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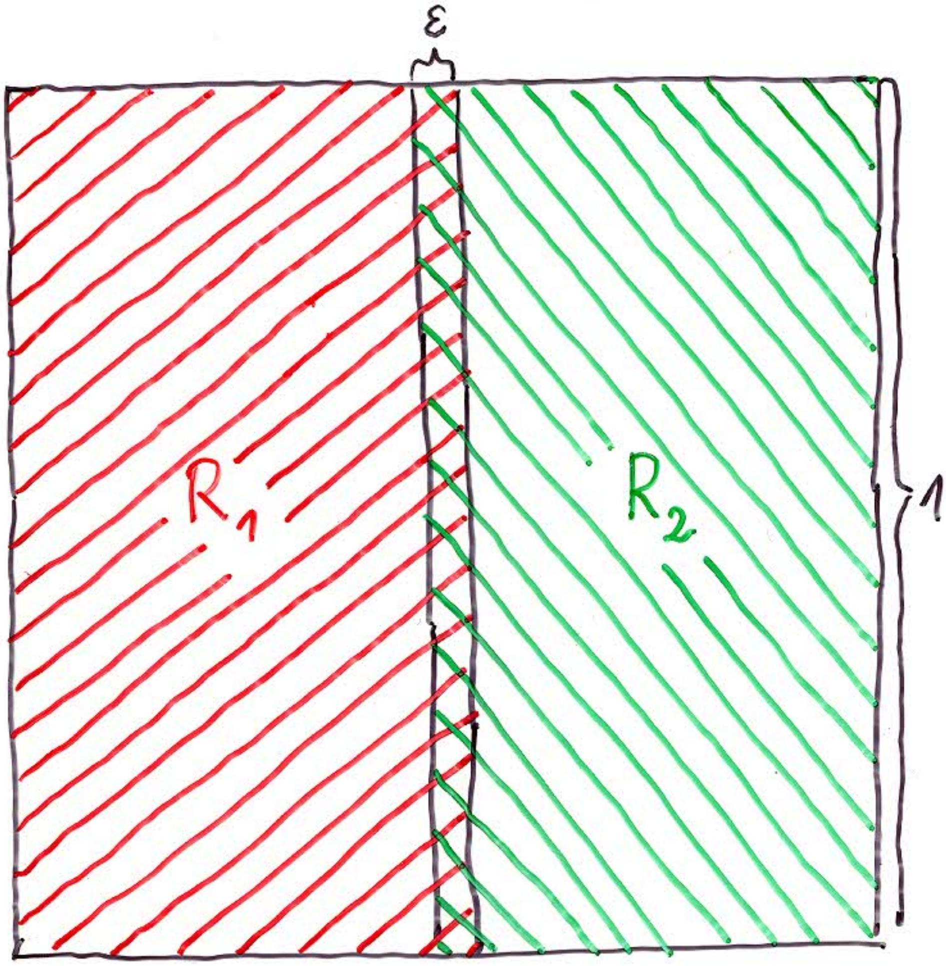
"Estimates for the splitting of holomorphic cocycles"



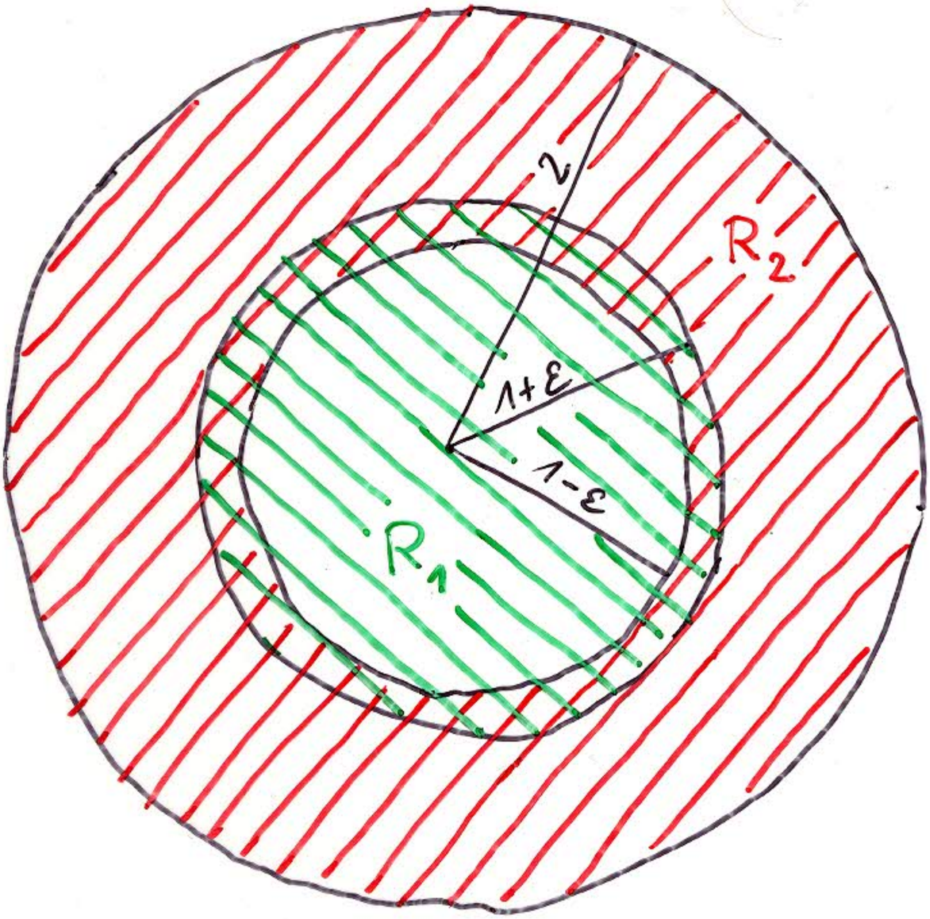
$GL(r, \mathbb{C})$ - invertible complex $r \times r$ matrices

I - the unit matrix

Theorem 1.

H
$$\begin{cases} f : R_1 \cap R_2 \xrightarrow{hol.} GL(r, \mathbb{C}) \\ \sup\limits_{z \in R_1 \cap R_2} \| f(z) - I \| < \dfrac{1}{32 |\log \varepsilon|} \end{cases}$$

C
$$\begin{cases} \exists\, f_j : R_j \xrightarrow{hol.} GL(r, \mathbb{C}) \text{ s.t.} \\ \qquad f = f_1 f_2^{-1} \\ \text{and} \\ \sup\limits_{z \in R_j} \| f_j(z) - I \| < 16 |\log \varepsilon| \sup\limits_{z \in R_1 \cap R_2} \| f(z) - I \| \end{cases}$$



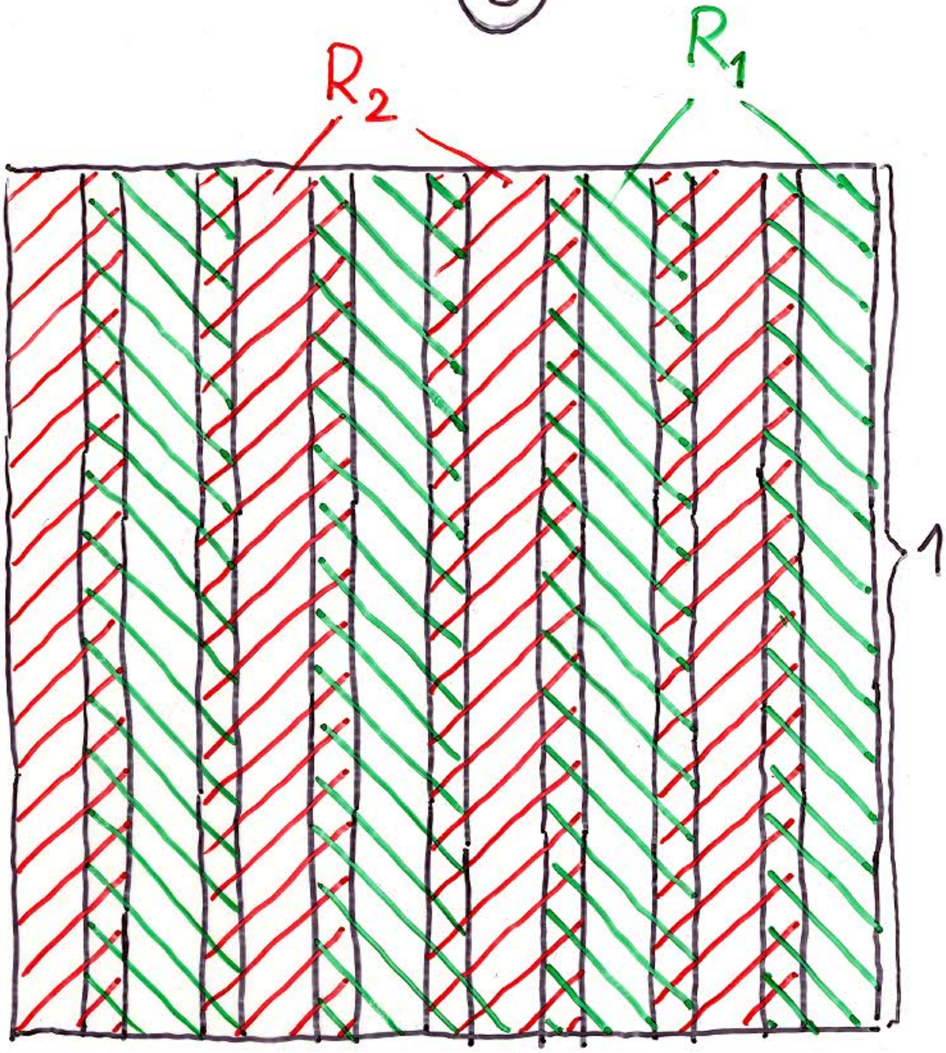
There is no constant $C<\infty$ such that:

$$\forall \varepsilon>0 \ \forall \text{ hol. } f: R_1 \cap R_2 \to \mathbb{C} \ \exists \text{ hol. } f_j: R_j \to \mathbb{C} \text{ s.t.}$$

$$f = f_1 - f_2$$

and

$$\sup_{z\in R_j} |f_j(z)| \le C \sup_{z \in R_1 \cap R_2} |f(z)| .$$



Theorem 2.

(H) $\begin{cases} f : R_1 \cap R_2 \xrightarrow{\text{hol.}} GL(r, \mathbb{C}) \\ \sup\limits_{z \in R_1 \cap R_2} \| f(z) - I \| < \dfrac{\varepsilon}{32} \end{cases}$

(C) $\begin{cases} \exists\, f_j : R_j \xrightarrow{\text{hol.}} GL(r, \mathbb{C}) \text{ s.t.} \\ \qquad f = f_1 f_2^{-1} \\ \text{and} \\ \quad \sup\limits_{z \in R_j} \| f_j(z) - I \| < \dfrac{16}{\varepsilon} \sup\limits_{z \in R_1 \cap R_2} \| f(z) - I \| \end{cases}$

4

Two definitions

Definition (in common use). Let $\mathcal{U} = \{U_\mu\}_{\mu \in I}$ be an open covering of an open set $D \subseteq \mathbb{C}$. Then

- $C^0\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$ is the set of families $\{f_\mu\}_{\mu \in I}$ of holomorphic functions $f_\mu : U_\mu \to GL(r, \mathbb{C})$, called **0-cochains**;
- $C^1\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$ is the set of families $\{f_{\mu\nu}\}_{\mu,\nu \in I}$ of holomorphic functions $f_{\mu\nu} : U_\mu \cap U_\nu \to GL(r, \mathbb{C})$, called **1-cochains**;
- $Z^1\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$ is the set of families $\{f_{\mu\nu}\}_{\mu,\nu \in I} \in C^1\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$ satisfying the **cocycle condition**

$$f_{\mu\nu} f_{\nu\kappa} = f_{\mu\kappa} \quad \text{on } U_\mu \cap U_\nu \cap U_\kappa, \qquad \mu, \nu, \kappa \in I.$$

The elements of $Z^1\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$ are called **1-cocycles**.

Definition (for this lecture). Let $\mathcal{U} = \{U_\mu\}_{\mu \in I}$ be an open covering of an open set $D \subseteq \mathbb{C}$. Then, for $f = \{f_\mu\}_{\mu \in I} \in C^0\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$, we define

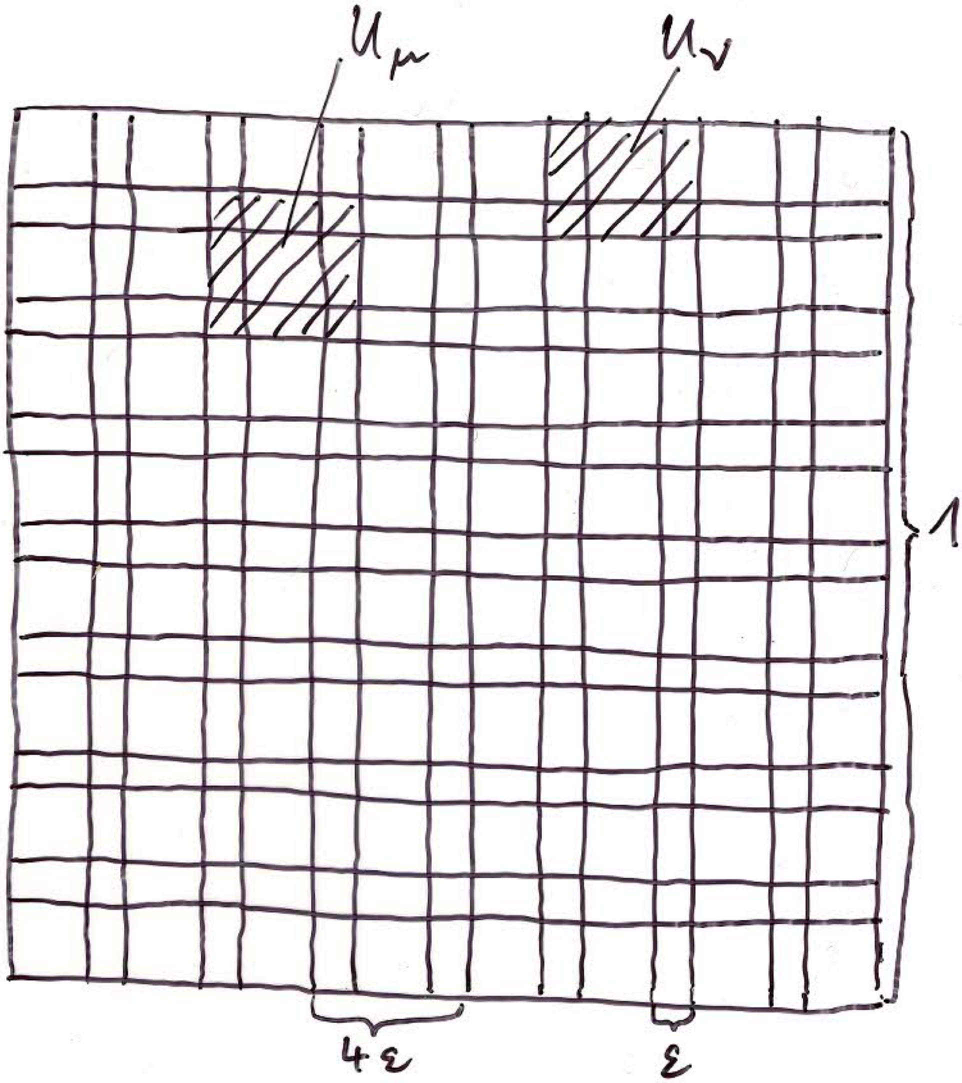
$$\|f - I\| = \sup_{\mu \in I} \sup_{z \in U_\mu} \Big(\|f_\mu(z) - I\| + \|f_\mu^{-1}(z) - I\| \Big)$$

and, for $f = \{f_{\mu\nu}\}_{\mu,\nu \in I} \in C^1\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$, we define

$$\|f - I\| = \sup_{\mu,\nu \in I} \sup_{z \in U_\mu \cap U_\nu} \Big(\|f_{\mu\nu}(z) - I\| + \|f_{\mu\nu}^{-1}(z) - I\| \Big).$$

Finally, for $g = \{g_\mu\}_{\mu \in I} \in C^0\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$, we denote by $\delta^* g$ the cocycle $\{f_{\mu\nu}\}_{\mu,\nu \in I} \in Z^1\big(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}\big)$ defined by

$$\boxed{f_{\mu\nu} = g_\mu g_\nu^{-1} \text{ on } U_\mu \cap U_\nu, \qquad \mu, \nu \in I.}$$



$\mathfrak{U} = \{U_\mu\}_{\mu \in I}$

Theorem 3.

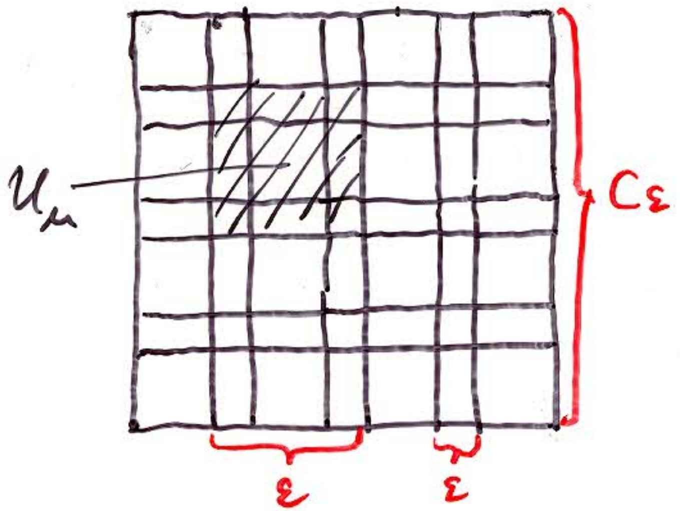
(H) $f \in Z^1(\mathfrak{U}, \mathcal{O}^{GL(r,\mathbb{C})})$ s.t. $\|f - I\| < \dfrac{\varepsilon^3}{2^{60}}$

(C) $\exists\, g \in C^0(\mathfrak{U}, \mathcal{O}^{GL(r,\mathbb{C})})$ s.t.

$$f_{\mu\nu} = g_\mu g_\nu^{-1}$$

and

$$\|g - I\| < \frac{2^{60}}{\varepsilon} \|f - I\|$$



$\mathcal{U} = \{U_\mu\}_{\mu \in I}$

C independ. of ε

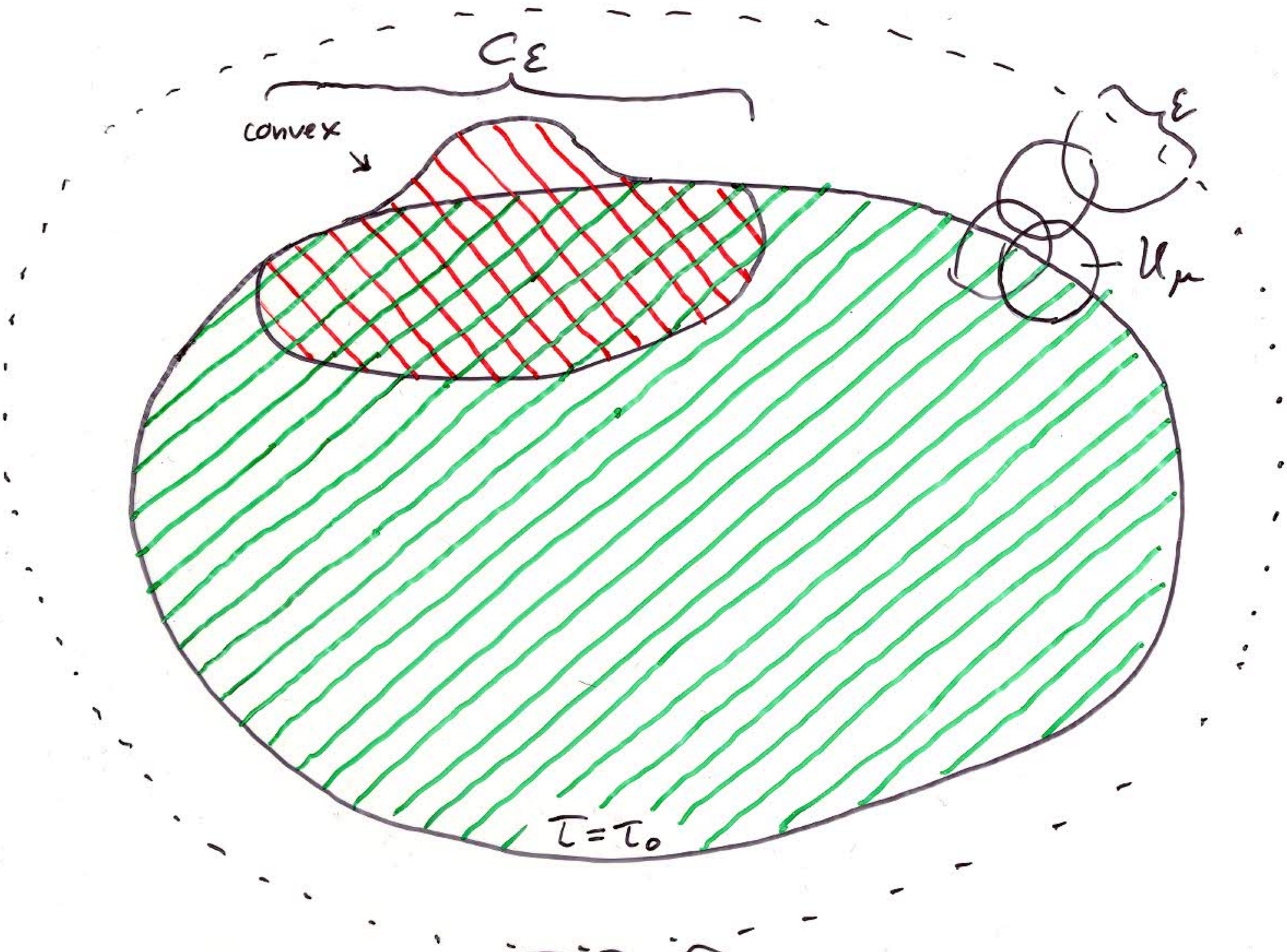
$\varepsilon \to 0$

Theorem 3 $\Longrightarrow$ Theorem 4.

(H) $f \in Z^1(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})})$ s.t. $\|f - I\| < \dfrac{1}{2^{60} C^3}$

(C) $\begin{cases} \exists\, g \in C^0(\mathcal{U}, \mathcal{O}^{GL(r,\mathbb{C})}) \text{ s.t.} \\ \qquad f_{\mu\nu} = g_\mu g_\nu^{-1} \\ \text{and} \\ \qquad \|g - I\| < 2^{60} C \|f - I\|. \end{cases}$

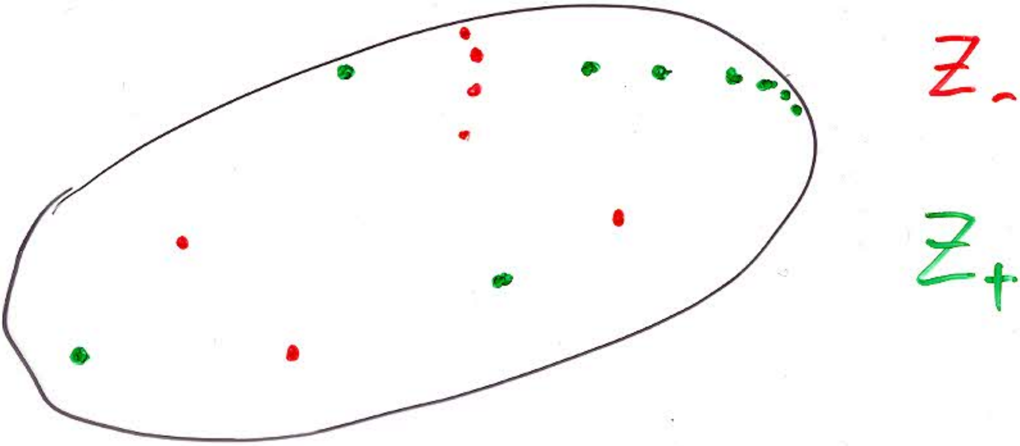
$\mathfrak{U} = \{U_\mu\}$ open covering of $\{\tau < \tau_0 + \sigma\}$



$(\{H_{\mu\nu}\}, \{A_\mu\}, \{S_{\mu\nu}\})$ near to Hermitian s.t.

$H_{\mu\nu} \,|\, U_\mu \cap U_\nu \cap \{\tau < \tau_0\}$ already holom.

The Weierstraß product theorem of Gohberg-Rodman



Theorem (Gohberg/Rodman 1983). Suppose:

- $D \subseteq \mathbb{C}$ open;
- $Z_-, Z_+ \subseteq D$ discrete and closed in D;
- $\forall\, w \in Z_+ \cup Z_-$, an integer $N_w \in \mathbb{N}$ is given;
- $\forall\, w \in Z_+ \cup Z_-$, a rational matrix-valued function A_w is given such that, for some neighborhood U_w of w,

$$A_w(z) = \sum_{j=-N_w}^{N_w} M_j (z-w)^j \in GL(r, \mathbb{C}), \qquad z \in U_w \setminus \{w\}.$$

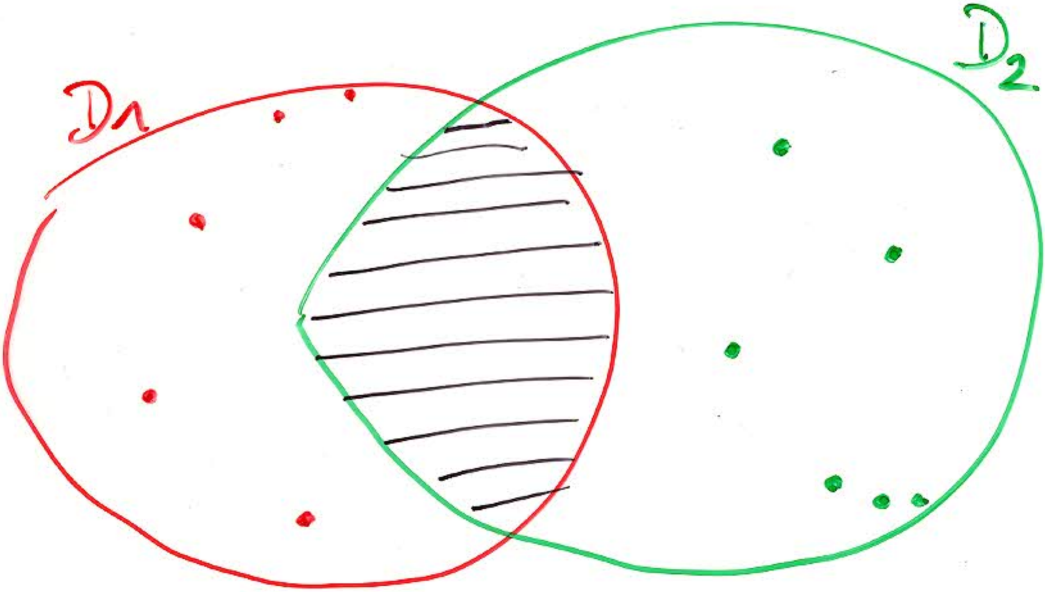
Then there exists a holomorphic function $A : D \setminus (Z_+ \cup Z_-) \to GL(r, \mathbb{C})$, such that

- if $w \in Z_+$, then
$$A(z) = M_w(z) + O\big(|z-w|^{N_w+1}\big) \quad \text{for } z \to w;$$
- if $w \in Z_+$, then
$$A^{-1}(z) = M_w(z) + O\big(|z-w|^{N_w+1}\big) \quad \text{for } z \to w.$$

The classical Weierstraß product theorem is contained as the special case $r = 1$ and $A_w(z) = (z-w)^{N_w}$.

#~~8~~ 9

This Gohberg/Rodman theorem can be proved using the following advanced version of the Cartan lemma:



Cartan lemma with prescribed units (Forster/Ramspott 1966). Suppose:

- $D_1, D_2 \subseteq \mathbb{C}$ open;
- $Z \subseteq (D_1 \cup D_2) \setminus (D_1 \cap D_2)$ discrete and closed in $D_1 \cup D_2$;
- $\forall\ w \in Z$, an integer $N_w \in \mathbb{N}$ is given.

Then each holomorphic function $f : D_1 \cap D_2 \to GL(r, \mathbb{C})$ can be factorized as $f = f_1 f_2^{-1}$, where $f_j : D_j \to GL(r, \mathbb{C})$ is holomorphic and, moreover, for each $w \in Z \cap D_j$,

$$f_1(z) = I + O\big(|z - w|^{N_w}\big) \quad \text{for } z \to w.$$