Thomas Bliem, June 10/2010 (Thu) 15:30–16:00.

$\boxed{\text{Demazure Modules of } \widehat{\mathfrak{sl}}_2}$ (with S. Kousidis)

§1. Problem & Result

Let $\mathfrak{g} = \mathfrak{g}\left(\overset{0}{\bullet} = \overset{1}{\bullet}\right) \simeq \widehat{\mathfrak{sl}}_2$. Λ_0, Λ_1 (fund. weights) α_0, α_1 (simple roots)

For $\Lambda = m\Lambda_0 + n\Lambda_1$ dominant integral, $w \in W$ (Weyl group):

$$\underset{\left(\text{integrable highest weight reps}\right)}{V(\Lambda)} \supset \underset{\left(\text{Demazure module}\right)}{V_w(\Lambda)} \overset{\text{def}}{=} \underset{(\text{Borel})}{\mathfrak{b}}\text{-module generated by } \underset{(\text{weight space})}{V(\Lambda)_{w\Lambda}}$$

Λ will be omitted from future notation (fixed).

Let
$$f_w(q) := \sum_{a=0}^{\infty} \dim \underbrace{\left(\bigoplus_{b=0}^{\infty} V_w(\Lambda)_{\Lambda - a\alpha_0 - b\alpha_1}\right)}_{\text{degree } a \text{ component}} q^a$$
$$= \text{basic specialization of } ch_{V_w(\Lambda)}.$$

Q.	A.
0. What is $f_w(1) = \dim(V_w(\Lambda))$?	— Sanderson '96 "Dimensions..." Obtain new proof here.
1. What is $f_w'(1)$?	— Our result
2. What is $f_w''(1)$?	— Don't know

Let $w_{N,0} = \underbrace{\cdots s_0 s_1 s_0}_{N}$, $w_{N,1} = \underbrace{\cdots s_1 s_0 s_1}_{N}$.

Theorem. Let $w = w_{N,0}$, $N \geq 1$. Then

0. $f_w(1) = (m+1)(m+n+1)^{N-1}$

1. $$\frac{f_w'(1)}{f_w(1)} = \frac{2(N-1)m(m+2) + (N-1)(N-2)(m+n)(m+n+2)}{12(m+n+1)} + \left\lfloor \frac{N-1}{2} \right\rfloor \frac{m+n}{2} + \frac{m}{2}.$$

For $w = w_{N,1}$: Similar.

§2. Proof

Define weight distribution $\mu_w := \sum_{\lambda \in \mathfrak{h}_{\mathbb{R}}^*} \dim(V_w(\Lambda)_\lambda) \cdot \delta_\lambda$.

μ_w is supported on $\Gamma := \Lambda + \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1$.

Coordinates $a, b : \Gamma \to \mathbb{Z}$ by $\lambda = \Lambda - a(\lambda)\alpha_0 - b(\lambda)\alpha_1$.

Can restate Thm as

0. $\underbrace{\mu_{N,0}}_{=\mu_{w_{N,0}}}(\Gamma) = (m+1)(m+n+1)$ (total mass)

1. $E_{\mu_{N,0}}[a] = \ldots$ (expected value)

Ingredients:
- Induction along Demazure's character formula.
- Must study both $E_\mu[a]$, $E_\mu[b]$.
- $a-b$ is (essentially) the finite weight, so $E_\mu[a-b]$ and $\mathrm{Var}_\mu(a-b)$ are known
  (elementary) (Sanderson '96 "Real characters")

Demazure operators act on $\mathrm{Meas}_c^{\mathbb{C}}(\Gamma)$ by

$$D_j \delta_\lambda = \sum_{i=0}^{\langle \lambda, \alpha_j^\vee \rangle} \delta_{\lambda - i\alpha_j} \qquad (j=0,1)$$

$$\rightsquigarrow \mu_{N,0} = \underbrace{\cdots D_0 D_1 D_0}_{N} \delta_\Lambda \qquad \text{(Demazure's character formula)}$$

<u>Observations:</u>

(1) $E_{D_0\mu}[a-b] = \frac{m}{2}$, $\quad E_{D_1\mu}[a-b] = -\frac{n}{2}$ (by symmetry)

arbitrary

(2) $E_{D_0\mu}[b^k] = \frac{\mu(\Gamma)}{D_0\mu(\Gamma)} E_\mu[b^k(m+1-2(a-b))]$, $\quad (k \geq 0)$

$E_{D_1\mu}[a^k] =$ similar (compute)

Here looks <u>bad</u> for induction, as reduce to higher moments!
But for $k=0$ can use (1):

(3) $$(D_0\mu)(\Gamma) = \mu(\Gamma)\underbrace{(m+1-2\underbrace{E_\mu[a-b]}_{=-\frac{n}{2}})}_{=m+n+1} \quad \text{if } \mu = D_1\tilde{\mu}$$

Hence Thm, O. holds!

Funny observation: The "magic moment reduction" also works for $2^{nd} \to 1^{st}$ moments.

$k=1$ in (2) yields:

$$(6)\quad E_{D_1 D_0 \mu}[a] = \frac{D_{0\mu}(\Gamma)}{(D_1 D_0 \mu)(\Gamma)} \Big((m+n+1) E_{D_0\mu}[a] + 2 \underbrace{\mathrm{Cov}_{D_0\mu}(a, a-b)}_{= \mathrm{Var}_{D_0\mu}(a-b)} \Big)$$

$$E_{D_0 D_1 \mu}[b] = \text{similar}$$

Observation: $\mathrm{Cov}_{D_0\mu}(b, a-b) = 0$ (by symmetry), hence

This is known by Landobson '96b

$\rightsquigarrow$ Obtain recurrence relation

$\rightsquigarrow$ Solve it

$\square$