On Certain Structured Fewnomials

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Khovanskii's Bound

- **Khovanskii's Bound:** A system of n polynomials in n variables having a total of $n+k+1$ distinct monomials has at most $2^{\binom{n+k}{2}}(n+1)^{n+k}$ positive solutions.

The Original Bound

- **Bihan and Sottile's Bound:** A system of n polynomials in n variables having a total of $n+k+1$ distinct monomials has at most $\frac{e^2+3}{4}2^{\binom{k}{2}}n^k$ positive solutions.

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- Method: reduce to a system of k equations in k variables having the form:

$$\sum_{i=1}^{n+k} \beta_{ij} \log(p_i),\ j = 1, \ldots, k,$$

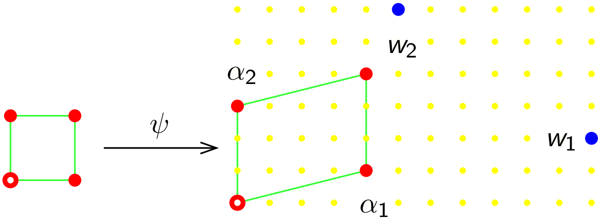
where $\deg(p_i) = 1$ and $\beta_{ij} \in \mathbb{R}$.

Structured Fewnomial

Let $P \subset \mathbb{R}^\ell$ be a lattice polytope. A fewnomial with structure P is a polynomial in n variables whose set of exponent vectors $\mathcal{A} \subset \mathbb{Z}^n$ decomposes as

$$\mathcal{A} \;=\; \mathcal{W} \bigcup \psi(P \cap \mathbb{Z}^\ell)\,, \tag{1}$$

where $\mathcal{W}$ consists of n linearly independent vectors and $\psi\colon \mathbb{Z}^\ell \to \mathbb{Z}^n$ is a linear map. For example:



Main Theorem

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A fewnomial with support $\mathcal{A}$ as defined earlier has fewer than

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To compare this to the original bound, observe that $k+1=\#(P\cap\mathbb{Z}^{\ell})$, which has order Volume(P), while ℓ is the dimension of the affine span of P. So $\ell \leq k$.

Example

$$0 = \frac{x_1^{2564} x_4^{3702}}{x_2^{1096} x_3^{3136}} - x_2^{4437} + \frac{23}{50} - \frac{9}{5} x_1^{4437} + x_1^{8874}$$

$$0 = \frac{x_2^{78} x_4^{2538}}{x_1^{3324} x_3^{1299}} - \frac{67}{50} - \frac{9}{5} x_1^{4437} + x_1^{8874} + x_2^{4437}$$

$$0 = x_4^{4437} - \frac{11}{20} + x_2^{8874} - \frac{7}{5} x_2^{4437} - x_1^{4437} x_2^{4437} + \frac{9}{10} x_1^{4437}$$

$$0 = x_3^{4437} - \frac{6}{5} + x_1^{8874} + \frac{1}{10} x_1^{4437} - x_1^{4437} x_2^{4437} + \frac{2}{5} x_2^{4437}.$$

Notice that $n = 4$.

- List of monomials:

$$1, x_4{}^{4437}, x_1{}^{4437}, x_1{}^{8874}, x_2{}^{4437}, x_2{}^{8874}, x_3{}^{4437},$$

$$x_1{}^{4437} x_2{}^{4437}, \frac{x_2{}^{78} x_4{}^{2538}}{x_1{}^{3324} x_3{}^{1299}}, \frac{x_1{}^{2564} x_4{}^{3702}}{x_2{}^{1096} x_3{}^{3136}}.$$

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- $n + \ell + 1 = 10$ so $\ell = 5$.
- Bihan and Sottile's bound ensures that we have no more than 3,145,728 positive roots.

If we row reduce, we get

$$p_1 = \frac{{x_1}^{2564}{x_4}^{3702}}{{x_2}^{1096}{x_3}^{3136}} = {x_2}^{4437} - \frac{23}{50} + \frac{9}{5}{x_1}^{4437} - {x_1}^{8874}$$

$$p_2 = \frac{{x_2}^{78}{x_4}^{2538}}{{x_1}^{3324}{x_3}^{1299}} = \frac{67}{50} + \frac{9}{5}{x_1}^{4437} - {x_1}^{8874} - {x_2}^{4437}$$

$$p_3 = {x_4}^{4437} = \frac{11}{20} - {x_2}^{8874} + \frac{7}{5}{x_2}^{4437} + {x_1}^{4437}{x_2}^{4437} - \frac{9}{10}{x_1}^{4437}$$

$$p_4 = {x_3}^{4437} = \frac{6}{5} + {x_1}^{8874} - \frac{1}{10}{x_1}^{4437} + {x_1}^{4437}{x_2}^{4437} - \frac{2}{5}{x_2}^{4437}$$

$$p_5 = {x_1}^{4437} =: u_1,\ \ p_6 = {x_2}^{4437} =: u_2,\ \ p_7 = {x_1}^{8874} =: u_3,$$

$$p_8 = {x_2}^{8874} =: u_4,\ \ p_9 = {x_1}^{4437}{x_2}^{4437} =: u_5,$$

of course $p_1, \ldots, p_9$ depend on $u_1, \ldots, u_5$.

Making substitutions we get linear equations:

$$p_1 = -\frac{23}{50} + \frac{9}{5}u_1 + u_2 - u_3$$

$$p_2 = \frac{67}{50} + \frac{9}{5}u_1 - u_2 - u_3$$

$$p_3 = \frac{11}{20} - \frac{9}{10}u_1 + \frac{7}{5}u_2 - u_4 + u_5$$

$$p_4 = \frac{6}{5} - \frac{1}{10}u_1 - \frac{2}{5}u_2 + u_3 + u_5, \text{ and}$$

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Then Bihan and Sotille find a bound based on the associated *Gale Dual System*, which will be defined later.

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$$u_3 = u_1^2$$

$$u_4 = u_2^2$$

$$u_5 = u_1 u_2.$$

This gives the new system of p_i's as:

$$
\begin{aligned}
p_1 &= -\frac{23}{50} + \frac{9}{5}u_1 + u_2 - u_1^2 \\
p_2 &= \frac{67}{50} + \frac{9}{5}u_1 - u_2 - u_1^2 \\
p_3 &= \frac{11}{20} - \frac{9}{10}u_1 + \frac{7}{5}u_2 - u_2^2 + u_1 u_2 \\
p_4 &= \frac{6}{5} - \frac{1}{10}u_1 - \frac{2}{5}u_2 + u_1^2 + u_1 u_2 \\
p_5 &= u_1 \\
p_6 &= u_2 .
\end{aligned}
$$

- Thus our system has structure P, where P has vertices $(0,0)$, $(2,0)$, $(0,2)$.

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- Thus our bound is $384 \ll 3{,}145{,}728$.

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$$A := \begin{bmatrix} 2564 & -3324 & 0 & 0 & 0 & 4437 \\ -1096 & 78 & 0 & 0 & 4437 & 0 \\ -3136 & -1299 & 0 & 4437 & 0 & 0 \\ 3702 & 2538 & 4437 & 0 & 0 & 0 \end{bmatrix}$$

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$$B := \begin{bmatrix} -\frac{51}{100} & -\frac{17}{50} & \frac{31}{50} & -\frac{23}{50} & -\frac{3}{25} & \frac{1}{25} \\ -\frac{9}{25} & \frac{63}{100} & -\frac{3}{50} & -\frac{7}{100} & -\frac{1}{10} & \frac{17}{25} \end{bmatrix}^T$$

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This gives us:

$$\prod_{i=1}^{6} x^{\beta_{j,i}\, \alpha_i} = 1, \;\; \textit{for } j = 1, 2,$$

for $x \in \mathbb{R}_+^4$. Replacing x^{α_i} with $p_i(u)$ we then have:

$$\prod_{i=1}^{6} p_i(u)^{\beta_{j,i}} = 1, \;\; \textit{for } j = 1, 2.$$

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Taking logs we have the System

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Theorem: There is a bijection between the positive solutions to the original system and the solutions to this Gale Dual variant inside the region

$$\Delta = \{y | p_i(y) > 0, \forall i\},$$

which restricts to a bijection between their nondegenerate solutions.

Δ

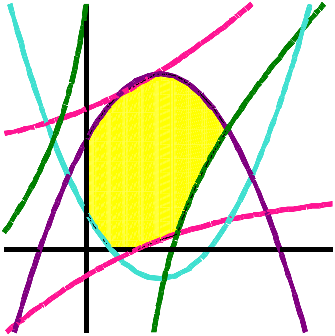

Figure: Δ is the yellow region

How it works

We want to find $|V(\psi_1, \psi_2)|$. It is not obvious what this bound should be, but...

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Khovanskii-Rolle Theorem: Let $f_1, \dots, f_k$ be smooth functions defined on $\Delta \subset \mathbb{R}^k$ which have finitely many common zeroes $V(f_1, \dots, f_k)$ in Δ, where $V(f_1, \dots, f_{k-1})$ is a smooth curve C in Δ. Let $\mathrm{ubc}(C)$ denote the number of unbounded components of C in Δ and let

$$\Gamma = \mathrm{Jac}(f_1, \dots, f_k) = \det\left(\frac{\partial f_j}{\partial y_l}\right)_{j,l=1,\dots,k}$$

be the Jacobian of $f_1, \dots, f_k$. Then

$$|V(f_1, \dots, f_k)| \leq \mathrm{ubc}(C) + |V(f_1, \dots, f_{k-1}, \Gamma)|.$$

Smooth curves

For $j = 1, \ldots, \ell$, define the curve

$$C_j = \{y \in \Delta | \psi_1(y) = \cdots = \psi_{j-1}(y) = \Gamma_{j+1}(y) = \cdots = \Gamma_\ell(y) = 0\}. \quad (3)$$

Iterating the Khovanskii-Rolle Theorem, we obtain

$$|V(\psi_1, \ldots, \psi_\ell)| \leq \text{ubc}(C_\ell) + \cdots + \text{ubc}(C_1) + |V(\Gamma_1, \ldots, \Gamma_\ell)|.$$

Lemma

Lemma:

- $J_{\ell-j} := \Gamma_{\ell-j}(y) \cdot (\prod_{i=1}^{n+\ell} p_i(y))^{2^j}$ is a polynomial of degree $2^\ell Dn$.
- C_j is a smooth algebraic curve and

$$\mathrm{ubc}(C_j) \leq \frac{1}{2} 2^{\binom{\ell-j}{2}} D^j n^{\ell-j} \binom{n+\ell+1}{j}.$$

- $|V(\Gamma_1, \ldots, \Gamma_\ell)| \leq 2^{\binom{\ell}{2}} (Dn)^\ell.$

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Iterating the Khovanskii-Rolle Theorem, we have

$$\begin{aligned} |V(\psi_1,\psi_2)| &\leq \mathrm{ubc}(\psi_1) + |V(\psi_1,\Gamma_2)| \\ &\leq \mathrm{ubc}(\psi_1) + |V(\psi_1,J_2)| \\ &\leq \mathrm{ubc}(\psi_1) + \mathrm{ubc}(J_2) + |V(\Gamma_1,J_2)| \\ &\leq \mathrm{ubc}(\psi_1) + \mathrm{ubc}(J_2) + |V(J_1,J_2)|. \end{aligned}$$

where $\Gamma_2 = \mathrm{Jac}(\psi_1,\psi_2)$, $\Gamma_1 = \mathrm{Jac}(\psi_1,J_2)$, and J_i is Γ_i after clearing the denominators.

$|V(\psi_1, J_2)|$

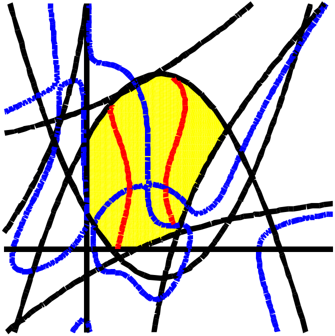

Figure: ψ_1 is red, J_2 is blue, $\mathrm{ubc}(\psi_1) = 4$, and $|V(\psi_1, J_2)| = 3$.

$|V(J_1, J_2)|$

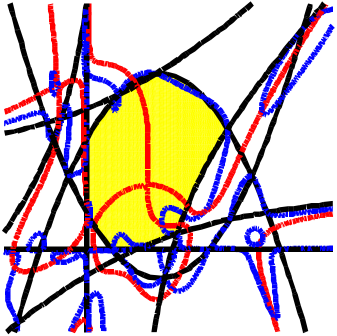

Figure: J_2 is red, J_1 is blue, $\text{ubc}(J_2) = 4$, and $|V(J_1, J_2)| = 1$.

$|V(\psi_1, \psi_2)|$

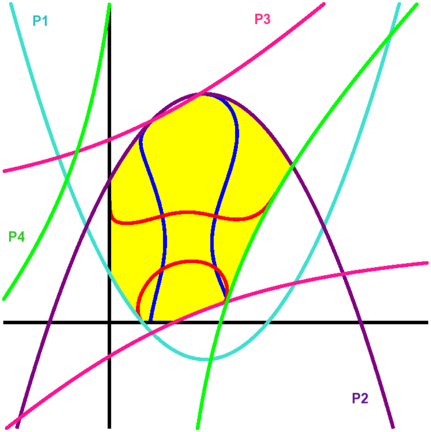

Figure: ψ_1 is blue, ψ_2 is red, and $|V(\psi_1, \psi_2)| = 4 \leq 9$.

Questions?