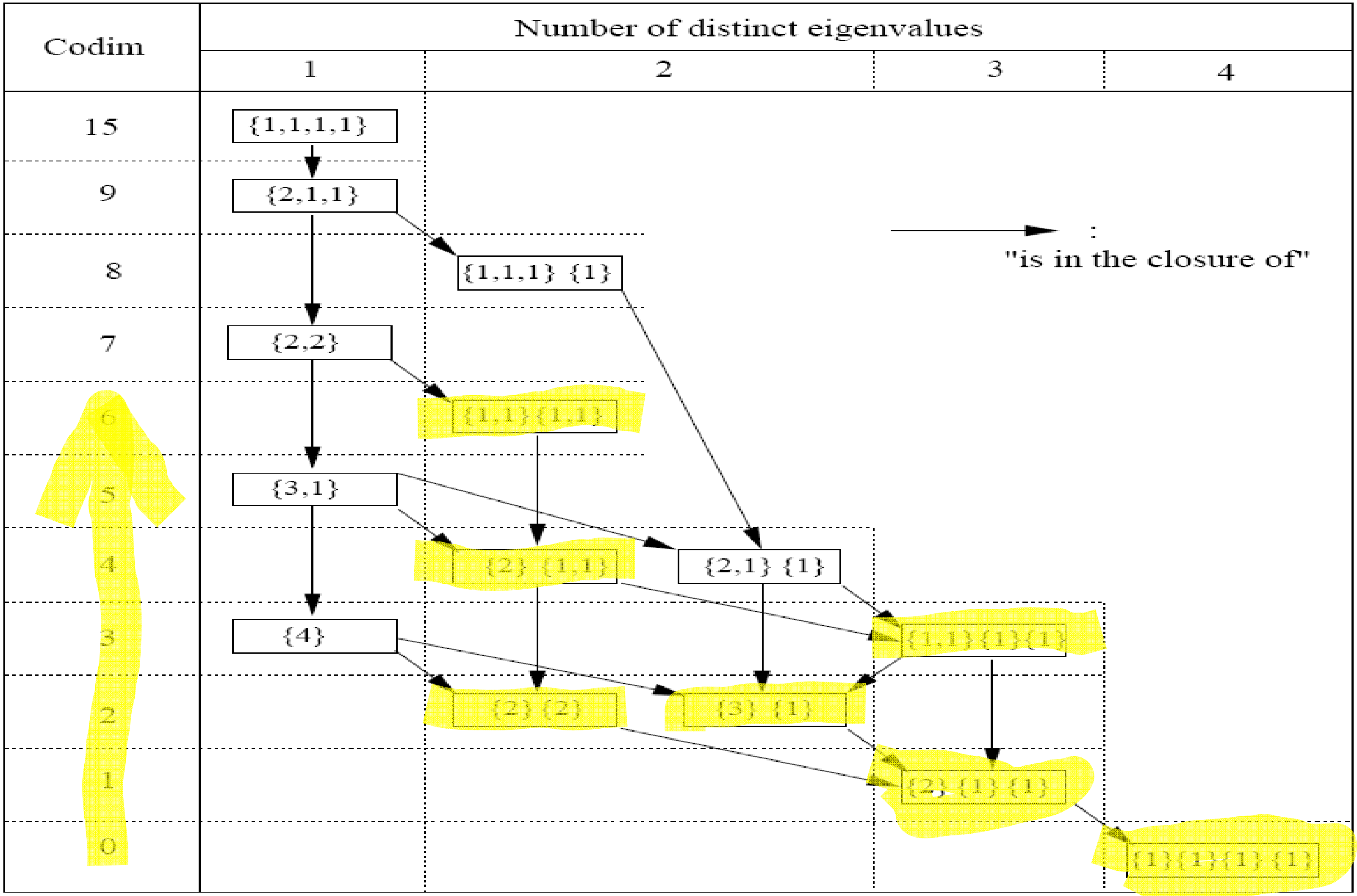


Problem A → perturbation → Problem B

The "nearest" manifold may not be the answer

The right manifold is of highest codimension within a certain distance

Manifolds of 4x4 matrices defined by Jordan structures



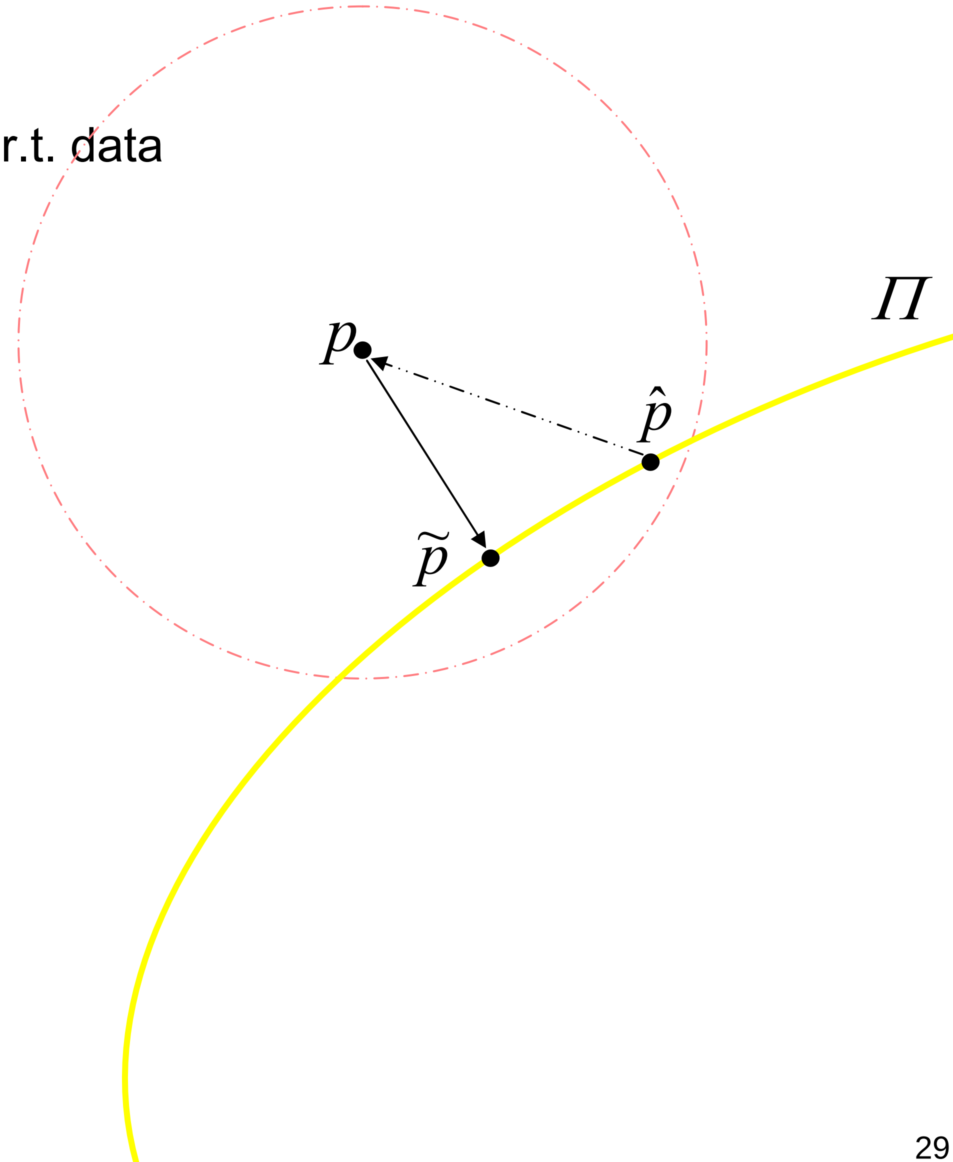
Ask the right question on polynomial factorization

I.e. Formulate a well-posed factorization problem, whose solution

- exists,
- is unique, and
- is Lipschitz continuous w.r.t. data

The approximate factorization of p is

- the exact factorization of $\tilde{p}$
- $\tilde{p}$ lies in the nearby manifold Π of the highest codimension
- $\tilde{p}$ is the nearest polynomial on Π from p



A “three-strikes” principle for formulating
a “numerical solution” to an ill-posed problem:

- **Backward nearness**: The numerical solution is the exact solution of a nearby problem
- **Maximum codimension**: The numerical solution is the exact solution of a problem on the nearby pejorative manifold of the highest codimension.
- **Minimum distance**: The numerical solution is the exact solution of the nearest problem on the nearby pejorative manifold of the highest codimension.

➢ Finding numerical solution becomes a well-posed problem

➢ Numerical solution is a generalization of exact solution.

Formulation of the numerical rank/kernel:

$$\forall\ A \in C^{m\times n} \ \text{ and } \ \forall\ \theta > 0$$

The numerical rank of *A* within θ:

$$rank_\theta(A) = \min_{\|B-A\|\le\theta} rank(B)$$

Backward nearness: num. rank of *A* is the exact rank of certain matrix *B* within θ.

Maximum codimension: That matrix *B* is on the rank manifold Π possessing the highest co-dimension and intersecting the θ–neighborhood of *A*.

The numerical kernel of *A* within θ:

$$Ker_\theta(A) = Ker(B) \quad \text{with}$$

$$\|B - A\|_2 = \min_{rank(C)=rank_\theta(A)} \|C - A\|_2$$

Minimum distance: That *B* is the nearest matrix on the rank manifold Π.

- An exact rank/kernel is the numerical rank/kernel within a small θ.
- Numerical rank/kernel is well-posed

$$Rank \begin{bmatrix} 12 & 17 & 11 & 7 & 9 & 15 \\ 6 & 6 & 9 & 0 & 3 & 6 \\ 13 & 15 & 11 & 10 & 8 & 14 \\ 18 & 13 & 19 & 13 & 6 & 14 \\ 14 & 7 & 11 & 11 & 7 & 11 \\ 19 & 10 & 15 & 13 & 11 & 16 \\ 14 & 3 & 13 & 11 & 4 & 8 \\ 19 & 6 & 17 & 13 & 8 & 13 \\ 10 & 7 & 12 & 7 & 2 & 7 \\ 6 & 7 & 4 & 7 & 3 & 6 \end{bmatrix} = 4$$

nullity = 2

$$+ \varepsilon E = 6$$

nullity = 0

$$\begin{bmatrix} -79 \\ -29 \\ 32 \\ 27 \\ 0 \\ 60 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ 4 \\ 3 \\ 4 \\ 0 \end{bmatrix} \xrightarrow[\text{basis}]{\text{kernel}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After reformulating the rank:

$$Rank_\theta \begin{bmatrix} 12 & 17 & 11 & 7 & 9 & 15 \\ 6 & 6 & 9 & 0 & 3 & 6 \\ 13 & 15 & 11 & 10 & 8 & 14 \\ 18 & 13 & 19 & 13 & 6 & 14 \\ 14 & 7 & 11 & 11 & 7 & 11 \\ 19 & 10 & 15 & 13 & 11 & 16 \\ 14 & 3 & 13 & 11 & 4 & 8 \\ 19 & 6 & 17 & 13 & 8 & 13 \\ 10 & 7 & 12 & 7 & 2 & 7 \\ 6 & 7 & 4 & 7 & 3 & 6 \end{bmatrix} = 4$$

$nullity_\theta = 2$

$$+ \varepsilon E = 4$$

$nullity_\theta = 2$

$$0 \le \varepsilon < \theta < 4.98$$

$$dist(Ker(A) - Ker_\theta(A + \varepsilon E)) < \frac{61.26}{1-\varepsilon}\varepsilon$$

Ill-posedness is removed successfully.

Numerical rank/kernel can be computed by SVD and other rank-revealing algorithms (e.g. Li-Zeng, Lee-Li-Zeng, SIMAX, 2005, 2009)

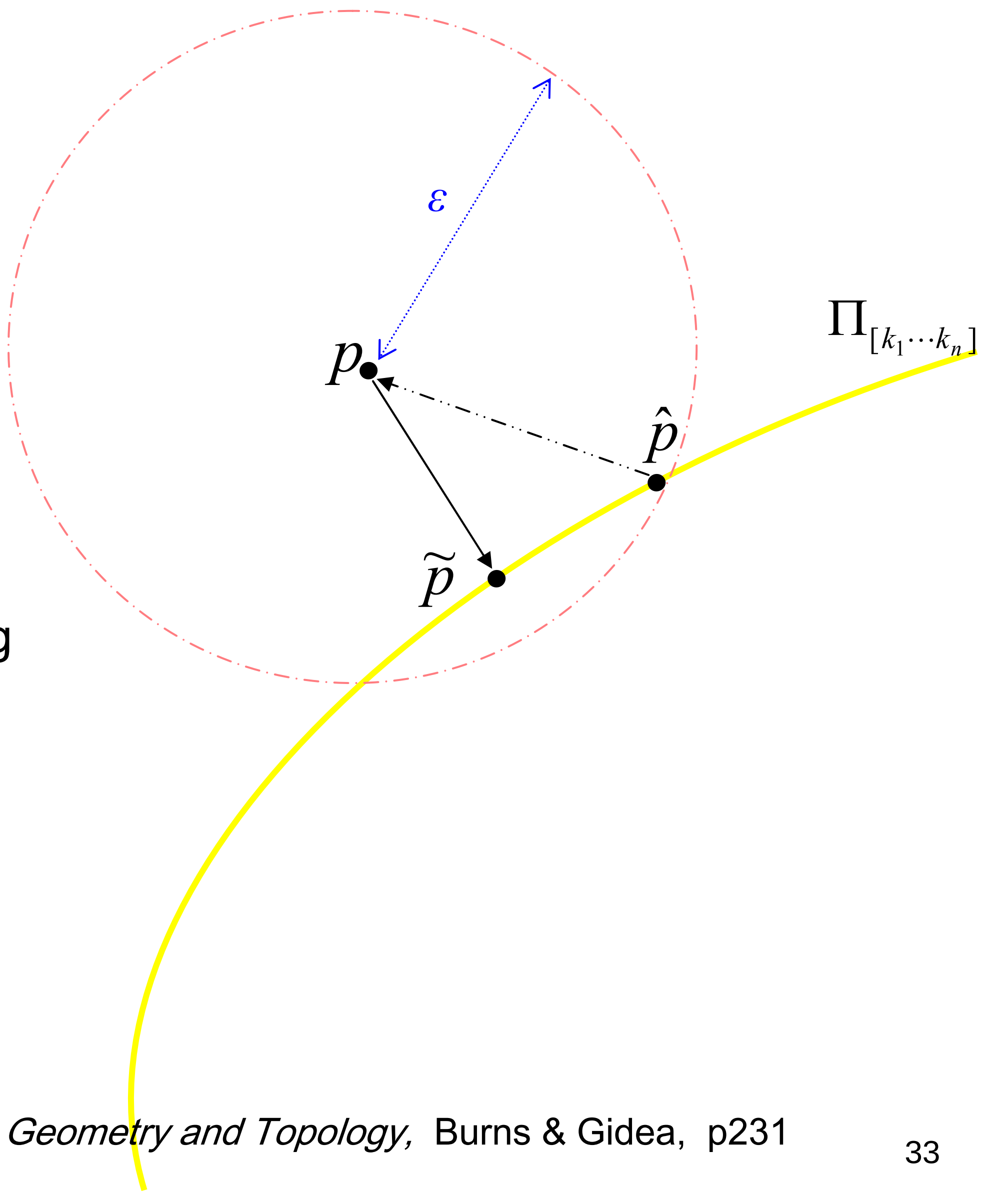
The Well-posedness Theorem of the Numerical Factorization (Zeng, 2009)

$$f(x) = a_0(a_1x+b_1)^{k_1}(a_2x+b_2)^{k_2}\cdots(a_nx+b_n)^{k_n}$$

Numerical factorization:
- exists
- is unique, and
- is Lipschitz continuous

Moreover:
- accurately approximates the underlying exact factorization

if the data is sufficiently accurate



c.f. Tubular Neighborhood Theorem, *Differential Geometry and Topology,* Burns & Gidea, p231 exact algebraic problem —regularization→ numerical problem

rank/kernel ----------------> numerical rank/kernel

root-finding ----------------> numerical factorization

GCD ----------------> numerical GCD

JCF ----------------> numerical staircase form

...

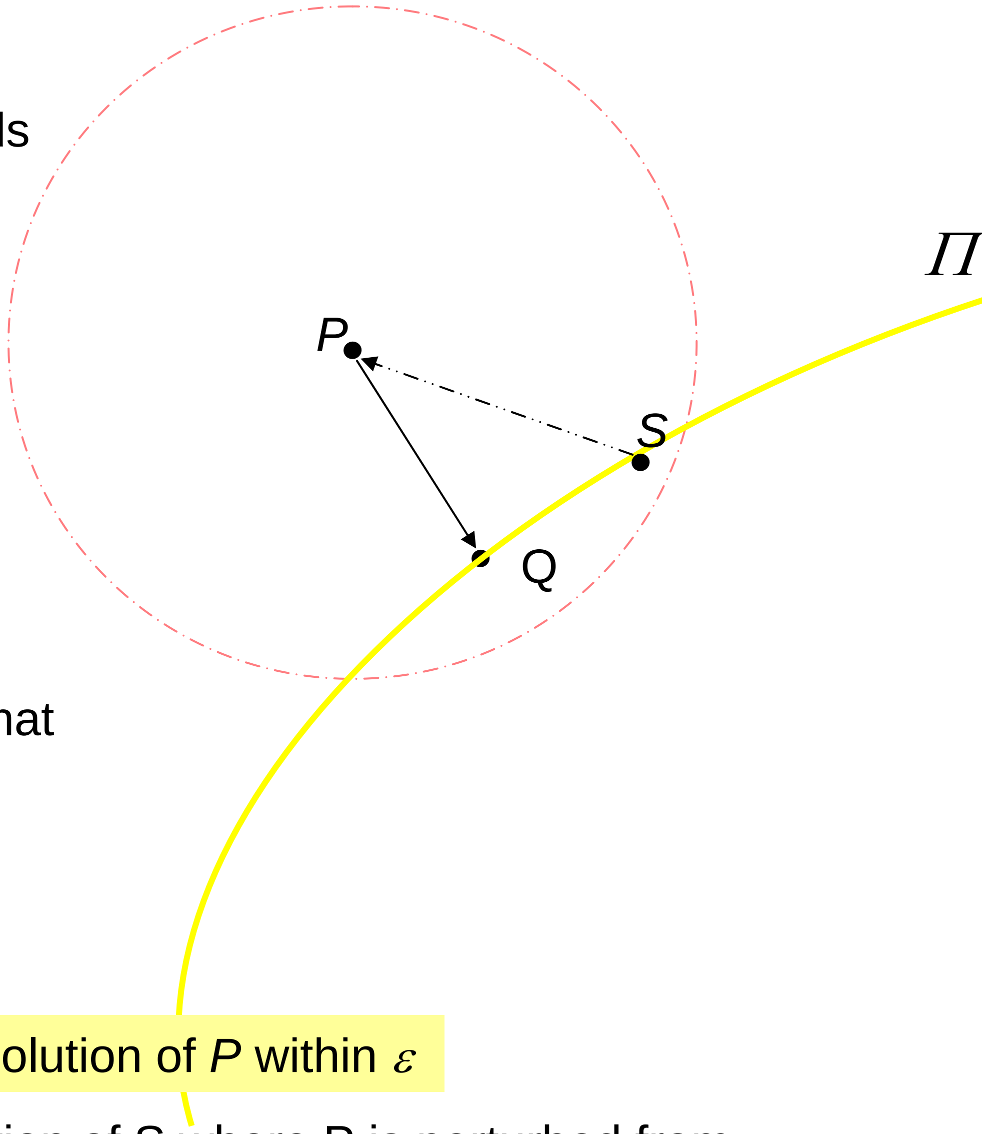
Ill-posed problem ⟶ Well posed problem

The two-staged algorithm

after formulating the numerical solution to problem *P* within ε

Stage I: Among all pejorative manifolds satisfy $dist(P, \Pi) < \varepsilon$

$$\text{Maximize } \operatorname{codim}(\Pi)$$



Stage II: Find/solve problem Q such that

$$\|P - Q\| = \min_{R \in \Pi} \|P - R\|$$

Exact solution of *Q* is the numerical solution of *P* within ε

which approximates the solution of S where P is perturbed from

How to identify the maximum codimension manifold?

Answer: Matrix computations

How to reach the minimum distance to the manifold?

Answer: Gauss-Newton iteration

GCD problem: $f = u \cdot v, \qquad g = u \cdot w$

$$[g, \ -f]\begin{bmatrix} x^j v \\ x^j w \end{bmatrix} = x^j \ [(u \cdot w)v - (u \cdot v)w] = \quad 0$$

The linear transformation

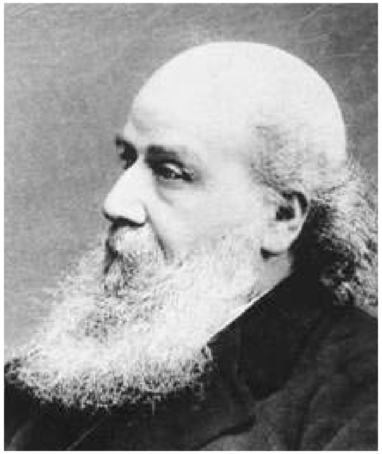
$$L : \begin{bmatrix} p \\ q \end{bmatrix} \rightarrow [g, \ -f]\begin{bmatrix} p \\ q \end{bmatrix}$$

On the vector space

$$\left\{ \begin{bmatrix} p \\ q \end{bmatrix} : \begin{matrix} \deg(p) < \deg(f) \\ \deg(q) < \deg(g) \end{matrix} \right\}$$

Has the kernel $span\left\{ x^0 \begin{bmatrix} v \\ w \end{bmatrix}, \ x^1 \begin{bmatrix} v \\ w \end{bmatrix}, x^2 \begin{bmatrix} v \\ w \end{bmatrix}, \cdots, x^{\deg(u)-1} \begin{bmatrix} v \\ w \end{bmatrix} \right\}$

Linear transformation $L \Rightarrow$ Sylverster matrix $S(f,g)$



James J. Sylvester

Numerical rank-deficiency = degree of the approx. GCD

Case study: univariate factorization: $\forall\, f \in C[x],\ \forall\, \varepsilon > 0,\ \deg(f) = n$

Stage I: Find the max-codimension pejorative manifold by applying univariate numerical GCD algorithm on (f, f')

$$\because\ f(x) \approx (x - z_1)^{m_1} \cdots (x - z_k)^{m_k}$$
$$\Rightarrow\ f'(x) \approx (x - z_1)^{m_1 - 1} \cdots (x - z_k)^{m_k - 1} q(x)$$
$$\Rightarrow\ NGCD(f, f') \approx (x - z_1)^{m_1 - 1} \cdots (x - z_k)^{m_k - 1}$$

Stage II: solve the (overdetermined) polynomial system $F(z_1, \ldots, z_k) = f$

$$(\bullet - z_1)^{m_1} \cdots (\bullet - z_k)^{m_k} = f(\bullet)$$

(in the form of coefficient vectors)

for a least squares solution $(z_1, \ldots, z_k)$ by Gauss-Newton iteration

(key theorem: The Jacobian is injective.)

Multivariate factorization structure: Matrix computations!

$\forall\ f \in \mathbb{C}[x,y]$ of bidegree $[m,n]$ $$\frac{\partial}{\partial y}\left(\frac{f_x}{f}\right) = \frac{\partial}{\partial x}\left(\frac{f_y}{f}\right)$$

Assume $f = f_1 f_2 f_3$ with distinct factors $f_1,\ f_2,$ and f_3

$$\frac{\partial}{\partial y}\frac{\mathbf{f_1} f_{2x}\, \mathbf{f_3}}{\mathbf{f_1}\ f_2\ \mathbf{f_3}} = \frac{\partial}{\partial x}\frac{\mathbf{f_1} f_{2y}\, \mathbf{f_3}}{\mathbf{f_1}\ f_2\ \mathbf{f_3}} \qquad \frac{\partial}{\partial y}\left(\frac{f_1 \cdot f_{2x} \cdot f_3}{f}\right) = \frac{\partial}{\partial x}\left(\frac{f_1 \cdot f_{2y} \cdot f_3}{f}\right)$$

The equation $$\frac{\partial}{\partial y}\frac{g}{f} = \frac{\partial}{\partial x}\frac{h}{f}$$ has three solutions

$$(g,h) = \left(f_{1x} f_2 f_3,\quad f_{1y} f_2 f_3\right), \qquad \left(f_1 f_{2x} f_3,\quad f_1 f_{2y} f_3\right) \qquad \left(f_1 f_2 f_{3x},\quad f_1 f_2 f_{3y}\right)$$

# of factors = # of solutions to $$\frac{\partial}{\partial y}\frac{g}{f} = \frac{\partial}{\partial x}\frac{h}{f}$$

Irreducibility condition (Ruppert '99, and Gao '03, Kaltofen-May '03,Gao-Kaltofen-May-Yang-Zhi'04)

A squarefree polynomial $f \in \mathbb{C}[x,y]$ of bidegree $[m,n]$ has k distince factors

$\Longleftrightarrow$ the homogeneous linear equation

$$f^2\left[\frac{\partial}{\partial y}\left(\frac{g}{f}\right)-\frac{\partial}{\partial x}\left(\frac{h}{f}\right)\right] = 0$$

has k linearly independent solutions (g,h) of bidegrees

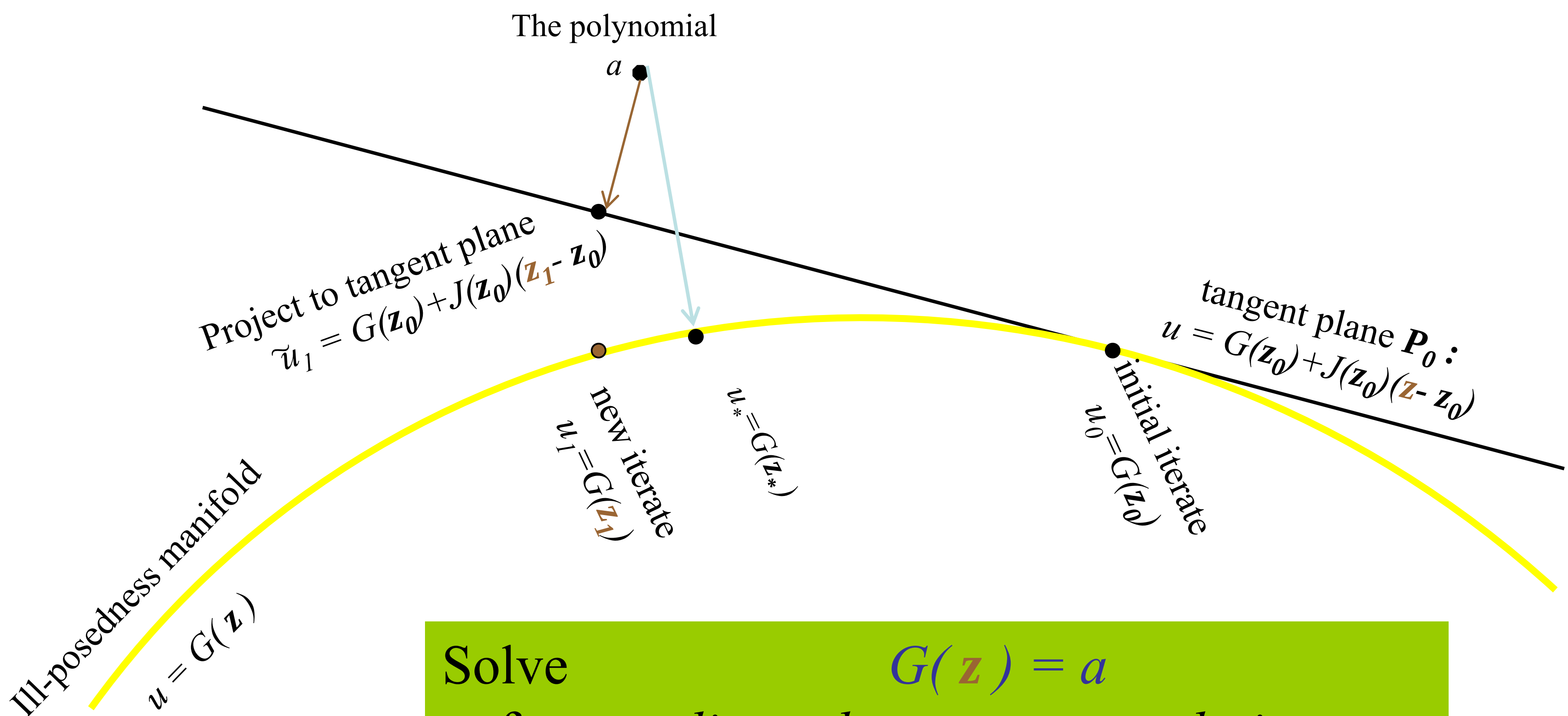
$$\deg(g) \le [m\text{-}1,n], \qquad \deg(h) \le [m,n\text{-}1].$$

$$L_f : (g,h) \rightarrow f^2\left[\frac{\partial}{\partial y}\frac{g}{f}-\frac{\partial}{\partial x}\frac{g}{f}\right]$$

is a linear transformation corresponding to a matrix R_f

Rank-deficiency = # of irreducible factors

Geometry of the Gauss-Newton iteration:



Solve $G(\mathbf{z}) = a$
for <u>*nonlinear least squares solution*</u> $\mathbf{z}=\mathbf{z}_*$

Solve $G(\mathbf{z}_0)+J(\mathbf{z}_0)(\mathbf{z} - \mathbf{z}_0) = a$
for <u>*linear least squares solution*</u> $\mathbf{z} = \mathbf{z}_1$

$$G(\mathbf{z}_0)+J(\mathbf{z}_0)(\mathbf{z} - \mathbf{z}_0) = a$$

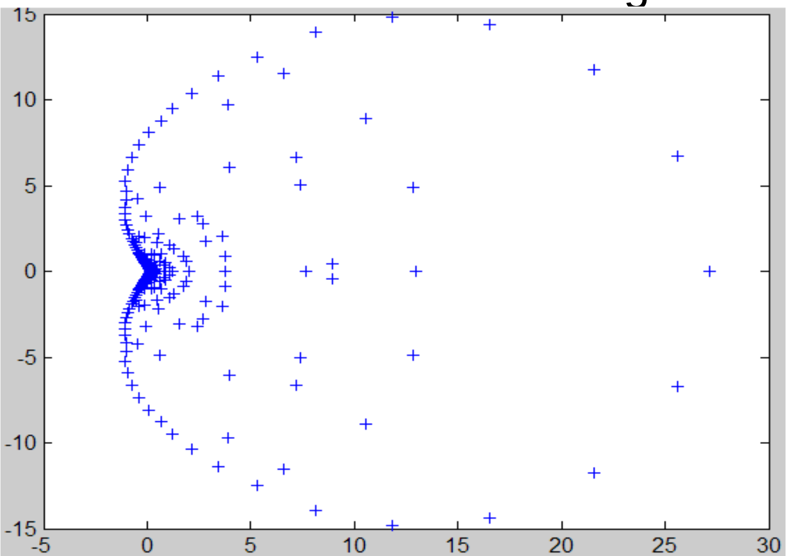
$$J(\mathbf{z}_0)(\mathbf{z} - \mathbf{z}_0) = - [G(\mathbf{z}_0) - a]$$

$$\mathbf{z}_1 = \mathbf{z}_0 - [J(\mathbf{z}_0)^+] [G(\mathbf{z}_0) - a]$$

Example: For polynomial $(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$

with (inexact) coefficients in hardware precision

Conventional root-finding



Numerical factorization:

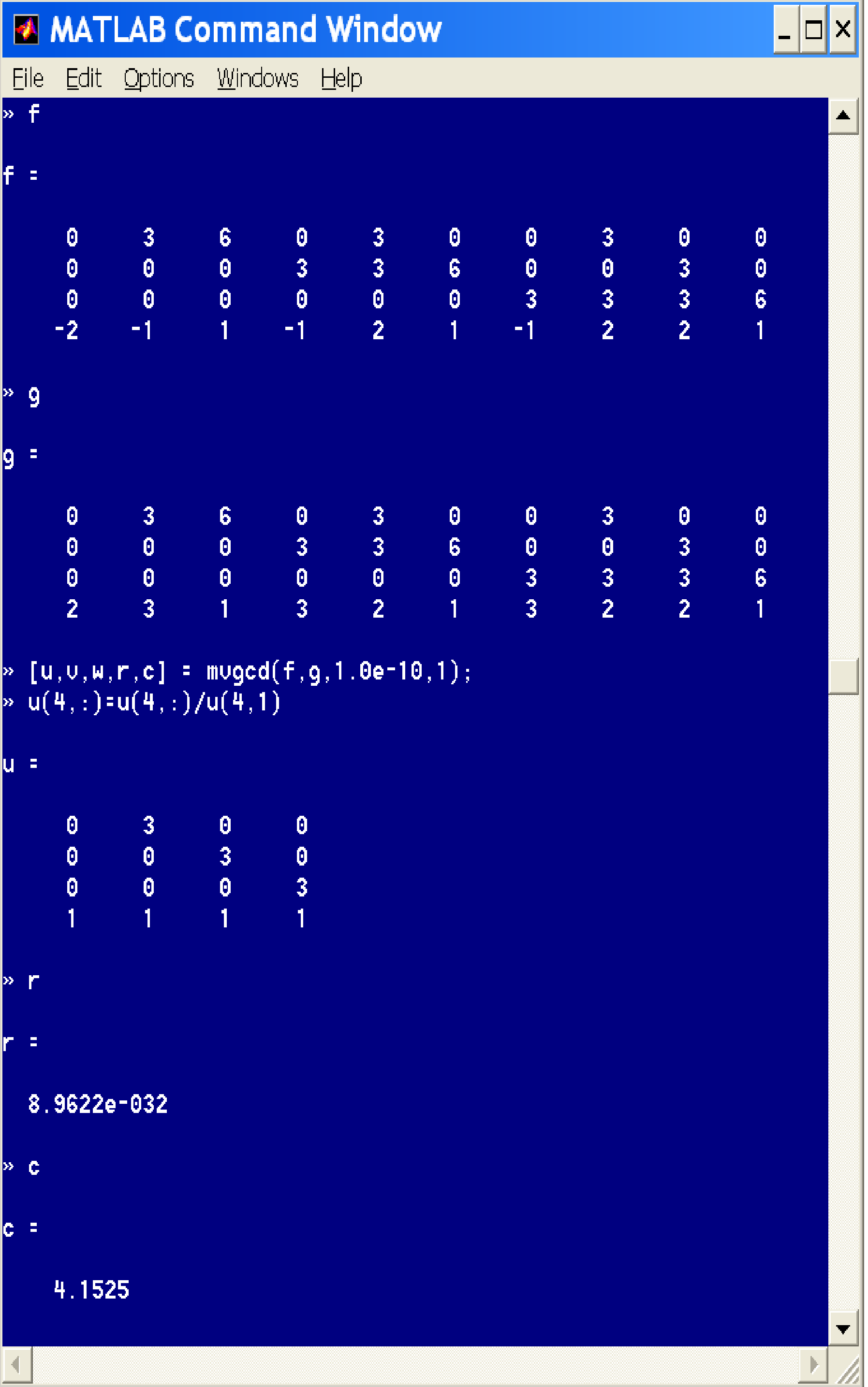
```
>> [F,res,fcnd]  =  uvFactor(f,1e-10,1);

THE CONDITION NUMBER:                          914.329
THE BACKWARD ERROR:                     5.71e-015
THE ESTIMATED FORWARD ROOT ERROR:       1.04e-011

  FACTORS

   ( x -      3.999999999999990 )^20
   ( x -      3.000000000000008 )^40
   ( x -      1.999999999999998 )^60
   ( x -      1.000000000000000 )^80
```

Z. Zeng, 2009



```
MATLAB Command Window
File  Edit  Options  Windows  Help
» f

f =

     0     3     6     0     3     0     0     3     0     0
     0     0     0     3     3     6     0     0     3     0
     0     0     0     0     0     0     3     3     3     6
    -2    -1     1    -1     2     1    -1     2     2     1

» g

g =

     0     3     6     0     3     0     0     3     0     0
     0     0     0     3     3     6     0     0     3     0
     0     0     0     0     0     0     3     3     3     6
     2     3     1     3     2     1     3     2     2     1

» [u,v,w,r,c] = mvgcd(f,g,1.0e-10,1);
» u(4,:)=u(4,:)/u(4,1)

u =

     0     3     0     0
     0     0     3     0
     0     0     0     3
     1     1     1     1

» r

r =

  8.9622e-032

» c

c =

    4.1525
```

Matlab demo:

$$f(x,y,z) = -2 - x^3 + x^6 - y^3 + 2x^3y^3 + y^6 - z^3 + 2x^3z^3 + z^6$$

$$g(x,y,z) = 2 + 3x^3 + x^6 + 3y^3 + 2x^3y^3 + y^6 + 3z^3 + 2x^3z^3 + 2y^3z^3 + z^6$$

$$GCD(f,g) = 1 + x^3 + y^3 + z^3$$

backward error = 8.9622×10^{-32}

condition number = 4.1525

MATLAB Command Window

File Edit Options Windows Help

```
g =

  Columns 1 through 7

         0    3.0000    6.0000         0    3.0000         0         0
         0         0         0    3.0000    3.0000    6.0000         0
         0         0         0         0         0         0    3.0000
    2.0000    3.0000    1.0050    3.0000    2.0000    1.0000    3.0000

  Columns 8 through 10

    3.0000         0         0
         0    3.0000         0
    3.0000    3.0000    6.0000
    2.0000    2.0000    1.0000

» [u,v,w,r,c] = mvgcd(f,g,1.0e-2,1);
» u(4,:) = u(4,:)/u(4,1)

u =

         0    3.0000         0         0
         0         0    3.0000         0
         0         0         0    3.0000
    1.0000    1.0015    0.9999    0.9999

» r

r =

    0.0031

» c

c =

    4.1530

»
```

$$\tilde{f}(x,y,z) = f(x,y,z) - 0.005x$$

$$\tilde{g}(x,y,z) = g(x,y,z) + 0.005y$$

$$AGCD\,(\tilde{f}, \tilde{g}) =$$

$$1 + 1.0015x^3 + 0.9999y^3 + 0.9999z^3$$

$$\approx 1 + x^3 + y^3 + z^3$$

backward error = 0.0031

condition number = 4.1530

Exact JCF is ill-posed (discontinuous)

Numerical JCF is strongly well-posed (uniquely exists and is Lipschitz continous) and can be computed with a two-staged algorithm (T.Y. Li and Z. Zeng)

Example: 100x100 matrix A with
multiple eigenvalues 1, -1, 2, -2
50 simple eigenvalues: random

distinct eigenvalues	Jordan block sizes	backward error	condition number
-2.00000000000010	6, 4	0.64e-12	4127.6
2.00000000000017	6, 3	2.96e-12	24554.3
-0.99999999999996	7, 4, 2	1.89e-12	6599.5
0.99999999999969	9, 6, 3	3.26e-12	7029.3
0.94798616906361	1	4.15e-12	635669.2
-0.23445335697101 - 0.08619618556166i	1	5.01e-11	552.7
-0.23445335697101 + 0.08619618556166i	1	5.01e-11	552.7
-0.35838446133613 - 1.08097722885608i	1	7.77e-13	435.5
-0.35838446133613 + 1.08097722885608i	1	7.77e-13	435.5
... ...	...	...	...

Summary:

- Ill-posed problems may indeed be wrong problems.
- To solve an ill-posed problem: Fix the problem, not the solution.
- Ill-posed problems are sensitive because they form manifolds of positive codimensions in strata.
- An ill-posed problem may be reformulated as a well-posed problem according to the “three-strikes” principle
- The reformulated problem can be solved via a two-staged strategy