Rate-independent systems: Modeling, Analysis, and Computations

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1 Introduction

The mathematical treatment of rate-independent systems has recently attracted an increasing deal of attention. For this 5-days meeting at BIRS in Banff we have succeeded in gathering some of the most active researchers in the field. Their presentations (recorded below) are documenting some of the current investigation lines for this community. We hope that this short informal report would help in order to document the current standpoint and outlook into some near-future research lines.

We take this opportunity in order to acknowledge the very kind hospitality of the BIRS and friendliness and effectiveness of its staff. The success of this meeting has been clearly linked to the delightful atmosphere of the Center.

2 An informal introduction to rate-independent systems

We shall present here some very introductory material. Our aim is that of introducing some basic keywords for this field in a sufficiently informal way so that to possibly be of use also for the non-specialist. Moreover, we will take the occasion to record some relevant references.

2.1 Basics

The term *rate-independent* is usually referred to time-dependent processes which are invariant under time rescaling. More precisely, assume to be given a process that maps a given time-dependent input function $t \in [0,T] \mapsto \ell(t)$ into a time-dependent output $t \in [0,T] \mapsto q(t)$. We say that the mapping $\ell \mapsto q$ is *rate-independent* if, given a sufficiently smooth time-reparametrization $s : [0,T] \to [0,T]$ we have that

$$\ell \mapsto q \quad \Longleftrightarrow \quad \ell \circ s \mapsto q \circ s.$$

This feature is responsible of the appearance of so-called *hysteresis*. In particular, it gives the right to illustrate the complex time-dependent mapping $\ell \mapsto q$ by identifying curves on the (ℓ, q)-plane which are then followed during the evolution, *independently from the rate at which the input changes* (= *rate-independently*).

The hysteretic behavior of physical systems is of a great importance, particularly in Mechanics. Examples of rate-independent systems are friction, damage, crack propagation, plasticity, delamination, solid-solid phase change, ferromagnetism, ferroelectricity, just to mention a few. The ubiquitous emergence of rate-independent behaviors in applications has triggered an intense mathematical research which has to be traced back at to the work by KRASNOSELŠKIĬ & A. V. POKROVSKIĬ which has been then formalized into the first monograph on this subject [23]. Reference monographs on the mathematical treatment of rate-independent evolutions are also those by MAYERGOYZ [33], VISINTIN [76], BROKATE & SPREKELS [7], and KREJČÍ [25]. The non-strictly-mathematical literature on hysteresis is obviously huge.

2.2 A toy rate-independent system

Let us illustrate here the easiest possible of such an examples: friction. Assume to be interested in sliding a heavy block on a rough surface by pulling it by means of a linearly elastic rope. The input of the system is the position $\ell(t)$ at time t of the free end of the linearly elastic rope within some given 1D frame whereas $q(t)$ is the position of the block (the insertion point of the rope, say). The roughness of the surface is such that the block will not move until the force exerted by the rope is (in modulus) less than some critical activation threshold $\tau > 0$. Above this threshold, the block slides rigidly with ℓ. In particular, no inertial effects are to be considered. By assuming that the tension of the rope equals $k(\ell - q)$ (with $k > 0$), the evolution of this system can described by the system of relations

$$|k(\ell - q)| \le \tau, \qquad \dot{q}(f - k(\ell - q)) \le 0 \quad \text{for every } f \in [-\tau, \tau]. \tag{1}$$

The first relation asserts that the tension of the rope is always below the threshold τ. The second relation says that if the tension is in modulus strictly less than τ then $\dot{q} = 0$ and the block does not move. If the tension is exactly at the threshold, then $\ell - q$ is constant. It is immediate to check that the latter system is rate-independent: by doubling the speed at which ℓ moves the effect on q is doubled in speed.

Rate-independent processes are indeed very common and often arise as limiting situations in cases when the relevant time scale of the input ℓ is much longer with respect to the intrinsic time scales in the system. This happens quite classically when *inertial* effects turn out to be negligible. In the latter example the inertia of the block has been neglected.

2.3 Energetic solutions

In the last ten years the mathematical theory of rate-independent systems has progressively grown as an effect of the new concept of *energetic solutions* introduced by MIELKE, THEIL, & LEVITAS [54, 55]. Assume to be given a system defined by its *state* $q(t) \in Q$ as a function of time where Q is some given state-space ($Q = \mathbb{R}^d$, for instance). We shall describe the evolution of the system by providing a time-dependent *energy* functional $E(t, \cdot) : Q \to \mathbb{R} \cup \{+\infty\}$ and a *dissipation* potential $D : TQ \to [0, +\infty]$ and imposing the relation

$$\partial D(\dot{q}(t)) + \partial_q E(t, q(t)) \ni 0 \quad t > 0, \qquad q(0) = q^0. \tag{2}$$

Here, q^0 is some initial state and the latter relation (settled in the cotangent space $T^*Q = (TQ)^*$) represents the balance of the system of conservative actions $\partial_q E(t, q(t))$ and that of dissipative actions $\partial D(\dot{q}(t))$. The symbol ∂ stands for some kind of differential, possibly suitably generalized for non-smooth situations: in this case on the left of (2) we might have a set and this justifies the inclusion notation. Note that, whenever the potential D is assumed to *positively* 1-*homogeneous*, the resulting differential problem (2) turns out to be rate-independent. Namely, if q solves (2) then $q \circ s$ solves (2) with $E(s, \cdot)$ instead of $E(t, \cdot)$.

The differential problem (2) is the reference for rate-independent systems: it appears quite naturally in all the above mentioned applications, in both the finite and infinite-dimensional setting. The toy situation of Subsection 2.2 can be included in the general frame of (2) by letting $Q = \mathbb{R}$, $E(t, q) = kq^2/2 - k\ell(t)q$, $D(\dot{q}) = \tau|\dot{q}|$. Indeed, in this case we have that relation (2) reads $\tau\partial|\dot{q}| + k(q - \ell) \ni 0$ which can be equivalently rewritten as

$$\dot{q} \in N(k(\ell - q)) \quad t > 0, \qquad q(0) = q^0$$

where $N(r) : \mathbb{R} \to 2^{\mathbb{R}}$ is the (necessarily multivalued) normal cone to the interval $[-\tau, \tau]$ at point r. The latter relation can be equivalently rewritten in form of a variational inequality as in (1).

Problem (2) is quite non-smooth and very often fails to admit strong solutions. Hence, some kind of weak solution notion has to be advanced. Energetic solutions respond quite effectively to this demand. An energetic solution $t \mapsto q(t) \in Q$ is an everywhere-defined function such that $q(0) = q^0$, $t \mapsto \partial_t E(t, q(t)) \in L^1(0,T)$, and, for all $t \in [0,T]$,

$$E(t,q(t)) \leq E(t,\hat{q}) + D(\hat{q} - q(t)) \quad \forall \hat{q} \in Q, \tag{3}$$

$$E(t,q(t)) + \int_0^t D(\dot{q})\mathrm{d}s = E(0,q^0) + \int_0^t \partial_t E(s,q(s))\mathrm{d}s. \tag{4}$$

These two relations are referred to as *global stability* (3) and *energy balance* (4). The global stability condition says that the state $q(t)$ is such that no competitor state $\hat{q}$ can be preferred in terms of energy gain vs. dissipation. The energy balance just reflects the belief that the energy at time t plus the energy dissipated in the time interval $[0,t]$ (the sum in the left-hand side of (4)) equal the initial energy plus the work supplied by external actions (the right-hand side of (4)).

In case the energy E is strictly convex with respect to q, the energetic formulation (3)-(4) and the differential formulation (2) are equivalent. This is particularly the case of the above toy problem. The energetic formulation is however more general as it is directly adapted to the case of non-smooth potentials (no differentiation of potentials actually shows up in (3)-(4)) and could be modified in order to encompass non-smooth in time evolutions $t \mapsto q(t)$. Moreover, it allows a very robust existence and approximation theory [36]. In particular, even in the non-convex energy frame, limits of time-discretized solutions are energetic solutions.

These desirable features have attracted a lot of attention and existence results in the energetic frame have been obtained for elasto-plasticity [24, 9, 29, 34, 32, 73, 52], damage [5, 37, 48], brittle fractures [12, 21], delamination [22, 67], ferro-electricity [56], shape-memory alloys [53, 41, 39, 47, 65, 1], materials models [19, 66, 4, 3, 30, 14, 71], and vortex pinning in superconductors [68]. Moreover, an extensive attention has been devoted to the extension of this concept to non-linear state-space settings [17, 31, 43], non-smooth evolutions [44], homogenization [57], optimal control [61, 62], approximation [42, 40, 49, 50], and variational characterization [28, 38, 51, 69, 70].

2.4 Beyond energetic solutions

Despite of its recent success, the concept of energetic solution seems nowadays not completely satisfactory out of the realm of convex energies. The crucial point is that global stability (3) turns out to be an excessively strong constraint for evolution. Indeed, when E is not convex it may happen that (2) admits a strong solution which is not an energetic solution.

This remark is not at all new and evidence of the unsatisfactory behavior of energetic solutions has been provided in [34, Ex. 6.1], [20, Ex. 6.3], [44, Ex. 7.1]. The global minimality in (3) is a central issue in the variational theory of fracture propagation and an example of some non-physical evolution via global minimization has to be traced back to BRAIDES, DAL MASO, & GARRONI [6]. More recently, local instead of global minimization in (3) has been considered by NEGRI & ORTNER [59], KNEES, MIELKE, & ZANINI [20], and TOADER & ZANINI [74]. See also the comparative analysis in NEGRI [58]. A second example in the frame of associative (linearized) elasto-plasticity with softening is due to DAL MASO, DESIMONE, MORA, & MORINI [10]: by requiring global stability (3), no softening actually occurs. Some additional 1D discussion on this topic is in [64, 70].

There is a general belief that a possible tool in order to move beyond energetic solutions is that of considering so-called *vanishing viscosity solutions*. In particular, these are small-viscosity limits $\epsilon \to 0$ of solutions q_ϵ to (compare with (2))

$$\epsilon A\dot{q}_\epsilon(t) + \partial D(\dot{q}_\epsilon(t)) + \partial_q E(t, q_\epsilon(t)) \ni 0 \quad t > 0, \qquad q_\epsilon(0) = q^0 \tag{5}$$

where A is a given linear operator (viscosity matrix). Solutions to (5) are better behaved than those of (2) and have been proved to exists and admit limits in many different concrete cases [8, 11, 15, 20, 74] as well as some abstract context [13, 44, 45, 46]. Note however that the vanishing-viscosity approach features the introduction of some additional *modeling* for the viscous behavior of the system. This choice is safe in the scalar case but it influences the vanishing-viscosity limit already in two dimensions [70, 45].

Let us mention that alternative (in some sense intermediate) ideas in order to *localize* stability are that of MIELKE [35] and LARSEN (see [27]). Both these approaches are intended to restrict minimization to neighboring states with respect to dissipation (the former) or smaller in energy up to a positive tolerance (the latter, in the frame of brittle fractures). Even more recent are the attempts to define rate-independent evolution by passing to the limit into a dynamic regularization.

Energetic formulations are specifically tailored for the weak solution of the differential system (2). Although the system (2) is quite general, it is far from encompassing *all* rate-independent evolutions. Results aimed at extending (and often suitably modifying) the energetic viewpoint to more general situations (non-associative plasticity, thermo-mechanics, magneto-mechanics, for instance) are currently in progress [2, 60, 63, 75].

3 Current research

The arguments of the conferences in our meeting were quite focused on the current trends in the mathematical research on rate-independent systems. As such, by recording here some keywords of what has been presented during the workshop we believe that we are indeed providing some reliable portrait of actual research as well.

The talks have been basically focusing on these main themes:

- Plasticity theories (5 presentations)
- Damage and crack propagation (10 presentations)
- Shape memory alloys (6 presentations)
- General/abstract results (5 presentations)

We shall devote separate subsections to these areas, with the understanding that this distinction is sometimes rather weak and basically motivated by the sake of presentation.

3.1 Plasticity

The effective description and prediction of the material behavior in presence of plastic evolution is clearly of extraordinary applicative interest. As such, the mathematical treatment of always refined plastic theories is constantly attracting attention.

A first result on plasticity was presented by G. Francfort in collaboration with F. Babadjian and M. G. Mora. The authors have considered some non-associative plasticity case. By passing to the limit in a viscous regularization of the problem, they proved the existence of suitably behaved solutions to the full 3D quasi-static evolution problem.

The presentation by D. Reddy has been focused on the variational analysis of so-called strain-gradient linearized elasto-plasticity. The second-order nature of strain-gradient plasticity allows for the inclusion of moderate softening behavior. Time-discretizations have also been commented.

G. Dolzmann presented a result in collaboration with S. Müller and C. Kreisbeck and a simple yet critical model for hardening effect in crystal plasticity. The idea here was to focus directly on some incremental energy minimization encoding the compatibility geometric requirements and pass then to a relevant Γ-limit.

The talk by B. Schweizer was targeting stochastic homogenization issues in dynamic plasticity and porous media. This talk has been slightly paralleled by the presentation by M. Veneroni who presented a periodic-homogenization result for classical Prandtl-Reuss plasticity including linear kinematic hardening effects.

The talk by P. Dondl in collaboration with N. Dirr and M. Scheutzow was focused on the emergence of rate-independent dynamics as an effect of the motion of a material interface through a lattice with defects. The related results might be related to material interfaces moving in materials.

3.2 Damage and crack propagation

These themes attracted the largest number of contributions to our workshop. The non-linear evolution of cracks in brittle material is not only a crucial applicative issue but also a quite demanding test-case for variational and non-variational functional methods. To some extent, damage theories represent a possible phase-filed-like view at the sharp-interface crack problem and are often more amenable for mathematical discussion, especially in combination with other complex materials behaviors. A direct connection between damage and brittle fracture via a scaling limit has been provided by M. Thomas in collaboration with A. Mielke and T. Roubíček.

A. Fiaschi presented some existence results in collaboration with D. Knees and U. Stefanelli for the case of non-regularized in space damage. Lacking a compactifying effect in space, the damage parameter is here allowed to develop as a Young measure [16].

In the talk presented by J. Zeman in collaboration with P. Gruber, some analysis and simulations for the mixed-mode evolution in delamination situation has been discussed. The case of mixed-mode crack propagation via approximate stress-intensity factors has then be analyzed by M. Negri. Some phase-field and adaptive finite-element methods for the prediction of crack growth have been presented by C. Ortner in collaboration with S. Burke and E. Süli.

G. Lazzaroni has presented an existence result for the finite-strain Griffith-crack propagation with non-interpenetration constraints (in collaboration with G. Dal Maso) and R. Toader reported on her work with G. Lazzaroni on the viscous approximation in the frame of crack growth. Finally, D. Knees presented some numerical convergence analysis for a vanishing viscosity approach to fracture.

Thermal effect in materials have also be mentioned in the presentations. In particular, a thermo-visco-elastic model in case of contact with adhesion friction has been presented by E. Bonetti in collaboration with G. Bonfanti and R. Rossi. Moreover, T. Roubíček in collaboration with S. Bartels presented the analysis of a Kelvin-Voigt visco-elastic model with internal variables (plastic, damage).

3.3 Shape memory alloys

A good deal of attention has been focused on the description of the complex thermomechanical behavior of shape memory alloys (SMAs) [18]. We had talks ranging from the more applied/experimental to the more analytic. We shall follow this path also here.

P. Šittner has reported on an extended experimental campaign targeting the complex thermomechanical evolution of microstructures in amorphous NiTi alloy wires and relating these microstructures with resistance to fatigue. Then a new plate finite tailored to the celebrated SMA model by Souza-Auricchio has been presented by E. Artioli in collaboration with S. Marfia, E. Sacco, and R.L. Taylor. Some numerical experiments on a finite-strain version of the Souza-Auricchio model have been illustrated by A. Reali with J. Arghavani, F. Auricchio, R. Naghdabadi. A general result on error-control for full space-time discretization of the Souza-Auricchio model by implicit-Euler in time and conformal finite-elements in space has been presented by A. Petrov with A. Mielke, L. Paoli, and U. Stefanelli [40]. J. Zimmer with M. Kružík advanced and analyze a model including both SMA and plastic behavior whereas J. Kopfová, in collaboration with M. Eleuteri and P. Krejčí, presented a novel modeling view at Thermomechanics based on a suitable use of Preisach-type hysteretic operators.

3.4 Abstract theory

A miscellanea of novel results at the abstract level have been presented. All of these basically tackle the understanding of rate-independent evolution and the outreach of the current theory in the direction of non-smooth, non-convex, or non-deterministic scenarios.

In collaboration with A. Mielke and G. Savaré, R. Rossi has presented a quite general result on convergence and characterization of the limit of viscous regularizations of rate-independent evolutions in the frame of Banach spaces [45]. With the same aim of overcoming the restrictive global minimality conditions in (3), C. Larsen presented his notion of ϵ-stability and applied it to Griffith-crack evolution. M. Liero presented his result with P. Krejčí[26] on rate-independent evolution in the frame of regulated (non-smooth) functions via Kurzweil integration and V. Recupero outlined his analysis on the possible extension of some rate-independent operators (including the sweeping process) from Lipschitz to discontinuous inputs.

Finally, T. Sullivan illustrated his results with M. Koslowski. M. Ortiz, and F. Theil on the effect of an heat bath on a rate-independent system. This *thermalization* of the system is actually interacting with the evolution and destroying the rate-independent character of the systems around equilibrium. This observations are connected with the so-called Andrade creep effect [72].

4 Outcome of the Meeting

Let us express again our gratitude to the BIRS for supporting this workshop, which all participants found very interesting and productive. Indeed, the familiar atmosphere at Corbett Hall has, we believe, facilitated the interaction of this small but highly focused community of researchers. Scientific relations have been boosted not only by the possibility of informally interacting during the scientific part of the meeting but also by some very rewarding (and in some sense inspirational) excursions in the stunning surroundings of the Centre. It was a real privilege for us to hold our meeting at BIRS.

To our view, a very nice first outcome of the meeting was a clear picture of future research lines for the rate-independent community. In particular, the interplay between a more theoretic/variational point of view with the exciting challenges coming from the applications in plasticity, shape-memory alloys and fracture will be a main trend of the future research and constitutes one of the strength point of the workshop, with the interaction of many specialists of the field and young promising researchers. In this respect, we regard the possibility of sketching this informal report as very appropriate in view of reporting on the current standpoint in this field.

Some collaborations have been initiated or renewed at the meeting and new results have already been obtained thanks to discussions at BIRS. Some of the presentations have been invited for publication in a special issue of the journal *Discrete and Continuous Dynamical Systems - Series S* edited by G. Dal Maso. A. Mielke, and U. Stefanelli. This issue, entitled *Rate-independent evolution*, should appear during 2012.

Plans for gathering the rate-independent community have been made at the BIRS. An important occasion will be the workshop on *Variational methods for evolution* at the *Mathematishes Forschungsinstitut Oberwolfach* in December 2011, which will offer the possibility of interaction with different kinds of evolutionary problems and variational techniques.

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