Multiscale mortar methods for Stokes-Darcy flow

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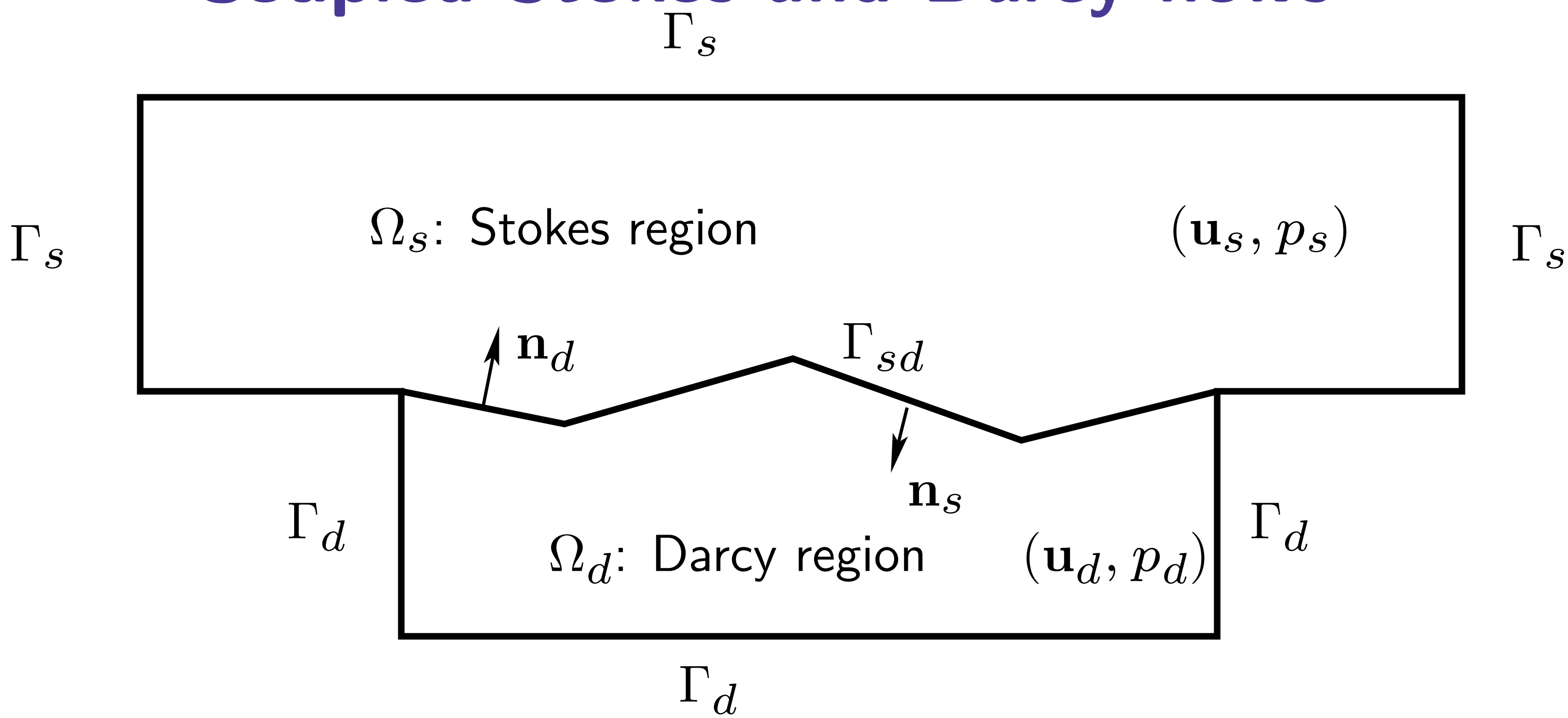
Coupled Stokes and Darcy flows



- surface water - groundwater flow
- flow in fractured porous media
- flow through vuggy rocks
- flow through industrial filters
- fuel cells
- blood flow

Outline

- Mathematical model for the coupled Stokes-Darcy flow problem
  - Interface conditions
  - Existence and uniqueness for a global weak formulation
  - Equivalence to a domain decomposition weak formulation

- Multiscale mortar finite element discretizations
  - Fine scale (h) conforming Stokes elements and mixed finite elements for Darcy
  - Coarse scale (H) mortar finite elements on subdomain interfaces
  - Discrete inf-sup condition
  - Existence and uniqueness of a discrete solution
  - Convergence analysis

- Non-overlapping domain decomposition
  - Reduction to a mortar interface problem
  - A multiscale flux basis

- Computational results

Flow equations

Deformation tensor $\mathbf{D}$ and stress tensor $\mathbf{T}$ in Ω_s:

$$\mathbf{D}(\mathbf{u}_s) := \frac{1}{2}(\nabla \mathbf{u}_s + (\nabla \mathbf{u}_s)^T), \quad \mathbf{T}(\mathbf{u}_s, p_s) := -p_s \mathbf{I} + 2\mu \mathbf{D}(\mathbf{u}_s).$$

Stokes flow in Ω_s:

$$\begin{aligned}
-\nabla \cdot \mathbf{T} \equiv -2\mu \nabla \cdot \mathbf{D}(\mathbf{u}_s) + \nabla p_s &= \mathbf{f_s} \quad \text{in } \Omega_s \quad \text{(conservation of momentum)}, \\
\nabla \cdot \mathbf{u}_s &= 0 \quad \text{in } \Omega_s \quad \text{(conservation of mass)}, \\
\mathbf{u}_s &= 0 \quad \text{on } \Gamma_s \quad \text{(no slip)}.
\end{aligned}$$

Darcy flow in Ω_d:

$$\begin{aligned}
\mu \mathbf{K}^{-1} \mathbf{u}_d + \nabla p_d &= \mathbf{f}_d \quad \text{in } \Omega_d \quad \text{(Darcy's law)}, \\
\nabla \cdot \mathbf{u}_d &= q_d \quad \text{in } \Omega_d \quad \text{(conservation of mass)}, \\
\mathbf{u}_d \cdot \mathbf{n}_d &= 0 \quad \text{on } \Gamma_d \quad \text{(no flow)}.
\end{aligned}$$

Interface conditions

Mass conservation across Γ_{sd}:

$$\mathbf{u}_s \cdot \mathbf{n}_s + \mathbf{u}_d \cdot \mathbf{n}_d = 0 \quad \text{on} \quad \Gamma_{sd}.$$

Continuity of normal stress on Γ_{sd}:

$$-\mathbf{n}_s \cdot \mathbf{T} \cdot \mathbf{n}_s \equiv p_s - 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_s) \cdot \mathbf{n}_s = p_d \quad \text{on} \quad \Gamma_{sd}.$$

Slip with friction interface condition:

(Beavers-Joseph (1967),Saffman (1971), Jones (1973), Jäger and Mikelić (2000))

$$-\mathbf{n}_s \cdot \mathbf{T} \cdot \boldsymbol{\tau}_j \equiv -2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_s) \cdot \boldsymbol{\tau}_j = \frac{\mu\alpha}{\sqrt{K_j}} \mathbf{u}_1 \cdot \boldsymbol{\tau}_j, \; j = 1, d-1, \; \text{on } \Gamma_{sd},$$

where $K_j = \boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j$.

Some previous results

- Existence and uniqueness of a weak solution
  - Discacciati, Miglio, Quarteroni 2002
  - Layton, Schieweck, Y. 2003
  - NSE-Darcy: Girault, Riviere 2009; Discacciati, Quarteroni 2009

- Numerical approximation with Stokes and Darcy elements
  - Discacciati, Miglio, Quarteroni 2002
  - Layton, Schieweck, Y. 2003
  - Riviere, Y. 2005
  - Galvis, Sarkis 2007
  - Babuska, Gatica 2010
  - Riviere, Kanschat 2010

- Numerical approximation with unified finite elements (Brinkman model)
  - Angot 1999
  - Mardal, Tai, Winther 2002
  - Arbogast, Lehr 2006
  - Burman, Hansbo 2005, 2007
  - Xie, Xu, Xue 2008

Global variational formulation

Stokes: $\mathbf{v}_s \in H^1(\Omega_s)^n$, $\mathbf{v}_s = 0$ on Γ_s,

$$2\,\mu \int_{\Omega_s} \mathbf{D}(\mathbf{u}_s) : \mathbf{D}(\mathbf{v}_s) - \int_{\Omega_s} p_s \,\mathrm{div}\,\mathbf{v}_s - \int_{\Gamma_{sd}} \mathbf{T}\mathbf{n}_s \cdot \mathbf{v}_s = \int_{\Omega_s} \mathbf{f}_s \cdot \mathbf{v}_s$$

Darcy: $\mathbf{v}_d \in H(\mathrm{div}; \Omega_d)$, $\mathbf{v}_d \cdot \mathbf{n}_d = 0$ on Γ_d

$$\mu \int_{\Omega_d} \mathbf{K}^{-1}\mathbf{u}_d \cdot \mathbf{v}_d - \int_{\Omega_d} p_d \,\mathrm{div}\,\mathbf{v}_d + \int_{\Gamma_{sd}} p_d \mathbf{v}_d \cdot \mathbf{n}_d = \int_{\Omega_d} \mathbf{f}_d \cdot \mathbf{v}_d$$

Interface term:

$$I = \sum_{j=1}^{n-1} \int_{\Gamma_{sd}} \frac{\mu\alpha}{\sqrt{K_j}} (\mathbf{u}_s \cdot \boldsymbol{\tau}_j)(\mathbf{v}_s \cdot \boldsymbol{\tau}_j) + \int_{\Gamma_{sd}} p_d [\mathbf{v}] \cdot \mathbf{n}_s$$

Global variational formulation, cont.

$$\tilde{X} = \{\mathbf{v} \in H(\mathrm{div};\Omega)\,;\, \mathbf{v}_s \in H^1(\Omega_s)^n, \mathbf{v}|_{\Gamma_s} = \mathbf{0}, (\mathbf{v}\cdot\mathbf{n})|_{\Gamma_d} = 0\}, \quad W = L^2_0(\Omega)$$

$$\|\mathbf{v}\|_{\tilde{X}} = \left(\|\mathbf{v}\|^2_{H(\mathrm{div};\Omega)} + |\mathbf{v}_s|^2_{H^1(\Omega_s)}\right)^{1/2}$$

Find $(\mathbf{u}, p) \in \tilde{X} \times W$ such that

$$\forall \mathbf{v} \in \tilde{X}, \quad \mu \int_{\Omega_d} \mathbf{K}^{-1}\mathbf{u}_d \cdot \mathbf{v}_d + 2\,\mu \int_{\Omega_s} \mathbf{D}(\mathbf{u}_s) : \mathbf{D}(\mathbf{v}_s) - \int_\Omega p \,\mathrm{div}\,\mathbf{v}$$
$$+ \sum_{j=1}^{n-1} \int_{\Gamma_{sd}} \frac{\mu\alpha}{\sqrt{K_j}} (\mathbf{u}_s \cdot \boldsymbol{\tau}_j)(\mathbf{v}_s \cdot \boldsymbol{\tau}_j) = \int_\Omega \mathbf{f}\cdot\mathbf{v},$$

$$\forall w \in W, \quad \int_\Omega w \,\mathrm{div}\,\mathbf{u} = \int_{\Omega_d} w\, q_d.$$

Global variational formulation, cont.

$$\tilde{a}(\mathbf{u},\mathbf{v}) = \mu \int_{\Omega_d} \mathbf{K}^{-1}\mathbf{u}_d \cdot \mathbf{v}_d + 2\,\mu \int_{\Omega_s} \mathbf{D}(\mathbf{u}_s) : \mathbf{D}(\mathbf{v}_s) + \sum_{j=1}^{d-1} \int_{\Gamma_{sd}} \frac{\mu\alpha}{\sqrt{K_j}} (\mathbf{u}_s \cdot \boldsymbol{\tau}_j)(\mathbf{v}_s \cdot \boldsymbol{\tau}_j)$$

$$\tilde{b}(\mathbf{v}, w) = -\int_{\Omega} w \,\mathrm{div}\,\mathbf{v}$$

Find $(\mathbf{u}, p) \in \tilde{X} \times W$ such that

$$\forall \mathbf{v} \in \tilde{X}, \quad \tilde{a}(\mathbf{u},\mathbf{v}) + \tilde{b}(\mathbf{v}, p) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v},$$

$$\forall w \in W, \quad \tilde{b}(\mathbf{u}, w) = -\int_{\Omega} w\, q_d.$$

Lemma: The variational formulation is equivalent to the PDE system.

Existence and uniqueness of a weak solution

Lemma:

$$\forall w \in W, \quad \sup_{\mathbf{v}\in\tilde{X}} \frac{\tilde{b}(\mathbf{v}, w)}{\|\mathbf{v}\|_{\tilde{X}}} \geq \beta \|w\|_W$$

Lemma:

$$\tilde{a}(\mathbf{v}, \mathbf{v}) \geq \gamma \|\mathbf{v}\|_{\tilde{X}}^2 \quad \forall\, \mathbf{v} \in \tilde{X}_0 = \{\mathbf{v} \in \tilde{X} : \mathrm{div} = 0\}$$

Proof: Korn's inequality:

$$\tilde{a}(\mathbf{v}, \mathbf{v}) \geq C(|\mathbf{v}_s|^2_{H^1(\Omega_s)} + \|\mathbf{v}_d\|^2_{L^2(\Omega_d)})$$

Poincare-type inequality:

$$\forall \mathbf{v} \in \tilde{X}_0, \quad \|\mathbf{v}_s\|_{L^2(\Omega_s)} \leq C\big(|\mathbf{v}_s|^2_{H^1(\Omega_s)} + \|\mathbf{v}_d\|^2_{L^2(\Omega_d)}\big)^{1/2} \quad \square$$

Lemma: The variational problem has a unique solution.

Domain decomposition variational formulation

Let $\overline{\Omega}_s = \cup \overline{\Omega}_{s,i}$, $\overline{\Omega}_d = \cup \overline{\Omega}_{d,i}$.

Interface conditions

Stokes-Stokes interfaces:

$$[\mathbf{v}_s] = 0, \ \ [\mathbf{T} \cdot \mathbf{n}] = 0 \quad \text{on } \Gamma_{ss}$$

Darcy-Darcy interfaces:

$$[\mathbf{u}_d \cdot \mathbf{n}] = 0, \ \ [p_d] = 0 \quad \text{on } \Gamma_{dd}$$

$$X = \{\mathbf{v}|_{\Omega_{s,i}} \in H^1(\Omega_{s,i})^n, \ \mathbf{v}|_{\Omega_{d,i}} \in H(\mathrm{div}, \Omega_{d,i}) + \text{ BCs},$$
$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma_{ij}} \in H^{-1/2}(\Gamma_{ij}) \ \forall \, \Gamma_{ij} \subset \Gamma_{dd} \cup \Gamma_{sd}\}$$

$$\Lambda = \{ \boldsymbol{\lambda}|_{\Gamma_{ij}} \in H^{-1/2}(\Gamma_{ij})^n \ \forall \, \Gamma_{ij} \subset \Gamma_{ss},$$
$$\lambda|_{\Gamma_{ij}} \in H^{1/2}(\Gamma_{ij}) \ \forall \, \Gamma_{ij} \subset \Gamma_{dd} \cup \Gamma_{sd}\}.$$

Domain decomposition variational formulation, cont.

$$a_{s,i}(\mathbf{u}_{s,i},\mathbf{v}_{s,i}) = 2\,\mu\int_{\Omega_{s,i}}\mathbf{D}(\mathbf{u}_{s,i}):\mathbf{D}(\mathbf{v}_{s,i}) + \sum_{j=1}^{d-1}\int_{\partial\Omega_{s,i}\cap\Gamma_{sd}}\frac{\mu\alpha}{\sqrt{K_j}}(\mathbf{u}_{s,i}\cdot\boldsymbol{\tau}_j)(\mathbf{v}_{s,i}\cdot\boldsymbol{\tau}_j)$$

$$a_{d,i}(\mathbf{u}_{d,i},\mathbf{v}_{d,i}) = \mu\int_{\Omega_{d,i}}\mathbf{K}^{-1}\mathbf{u}_{d,i}\cdot\mathbf{v}_{d,i}, \quad b_i(\mathbf{v}_i,w_i) = -\int_{\Omega_i} w_i\,\mathrm{div}\,\mathbf{v}_i$$

$$a(\cdot,\cdot) = \sum a_{s,i}(\cdot,\cdot) + \sum a_{d,i}(\cdot,\cdot), \quad b(\cdot,\cdot) = \sum b_i(\cdot,\cdot)$$

$$b_\Gamma(\mathbf{v},\tilde{\mu}) = \int_{\Gamma_{ss}}[\mathbf{v}]\,\boldsymbol{\mu} + \int_{\Gamma_{dd}}[\mathbf{v}\cdot\mathbf{n}]\,\mu + \int_{\Gamma_{sd}}[\mathbf{v}\cdot\mathbf{n}]\,\mu$$

Domain decomposition variational formulation, cont.

Find $\mathbf{u} \in X$, $p \in W$, $\tilde{\lambda} \in \Lambda$:

$$\forall \mathbf{v} \in X,\ a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b_\Gamma(\mathbf{v}, \tilde{\lambda}) = \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

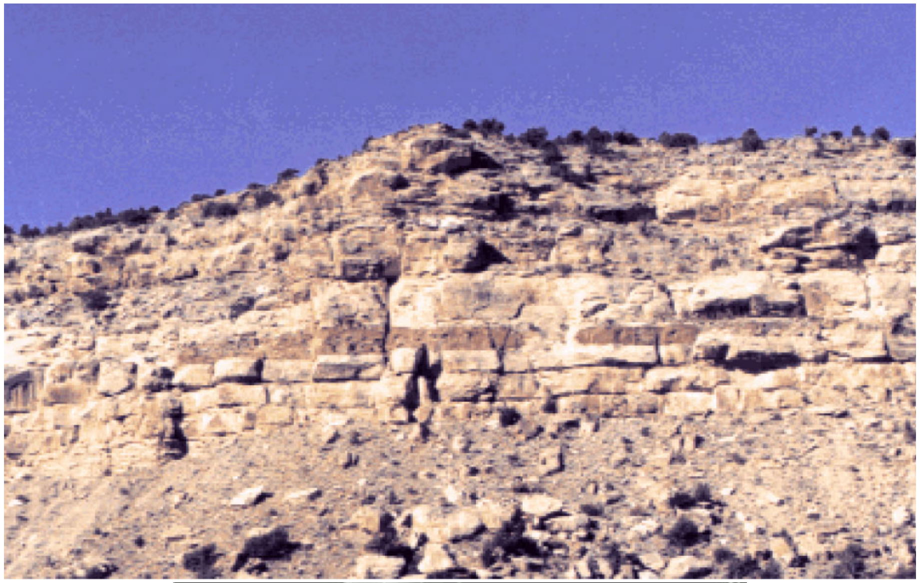
$$\forall w \in W,\ b(\mathbf{u}, w) = -\int_\Omega w\, q_d,$$

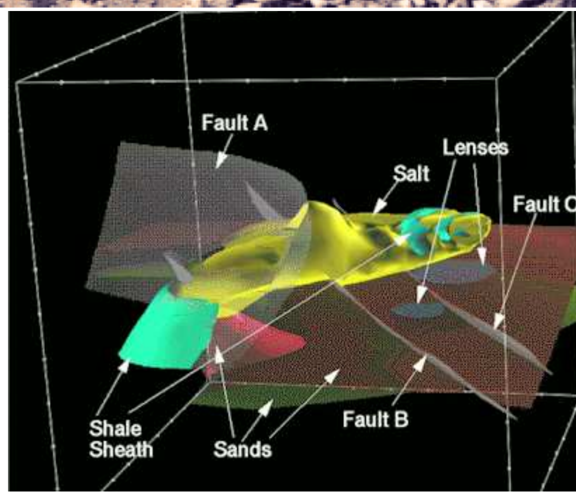
$$\forall \tilde{\mu} \in \Lambda,\ b_\Gamma(\mathbf{u}, \tilde{\mu}) = 0.$$

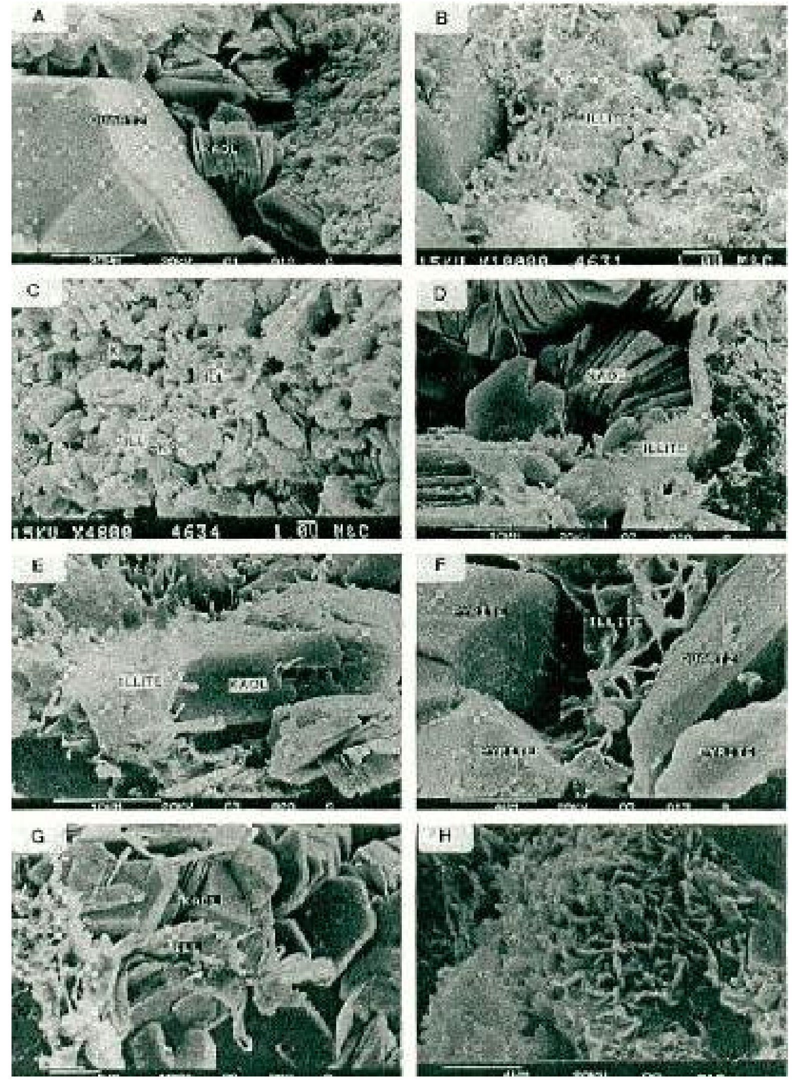
Lemma: The two variational formulations are equivalent.

$$\tilde{\lambda} = -\mathbf{T} \cdot \mathbf{n} \quad \text{on } \Gamma_{ss}, \quad \tilde{\lambda} = p_d \quad \text{on } \Gamma_{sd} \cup \Gamma_{dd}$$

Porous media scales







Full fine scale grid resolution $\Rightarrow$ large, highly coupled system of equations $\Rightarrow$ solution is computationally intractable

Multiscale methods

- Variational Multiscale Method
  - Galerkin FEM: Hughes et al; Brezzi
  - Mixed FEM: Arbogast et al
- Multiscale Finite Elements
  - Galerkin FEM: Hou, Wu, Cai, Efendiev et al
  - Mixed FEM: Chen and Hou; Aarnes et al
- Multiscale Mortar Methods: based on domain decomposition and mortar finite elements
  - Mixed FEM: Arbogast, Pencheva, Wheeler, Y.
  - DG-Mixed: Girault, Sun, Wheeler, Y.

More flexible - easy to improve global accuracy by adapting the local mortar grids

Allows for multiphysics subdomain models

Multiscale finite element/subgrid upscaling methods

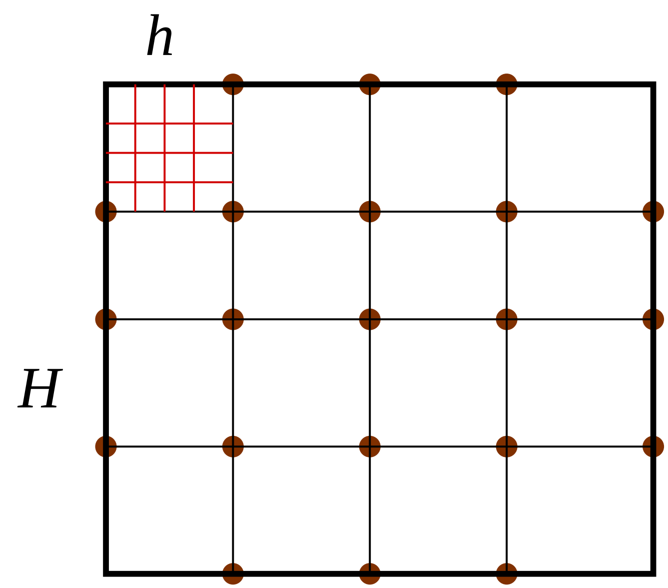
$$L_\epsilon u = f \quad \Rightarrow \quad u \in V : \ a(u,v) = (f,v) \ \forall\, v \in V.$$

Multiscale approximation: H - coarse grid, $h \approx \epsilon$ - fine grid (subgrid)

$$V_{H,h} = V_H + V_h'$$

Basis for $V_h'(E)$: $\phi_{h,i}^E$, $i = 1,\ldots,N_E$,

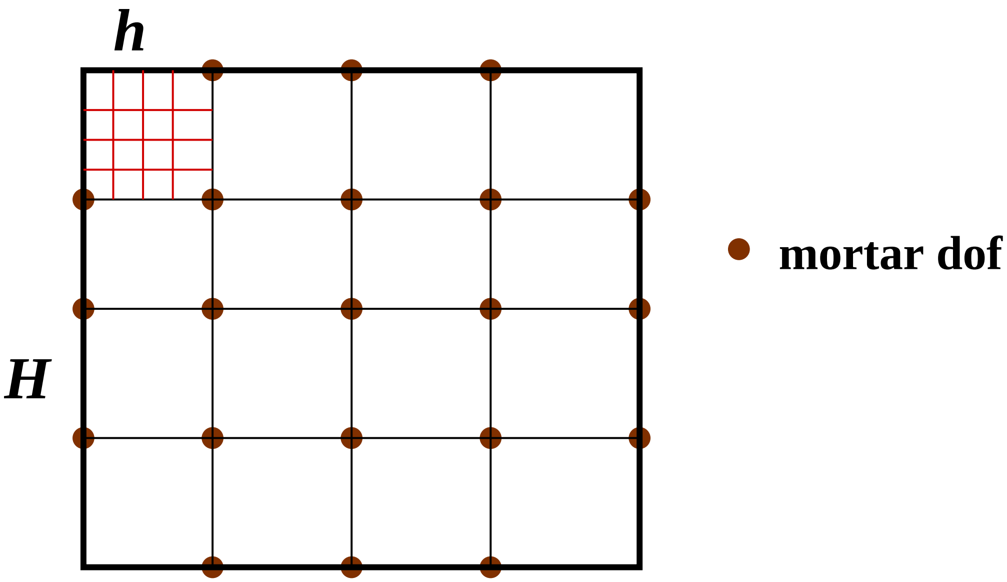
$$a_E(\phi_{H,i}^E + \phi_{h,i}^E, v_h) = 0 \quad \forall\, v_h \in V_h(E)$$

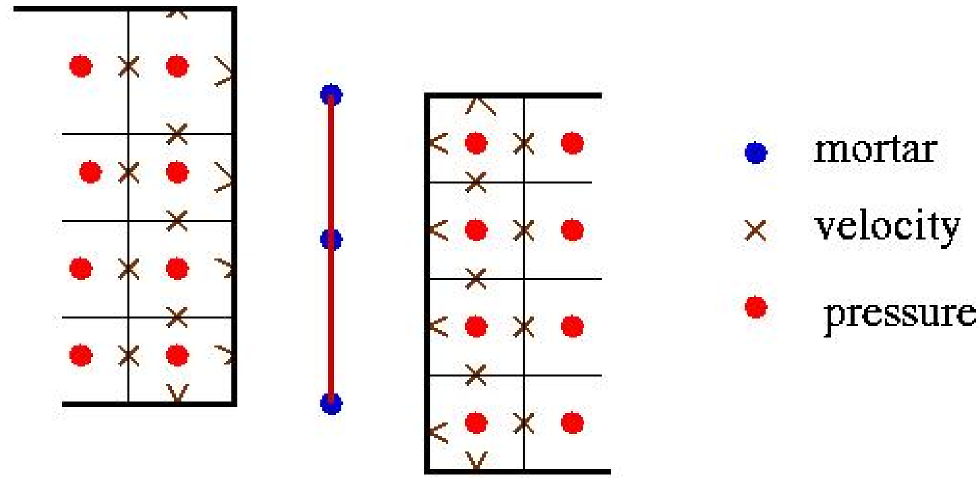


Multiscale solution: $u_{H,h} \in V_{H,h}$,

$$a(u_{H,h}, v_{H,h}) = (f, v_{H,h}) \quad \forall\, v_{H,h} \in V_{H,h}$$

Multiscale mortar approximation



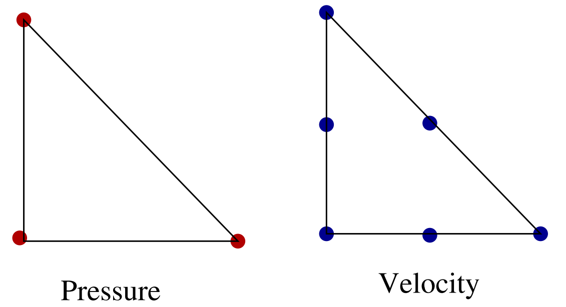


- Each block is an element of the coarse grid.
- Each block is discretized on the fine scale.
- A coarse mortar space on each interface.
- Result: Multiscale solution, fine scale on subdomains, coarse scale flux matching

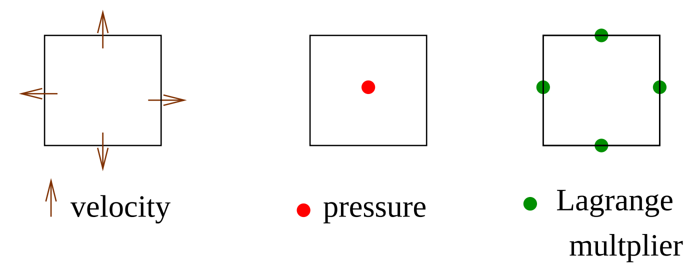
Finite element discretization

Partition $\mathcal{T}_i^h$ on Ω_i; $\mathcal{T}_i^h$ and $\mathcal{T}_j^h$ need not match at Γ_{ij}.

Stokes elements $X_{s,i}^h \times W_1^h$ in $\Omega_{s,i}$: **MINI (Arnold-Brezzi-Fortin), Taylor-Hood, Bernardi-Raugel**; contain at least polynomials of degree r_s and $r_s - 1$ resp.



Mixed finite elements $X_{d,i}^h \times W_{d,i}^h$ in $\Omega_{d,i}$: **RT, BDM, BDFM, BDDF**; contain at least polynomials of degree r_d



$$X^h := \bigoplus X_i^h, \; W^h := \bigoplus W_i^h \cap L_0^2(\Omega)$$

$\mathcal{T}_{ij}^H$ - partition of Γ_{ij}, possibly different from the traces of $\mathcal{T}_i^h$ and $\mathcal{T}_j^h$

Λ_{ij}^H: continuous or discontinuous piecewise polynomials of degree at least m_s on Γ_{ss} or m_d on Γ_{dd} and Γ_{sd}

$$\Lambda^H := \bigoplus \Lambda_{ij}^H$$

Nonconforming approximation: $\Lambda^h \not\subset \Lambda$

Multiscale mortar finite element method

Find $\mathbf{u}^h \in X^h$, $p^h \in W^h$, $\lambda^H \in \Lambda^H$:

$$\forall \mathbf{v}^h \in X^h,\ a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) + b_\Gamma(\mathbf{v}^h, \lambda^H) = \int_\Omega \mathbf{f} \cdot \mathbf{v}^h,$$

$$\forall w^h \in W^h,\ b(\mathbf{u}^h, w^h) = -\int_\Omega w^h\, q_d,$$

$$\forall \mu^H \in \Lambda^H,\ b_\Gamma(\mathbf{u}^h, \mu^H) = 0.$$

Equivalently, letting

$$V^h = \{\mathbf{v}^h \in X^h : b_\Gamma(\mathbf{v}^h, \mu^H) = 0\ \forall\, \mu^H \in \Lambda^H\},$$

Find $\mathbf{u}^h \in V^h$, $p^h \in W^h$:

$$\forall \mathbf{v}^h \in V^h,\ a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) = \int_\Omega \mathbf{f} \cdot \mathbf{v}^h,$$

$$\forall w^h \in W^h,\ b(\mathbf{u}^h, w^h) = -\int_\Omega w^h\, q_d.$$

Mortar compatibility conditions

On $\Gamma_{ij} \subset \Gamma_{dd} \cup \Gamma_{sd}$, $i < j$ (assume that Ω_j is a Darcy domain),

$$\forall\, \mu^H \in \Lambda^H, \quad \sup_{\varphi_j^h \in X_j^h \cdot \mathbf{n}} \frac{\langle \varphi_j^h, \mu^H \rangle_{\Gamma_{ij}}}{\|\varphi_j^h\|_{L^2(\Gamma_{ij})}} \geq \beta_d \|\mu^H\|_{L^2(\Gamma_{ij})}.$$

On $\Gamma_{ij} \subset \Gamma_{ss}$, $i < j$,

$$\forall\, \boldsymbol{\mu}^H \in \Lambda^H, \quad \sup_{\boldsymbol{\varphi}_j^h \in X_j^h|_{\Gamma_{ij}} \cap H_{00}^{1/2}(\Gamma_{ij})^n} \frac{\langle \boldsymbol{\varphi}_j^h, \boldsymbol{\mu}^H \rangle_{\Gamma_{ij}}}{\|\boldsymbol{\varphi}_j^h\|_{H^{1/2}(\Gamma_{ij})}} \geq \beta_s \|\boldsymbol{\mu}^H\|_{H^{-1/2}(\Gamma_{ij})}.$$

These are much more general than the mortar conditions used in [Bernardi-Maday-Patera], [Ben Belgacem] for Laplace and Stokes, and in [Layton-Schieweck-Y.], [Riviere-Y.], [Galvis-Sarkis] for Stokes-Darcy.

The above condition on Γ_{dd} is similar to the one used in [Arbogast-Cowsar-Wheeler-Y.] and [Arbogast-Pencheva-Wheeler-Y.].

Weakly continuous interpolants

$$V_s^h = \{\mathbf{v}^h \in X_s^h : \langle [\mathbf{v}^h], \boldsymbol{\mu}^H \rangle_{\Gamma_{ss}} = 0 \;\forall\, \boldsymbol{\mu}^H \in \Lambda^H\}$$

$$V_d^h = \{\mathbf{v}^h \in X_d^h : \langle [\mathbf{v}^h \cdot \mathbf{n}], \mu^H \rangle_{\Gamma_{dd}} = 0 \;\forall\, \mu^H \in \Lambda^H\}$$

There exists $\Pi_s^h : H^1(\Omega_s) \to V_s^h$ such that

$$(\mathrm{div}(\mathbf{v} - \Pi_s^h \mathbf{v}), w^h)_{\Omega_{s,i}} = 0 \;\forall w^h \in W^h,\ \forall \Omega_{s,i} \subset \Omega_s,$$

$$\sum \|\Pi_s^h \mathbf{v}\|_{H^1(\Omega_{s,i})} \le C \sum \|\mathbf{v}\|_{H^1(\Omega_{s,i})}$$

There exists $\Pi_d^h : H^1(\Omega_d) \to V_d^h$ such that

$$(\mathrm{div}(\mathbf{v} - \Pi_d^h \mathbf{v}), w^h)_{\Omega_{d,i}} = 0 \;\forall w^h \in W^h,\ \forall \Omega_{d,i} \subset \Omega_d,$$

$$\forall \Gamma_{ij} \subset \Gamma_{sd}, \quad \forall \mu^H \in \Lambda_{sd}^H, \int_{\Gamma_{ij}} \mu^H \big(\Pi_d^h(\mathbf{v}) - \Pi_s^h(\mathbf{v})\big) \cdot \mathbf{n}_s = 0,$$

$$\sum \|\Pi_d^h \mathbf{v}\|_{H(\mathrm{div};\Omega_{s,i})} \le C \sum \|\mathbf{v}\|_{H^1(\Omega_{s,i})}$$

Discrete inf-sup condition

Global weakly continuous interpolant:

$$\Pi^h : H_0^1(\Omega) \to V^h, \quad \Pi^h|_{\Omega_s} = \Pi_s^h,\ \Pi^h|_{\Omega_d} = \Pi_d^h$$

$$b(\Pi^h \mathbf{v} - \mathbf{v}, w^h) = 0 \ \forall\, w^h \in W^h, \quad \|\Pi^h \mathbf{v}\|_X \leq C\|\mathbf{v}\|_{H^1(\Omega)}.$$

Lemma:

$$\forall w^h \in W^h, \ \sup_{\mathbf{v}^h \in V^h} \frac{b(\mathbf{v}^h, w^h)}{\|\mathbf{v}^h\|_X} \geq \beta \|w^h\|_W$$

Proof: Given $w^h \in W^h$, let $\mathbf{v} \in H^1(\Omega)^n$:

$$\operatorname{div} \mathbf{v} = -w_h, \quad \|\mathbf{v}\|_{H^1(\Omega)} \leq C\|w_h\|_{L^2(\Omega)}.$$

Take $\mathbf{v}^h = \Pi^h \mathbf{v}$. $\square$

Coercivity

$$Z^h = \{\mathbf{v} \in V^h : \forall w^h \in W^h,\, b(\mathbf{v}^h, w^h) = 0\}$$

Lemma:

$$\forall \mathbf{v}^h \in Z^h\,,\; a(\mathbf{v}^h, \mathbf{v}^h) \geq \alpha \|\mathbf{v}^h\|_X^2$$

Proof: On Ω_s, discrete Korn and Poincare inequalities (Brenner):

$$\sum a_{s,i}(\mathbf{v}^h, \mathbf{v}^h) \geq \alpha_s \sum \|\mathbf{v}^h\|^2_{H^1(\Omega_{s,i})}$$

On Ω_d,

$$\sum a_{d,i}(\mathbf{v}^h, \mathbf{v}^h) \geq \alpha_d \sum \|\mathbf{v}^h\|^2_{H(\mathrm{div};\Omega_{d,i})} \;\forall \mathbf{v}^h \in Z^h \qquad \square$$

Existence and uniqueness of a discrete solution

Lemma: There exists a unique solution to: Find $\mathbf{u}^h \in V^h$, $p^h \in W^h$:

$$\forall \mathbf{v}^h \in V^h,\ a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) = (\mathbf{f}, \mathbf{v}^h),$$

$$\forall w^h \in W^h,\ b(\mathbf{u}^h, w^h) = -(q_d, w^h).$$

Proof: Follows from coercivity and discrete inf-sup condition. $\square$

Constants do not depend on the size of the subdomains.

Approximation properties of V^h

Construct an interpolant $I^h : H_0^1(\Omega) \to V^h$, $I^h\mathbf{v} = (I_s^h\mathbf{v}, I_d^h\mathbf{v})$,

$$\sum \|\mathbf{v} - I_s^h\mathbf{v}\|_{H^1(\Omega_{s,i})} \leq C \sum h^{r_s} \|\mathbf{v}\|_{H^{r_s+1}(\Omega_{s,i})},$$

$$\sum \|\mathbf{v} - I_d^h\mathbf{v}\|_{H(\mathrm{div};\Omega_{d,i})} \leq C \sum (h^{r_d+1} \|\mathbf{v}\|_{H^{r_d+1}(\Omega_{d,i})} + h^{r_s} \|\mathbf{v}\|_{H^{r_s+1}(\Omega_{s,i})}).$$

Convergence analysis

Theorem:

$$\|\mathbf{u} - \mathbf{u}^h\|_X + \|p - p_h\|_W \le C(h^{r_s} + h^{r_d+1} + H^{m_s+1/2} + H^{m_d+1/2})$$

Proof:

$$\|\mathbf{u} - \mathbf{u}^h\|_X + \|p - p_h\|_W \le C(\inf_{\mathbf{v}^h \in V^h} \|\mathbf{u} - \mathbf{v}^h\|_X + \inf_{w^h \in W^h} \|p - w_h\|_W) + \mathcal{R}^h,$$

where

$$\mathcal{R}^h = \sup_{\mathbf{v}^h \in V^h} \frac{|a(\mathbf{u}, \mathbf{v}^h) + b(\mathbf{v}^h, p) - (\mathbf{f}, \mathbf{v}^h)|}{\|\mathbf{v}^h\|_X}$$

Bound on the non-conforming error:

$$\Theta(\mathbf{v}^h) := a(\mathbf{u}, \mathbf{v}^h) + b(\mathbf{v}^h, p) - (\mathbf{f}, \mathbf{v}^h) = -b_\Gamma(\mathbf{v}_h, \lambda)$$

$$|\langle [\mathbf{v}^h \cdot \mathbf{n}], \lambda \rangle_{\Gamma_{sd}}| \le C H^{m_d+1} \|\mathbf{v}^h\|_X$$
$$|\langle [\mathbf{v}^h \cdot \mathbf{n}], \lambda \rangle_{\Gamma_{dd}}| \le C H^{m_d+1/2} \|\mathbf{v}^h\|_X$$
$$|\langle [\mathbf{v}^h], \lambda \rangle_{\Gamma_{ss}}| \le C H^{m_s+1/2} \|\mathbf{v}^h\|_X \quad \Box$$

Convergence test

$\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = [0,1] \times [\frac{1}{2}, 1]$ and $\Omega_2 = [0,1] \times [0, \frac{1}{2}]$

Taylor-Hood on triangles in Ω_1; lowest order Raviart-Thomas on rectangules in Ω_2.

$$\mathbf{u}_1 = \left[\begin{array}{l} (2-x)(1.5-y)(y-\xi) \\ -\frac{y^3}{3} + \frac{y^2}{2}(\xi + 1.5) - 1.5\xi y - 0.5 + \sin(\omega x) \end{array} \right],$$

$$\mathbf{u}_2 = \left[\begin{array}{l} \omega \cos(\omega x) y \\ \chi(y + 0.5) + \sin(\omega x) \end{array} \right],$$

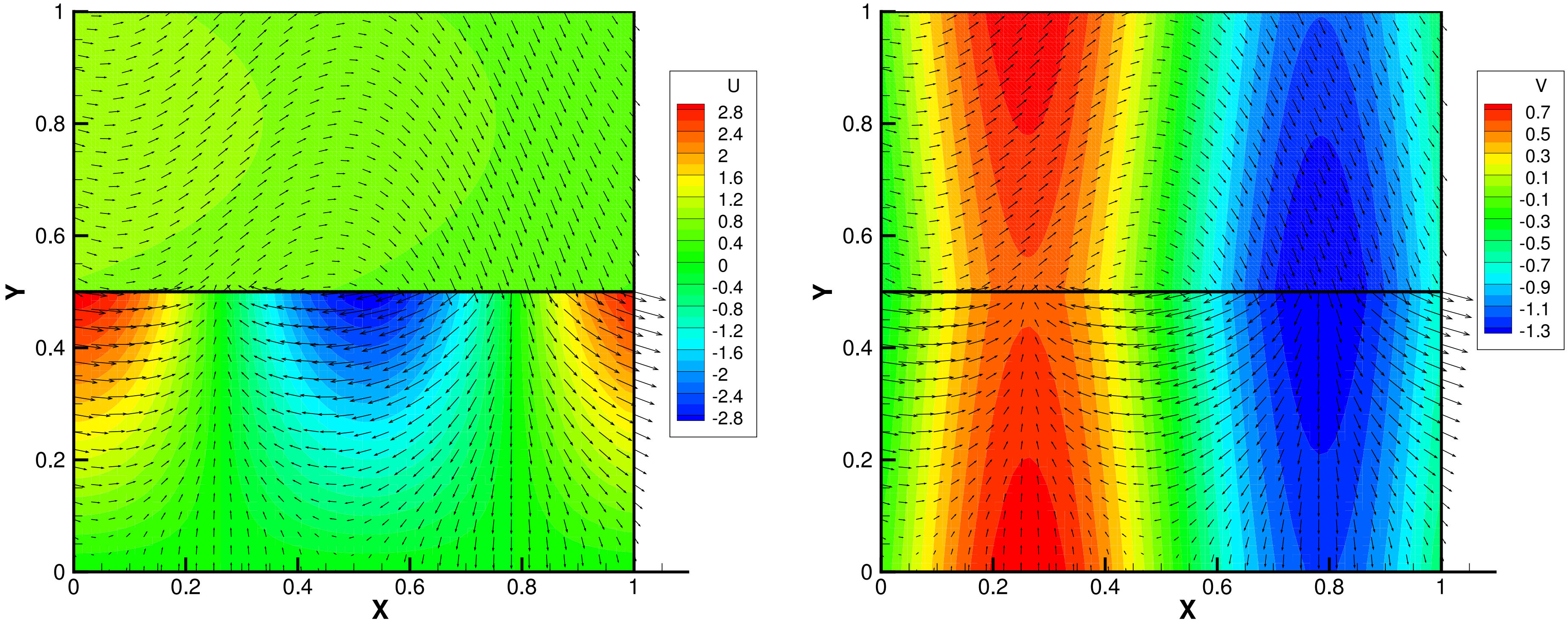
$$p_1 = -\frac{\sin(\omega x) + \chi}{2K} + \mu(0.5 - \xi) + \cos(\pi y),$$

$$p_2 = -\frac{\chi}{K}\frac{(y+0.5)^2}{2} - \frac{\sin(\omega x) y}{K},$$

where

$$\mu = 0.1,\ K = 1,\ \alpha = 0.5,\ G = \frac{\sqrt{\mu K}}{\alpha},\ \text{and } \omega = 6.$$

Computed pressure and velocity



Left: horizontal velocity; Right: vertical velocity

Convergence rates

$mesh$	$\|\mathbf{u}_1 - \mathbf{u}_{1,h}\|_{1,\Omega_1}$	$rate$	$\|p - p_{1,h}\|_{0,\Omega_1}$	$rate$
4x4	3.54e-01		3.00e-02	
8x8	8.60e-02	2.04	7.09e-03	2.08
16x16	2.15e-02	2.00	1.76e-03	2.01
32x32	5.47e-03	1.97	4.44e-04	1.99
64x64	1.40e-03	1.97	1.12e-04	1.99

Table 1: Numerical errors and convergence rates in Ω_1.

$mesh$	$\|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,\Omega_2}$	$rate$	$\|p - p_{2,h}\|_{0,\Omega_2}$	$rate$
4x4	2.16e-01		1.18e-01	
8x8	5.79e-02	1.90	2.87e-02	2.04
16x16	1.47e-02	1.98	7.13e-03	2.01
32x32	3.70e-03	1.99	1.78e-03	2.00
64x64	9.27e-04	2.00	4.45e-04	2.00

Table 2: Numerical errors and convergence rates in Ω_2.

Solution of the algebraic Stokes-Darcy system

$$\begin{pmatrix} A & B & L \\ B^T & 0 & 0 \\ L^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} R & L \\ L^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$\mathbf{x}$ - subdomain unknowns; λ - interface unknowns

Matrix is symmetric but indefinite. Form the Shur complement system:

$$L^T R^{-1} L \lambda = L^T R^{-1} f \tag{1}$$

Iterative method for (1) (e.g. CG). Each iteration requires evaluating

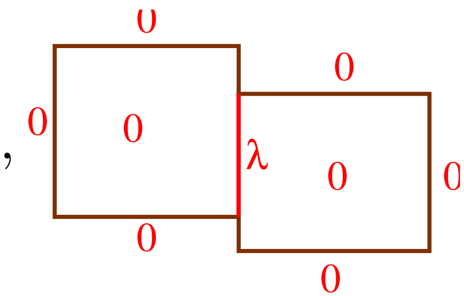
$$R^{-1} = \begin{pmatrix} R_1^{-1} & & \\ & \ldots & \\ & & R_n^{-1} \end{pmatrix},$$ i.e., solving subdomain problems.

Advantages: subdomain solves in parallel; reuse existing Stokes and Darcy codes

Domain decomposition

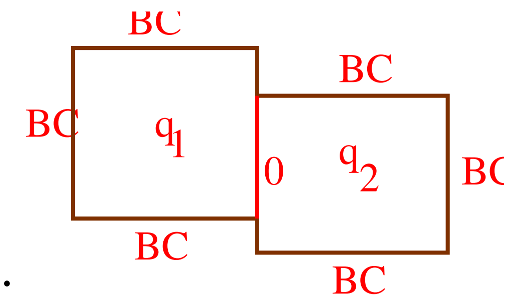
Subdomain problems with specified interface normal stress $(R^{-1}L\lambda)$:
find $(\mathbf{u}_i^{h,*}(\lambda), p_i^{h,*}(\lambda)) \in X_i^h \times W_i^h$ such that

$$\begin{aligned} a_i(\mathbf{u}_i^{h,*}(\lambda), \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i^{h,*}(\lambda)) &= -\int_\Gamma \lambda\, \mathbf{v}_i \cdot \mathbf{n}_i, \quad \mathbf{v}_i \in X_i^h, \\ b_i(\mathbf{u}_i^{h,*}(\lambda), w_i) &= 0, \quad w_i \in W_i^h. \end{aligned}$$



Complementary subdomain problems $(R^{-1}f)$: find $(\bar{\mathbf{u}}_h, \bar{p}_h) \in X_i^h \times W_i^h$ such that

$$\begin{aligned} a_i(\bar{\mathbf{u}}_i^h, \mathbf{v}_i) + b_i(\mathbf{v}_i, \bar{p}_i^h) &= \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_i, \quad \mathbf{v}_i \in X_i^h, \\ b_i(\bar{\mathbf{u}}_i^h, w_i) &= -\int_{\Omega_i} q_i\, w_i, \quad w_i \in W_i^h. \end{aligned}$$



Interface problem: find $\lambda^H \in \Lambda^H$ such that

$$s^H(\lambda^H, \mu) \equiv -b_\Gamma(\mathbf{u}^{h,*}(\lambda^H), \mu) = b_\Gamma(\bar{\mathbf{u}}^h, \mu), \quad \mu \in \Lambda^H,$$

Recover global velocity and pressure: $\mathbf{u}^h = \mathbf{u}^{h,*}(\lambda^H) + \bar{\mathbf{u}}^h$, $p^h = p^{h,*}(\lambda^H) + \bar{p}^h$.

Interface operator

Steklov–Poincaré operator: $S^H : \Lambda^H \to \Lambda^H$,

$$(S^H \lambda, \mu) = s^H(\lambda, \mu) \quad \forall\ \lambda, \mu \in \Lambda^H$$

Interface problem: Find $\lambda^H \in \Lambda^H$ such that

$$S^H \lambda^H = g^H.$$

On Γ_{sd} and Γ_{dd}:

$$S^H : \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n}\ (= \lambda^H) \to [\mathbf{u}^{h,*}(\lambda^H) \cdot \mathbf{n}]$$

On Γ_{ss}:

$$S^H : \mathbf{n} \cdot \mathbf{T}\ (= \lambda^H) \to [\mathbf{u}^{h,*}(\lambda^H)]$$

S^H is Dirichlet-to-Neumann in Darcy

S^H is Neumann-to-Dirichlet in Stokes

Different number of primary variables on different interfaces.

Interface algorithm

Apply the Conjugate Gradient method for $S^H \lambda^H = g^H$.

Computing the action of the operator (needed at each CG iteration):

- Given mortar data $\lambda^H \in \Lambda^H$, project onto subdomain grids:

$$\lambda^H \rightarrow Q_{h,i} \lambda^H$$

- Solve local Stokes and Darcy problems in parallel with boundary data $Q_{h,i} \lambda^H$

- Project velocities onto the mortar space and compute the jump:

$$\mathbf{u}_{h,i} \rightarrow Q_{h_i}^T \mathbf{u}_{h,i}, \quad S^H \lambda^H = [Q_h^T \mathbf{u}_h] \text{ on } \Gamma_{ss}; \quad S^H \lambda^H = [Q_h^T \mathbf{u}_h \cdot \mathbf{n}] \text{ on } \Gamma_{sd} \text{ and } \Gamma_{dd}$$

Condition number for two subdomains

Lemma: For all $\lambda \in \Lambda^H$,

$$C_{1,1} h \|\lambda\|^2_{\Gamma_{sd}} \leq s_1(\lambda, \lambda) \leq C_{1,2} \|\lambda\|^2_{\Gamma_{sd}} \quad (S_1 : H^{-1/2}(\Gamma_{sd}) \rightarrow H^{1/2}(\Gamma_{sd}))$$

$$C_{2,1} \|\lambda\|^2_{\Gamma_{sd}} \leq s_2(\lambda, \lambda) \leq C_{2,2} h^{-1} \|\lambda\|^2_{\Gamma_{sd}} \quad (S_2 : H^{1/2}(\Gamma_{sd}) \rightarrow H^{-1/2}(\Gamma_{sd}))$$

Theorem: $\mathrm{cond}(S_h) = O(h^{-1})$

Proof:

$$(C_{1,1} h + C_{2,1}) \|\lambda\|^2_{\Gamma_I} \leq s(\lambda, \lambda) \leq (C_{1,2} + C_{2,2} h^{-1}) \|\lambda\|^2_{\Gamma},$$

Condition number for multiple subdomains

$$C_1(h\|\lambda\|^2_{\Gamma_{ss}} + \|\lambda\|^2_{\Gamma_{dd}\cup\Gamma_{sd}}) \le s(\lambda,\lambda) \le C_2(\|\lambda\|^2_{\Gamma_{ss}} + h^{-1}\|\lambda\|^2_{\Gamma_{dd}\cup\Gamma_{sd}})$$

$$C_1 h\|\lambda\|^2_{\Gamma} \le s(\lambda,\lambda) \le C_2 h^{-1}\|\lambda\|^2_{\Gamma}$$

Theorem: $\mathsf{cond}(S_h) = O(h^{-2})$

Condition number studies: two subdomains

$1/h$	eigmin	eigmax	iter
5	2.41	4.505	9
10	1.977	7.392	14
20	1.980	14.259	21
40	1.981	28.284	28

Condition number studies: multiple subdomains

4 subdomains

$1/h$	eigmin	eigmax	iter
4	0.376	12.528	22
8	0.247	24.683	34
16	0.121	50.178	70
32	0.056	102.63	152
64	0.023	209.98	337

Varying number of subdomains

$1/H$	eigmin	eigmax	iter
1	0.443	1.830	12
2	0.061	4.788	41
4	0.061	12.221	68
8	0.061	47.639	111
16	0.042	75.640	184

Multiscale flux basis

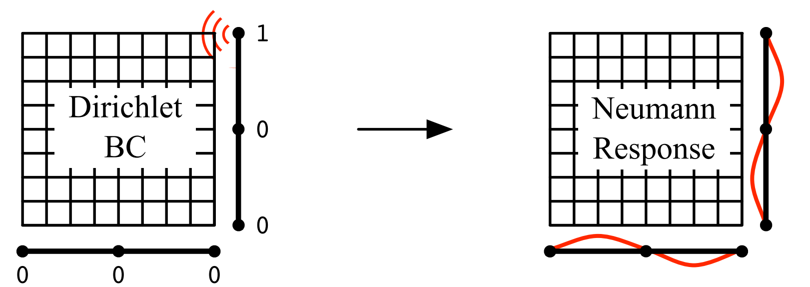
$\left\{\phi_{H,j}^{(k)}\right\}_{k=1}^{N_{H,j}}$: basis for the mortar space $M_{H,j}$ on Γ_j.

$$\lambda_{H,j} = \sum_{k=1}^{N_{H,j}} \lambda_{H,j}^{(k)} \phi_{H,j}^{(k)}$$

Multiscale flux basis:

$$\psi_{H,j}^{(k)} = A_{H,j}\phi_{H,j}^{(k)}, \quad k = 1, \dots, N_{H,j}$$

Computing $\psi_{H,j}^{(k)}$ requires solving a subdomain problem in Ω_j with Dirichlet boundary data $\phi_{H,j}^{(k)}$.



Using the pre-computed multiscale flux basis in the CG iteration:

$$A_{H,j}\lambda_{H,j} = A_{H,j}\left(\sum_{k=1}^{N_{H,j}} \lambda_{H,j}^{(k)} \phi_{H,j}^{(k)}\right) = \sum_{k=1}^{N_{H,j}} \lambda_{H,j}^{(k)} A_{H,j}\phi_{H,j}^{(k)} = \sum_{k=1}^{N_{H,j}} \lambda_{H,j}^{(k)} \psi_{H,j}^{(k)}.$$

Coupling Stokes-Darcy flow with transport

concentration of a chemical $c(x,t)$
porosity of the medium $\phi(x)$
diffusion - dispersion tensor $D_t(x,t)$ (symmetric and positive definite)
source Q

$$\phi c_t + \nabla \cdot (\mathbf{u}c - D_t \nabla c) = \phi Q \quad , \quad \forall (x,t) \in \Omega \times (0,T)$$

IC: $c(x,0) = c^0(x) \ , \ \forall x \in \Omega$

BC: $\begin{cases} (\mathbf{u}c - D_t \nabla c) \cdot \mathbf{n} = \mathbf{u}c_I \cdot \mathbf{n} \quad \text{on} \ \Gamma_{in} \\ (D_t \nabla c) \cdot \mathbf{n} = 0 \quad \text{on} \ \Gamma_{out} \end{cases}$

where $\Gamma_{in} := \{x \in \partial\Omega : \mathbf{u} \cdot \mathbf{n} < 0\}$ and $\Gamma_{out} := \{x \in \partial\Omega : \mathbf{u} \cdot \mathbf{n} \geq 0\}$.

Stability and accuracy of a discontinuous Galerkin method

$\mathbf{z} = -D_t \nabla c$; $(c_h, \mathbf{z}_h)$: approximation of $(c, \mathbf{z})$; Norm for $(c_h, \mathbf{z}_h)$:

$$|||(c_h, \mathbf{z}_h)|||^2 := \|\phi^{1/2} c_h(T)\|^2_{0,\Omega} + \int_0^T \|D_t^{-1/2} \mathbf{z}_h\|^2_{0,\Omega}\, dt$$
$$+ \int_0^T \left\{ \langle |\mathbf{u}_h \cdot \mathbf{n}|, (c_h^-)^2 \rangle_\Gamma + \sum_l \langle |\mathbf{u}_h \cdot n_l|, [c_h]^2 \rangle_{\gamma_l} \right\} dt$$

where $[c_h]$ is the jump across the interior boundary edge γ_l.

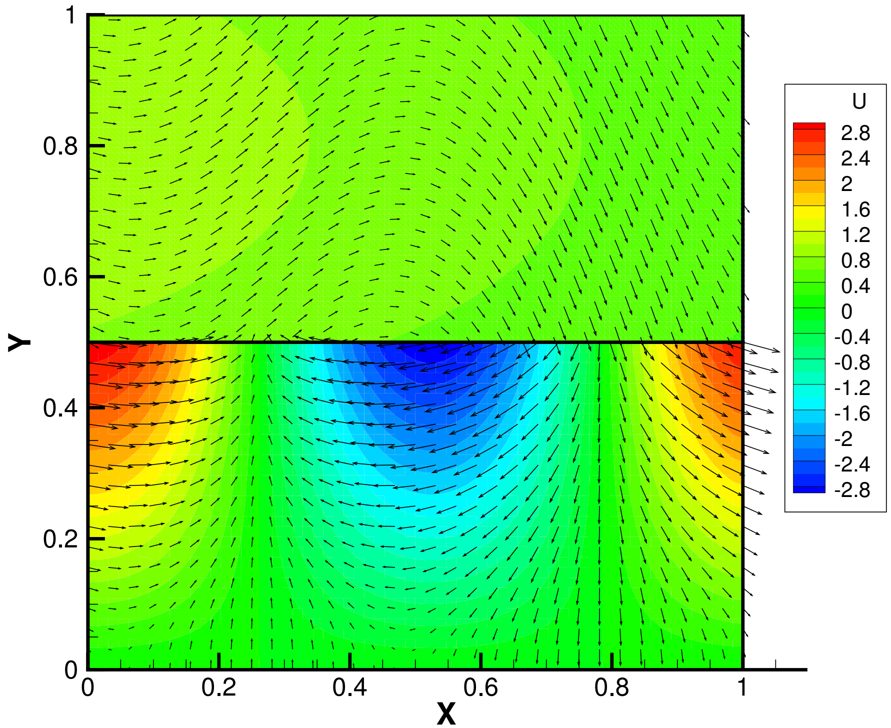
Theorem: (Vassilev and Y.)

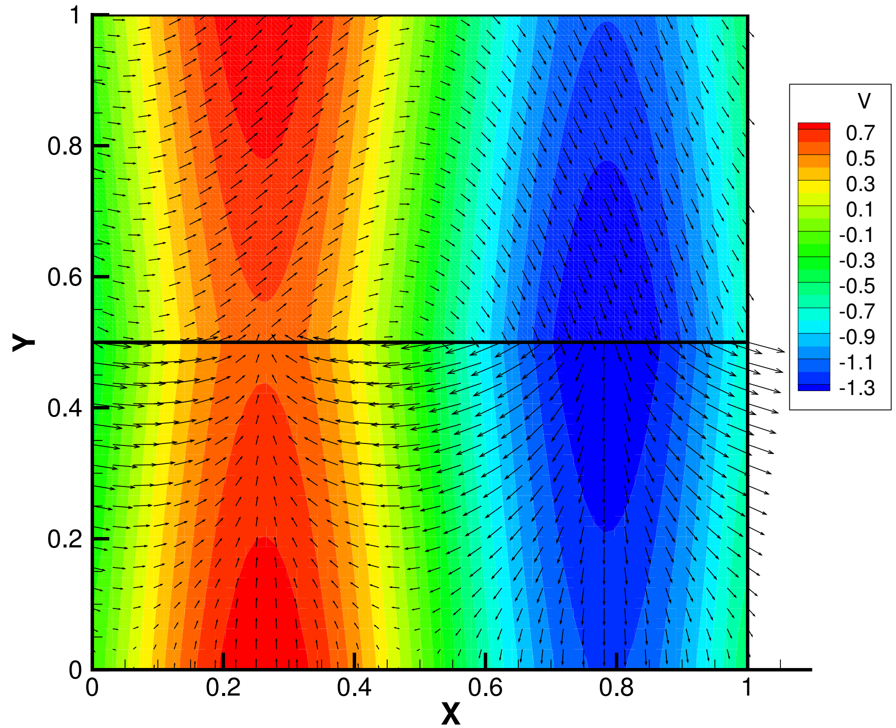
$$|||(c_h, \mathbf{z}_h)||| \le \left\{ \|\phi^{1/2} c^0\|^2_{0,\Omega} + \int_0^T \|\, c_I |\mathbf{u} \cdot \mathbf{n}|^{1/2}\|^2_{0,\Gamma_{in}}\, dt \right\}^{1/2} + \int_0^T \|\phi^{1/2} Q\|_{0,\Omega}\, dt$$

Theorem: (Vassilev and Y.)

$$|||(c - c_h, \mathbf{z} - \mathbf{z}_h)||| \le C(h^k + \|\mathbf{u} - \mathbf{u}_h\|_X)$$

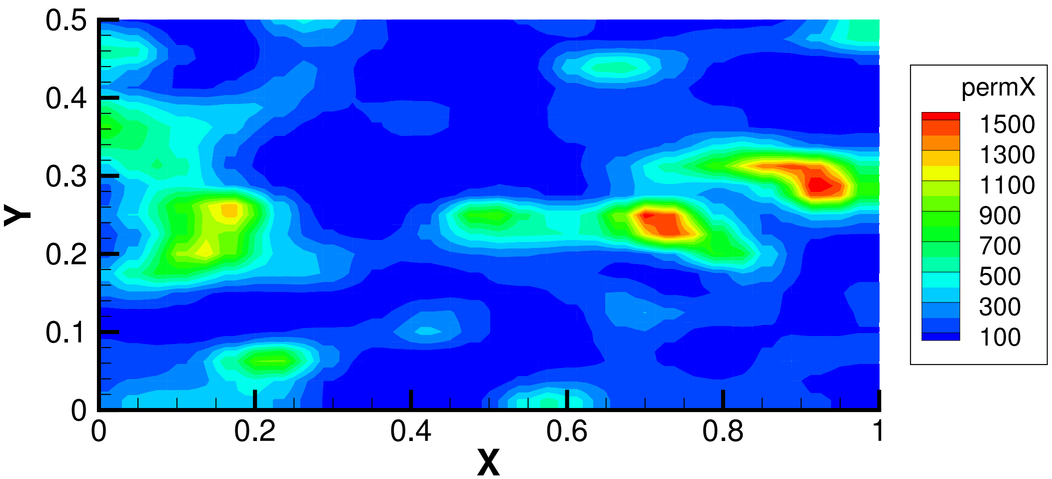
Convergence test



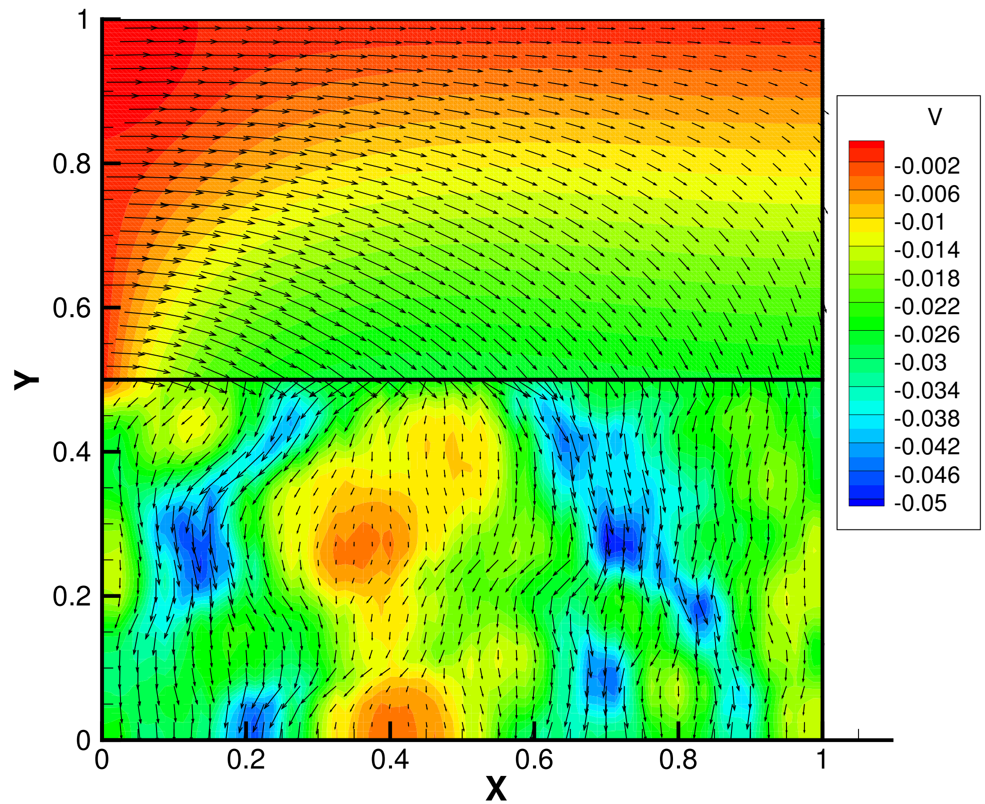


	$\mathcal{D} = 10^{-3}\,\mathbf{I}$				$\mathcal{D} = 0$	
mesh	$\|c - c_h\|_{L^\infty(L^2)}$	*rate*	$\|\mathbf{z} - \mathbf{z}_h\|_{L^2(L^2)}$	*rate*	$\|c - c_h\|_{L^\infty(L^2)}$	*rate*
4x4	1.99e+00		8.95e-03		2.07e+00	
8x8	3.27e-01	2.60	2.71e-03	1.72	3.39e-01	2.61
16x16	8.48e-02	1.95	1.20e-03	1.18	9.04e-02	1.91
32x32	2.23e-02	1.93	5.33e-04	1.17	2.59e-02	1.80
64x64	5.60e-03	2.00	1.77e-04	1.59	7.76e-03	1.74

Contaminant transport example



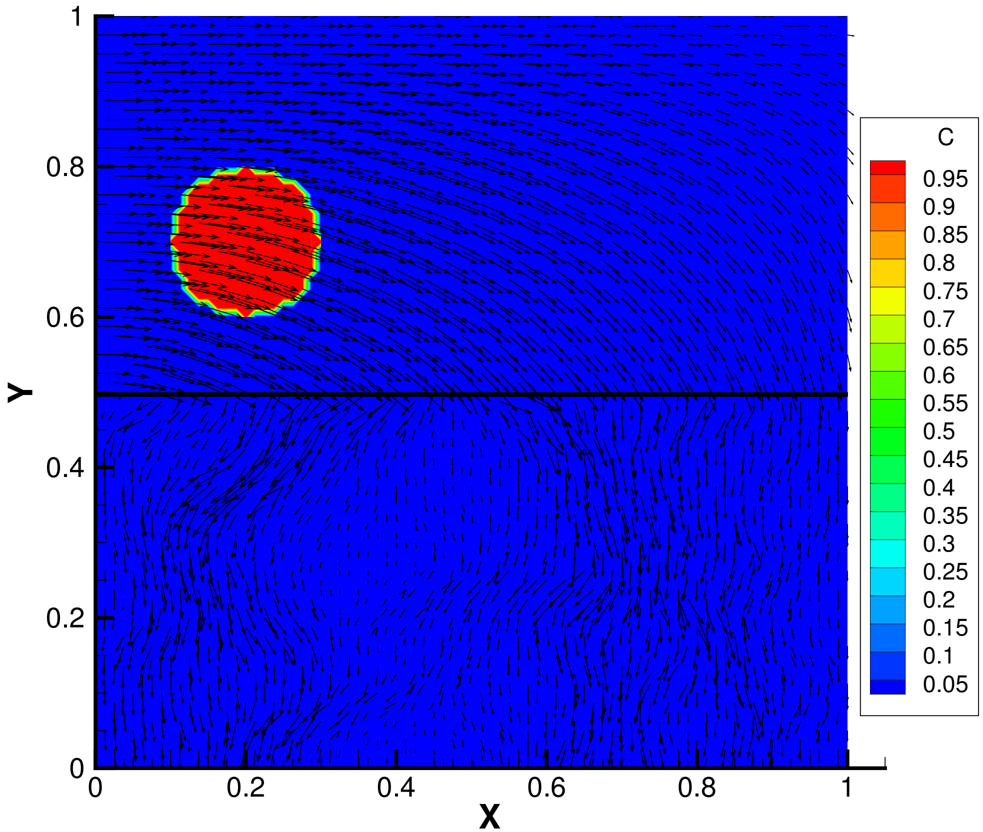
Permeability



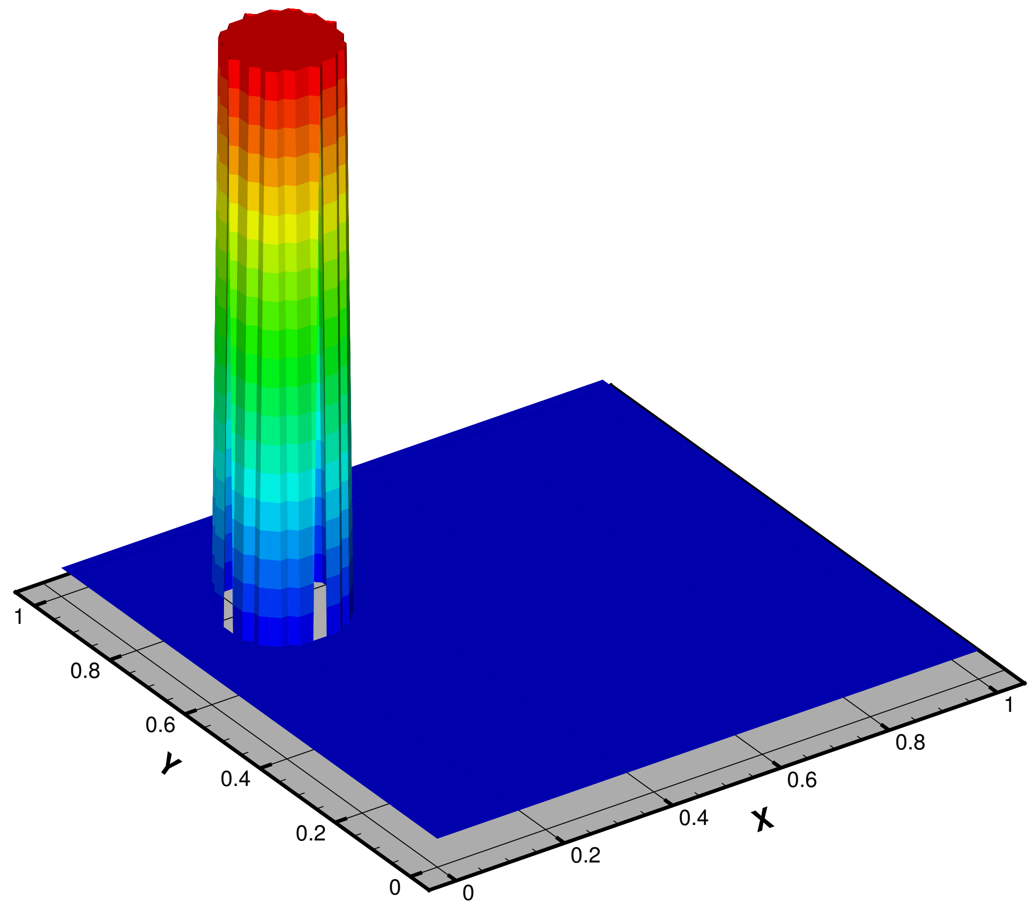
Computed velocity

$$\mathcal{D}_{\Omega_2} = \phi d_m \mathbf{I} + d_l |\mathbf{u}| \mathbf{M} + d_t |\mathbf{u}| (\mathbf{I} - \mathbf{M}), \quad \mathbf{M} = \frac{\mathbf{u}\mathbf{u}}{|\mathbf{u}|^2}$$

Transport: initial plume

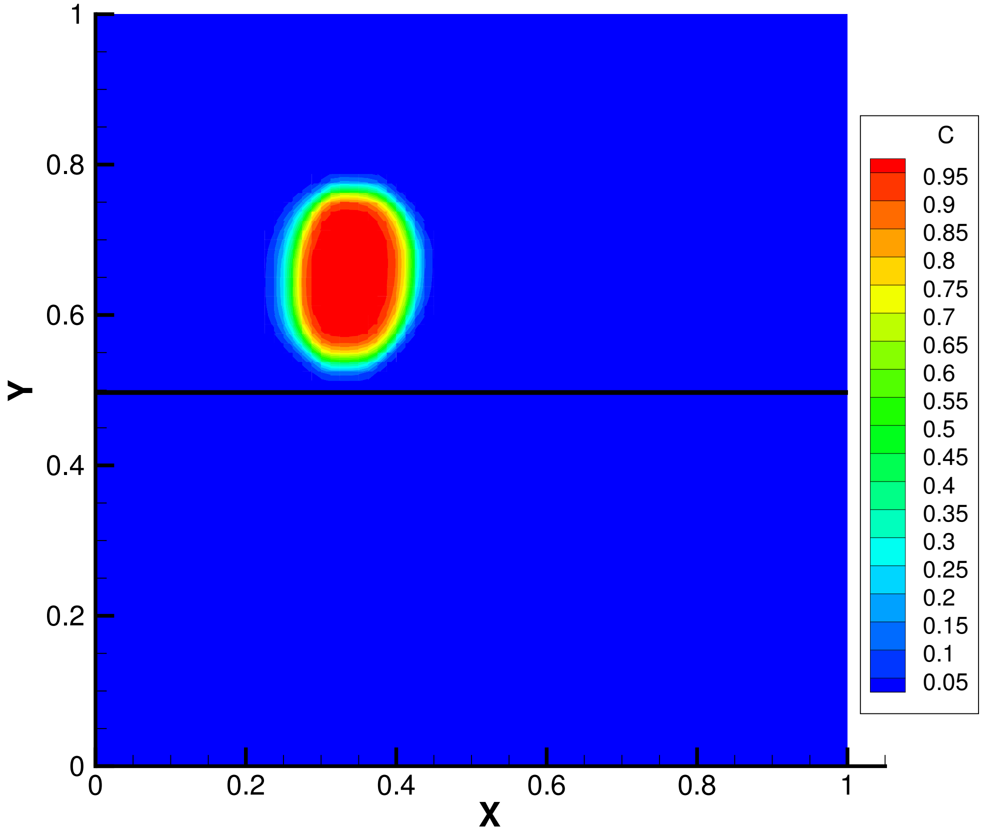


Time = 0

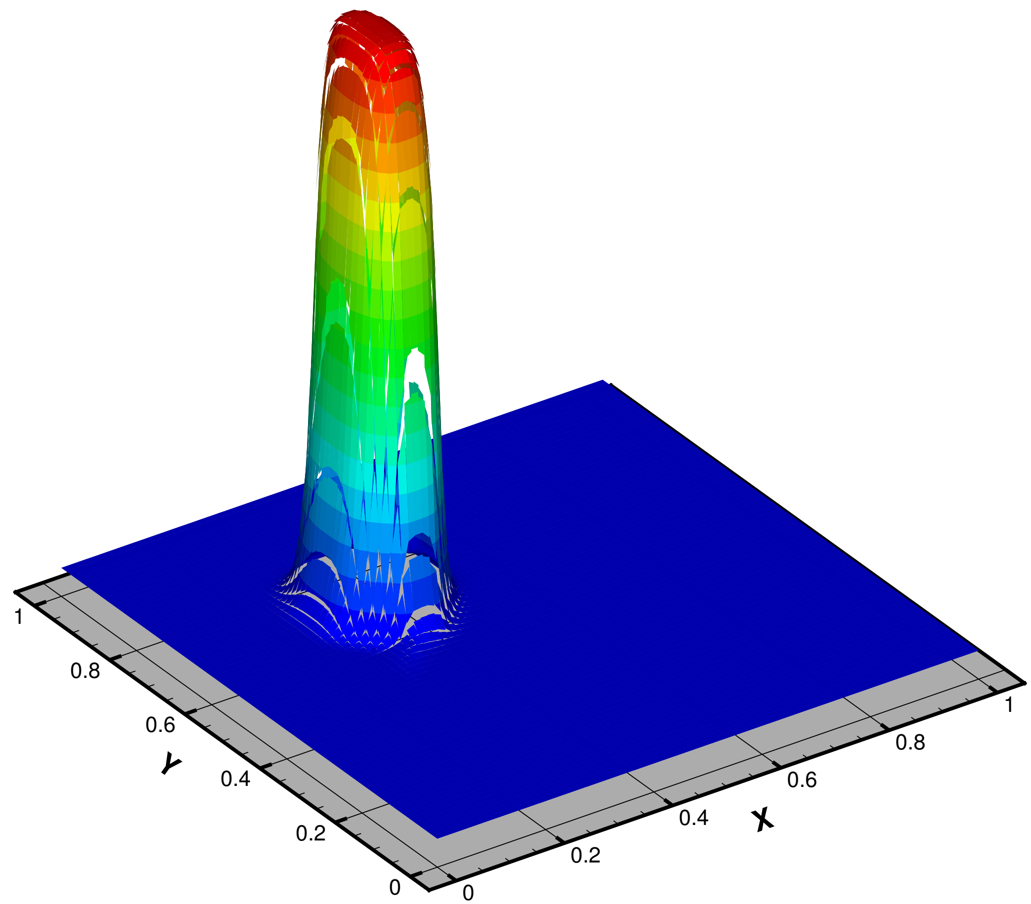


Time = 0

Transport: initial plume

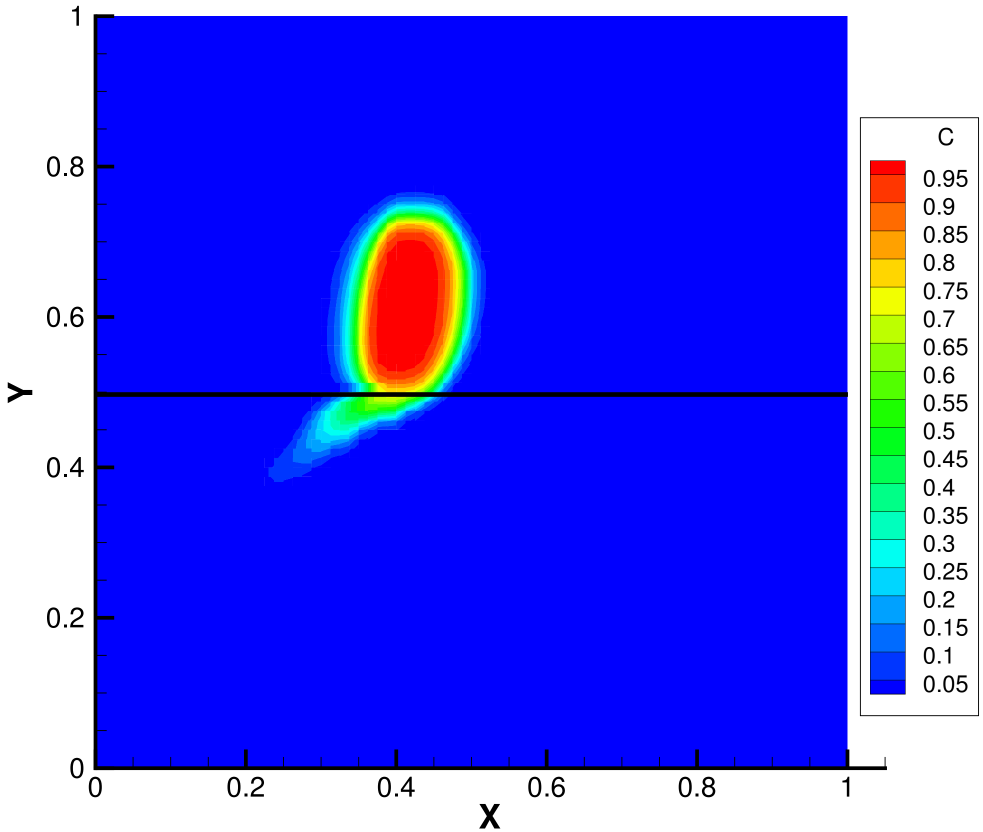


Time = 3

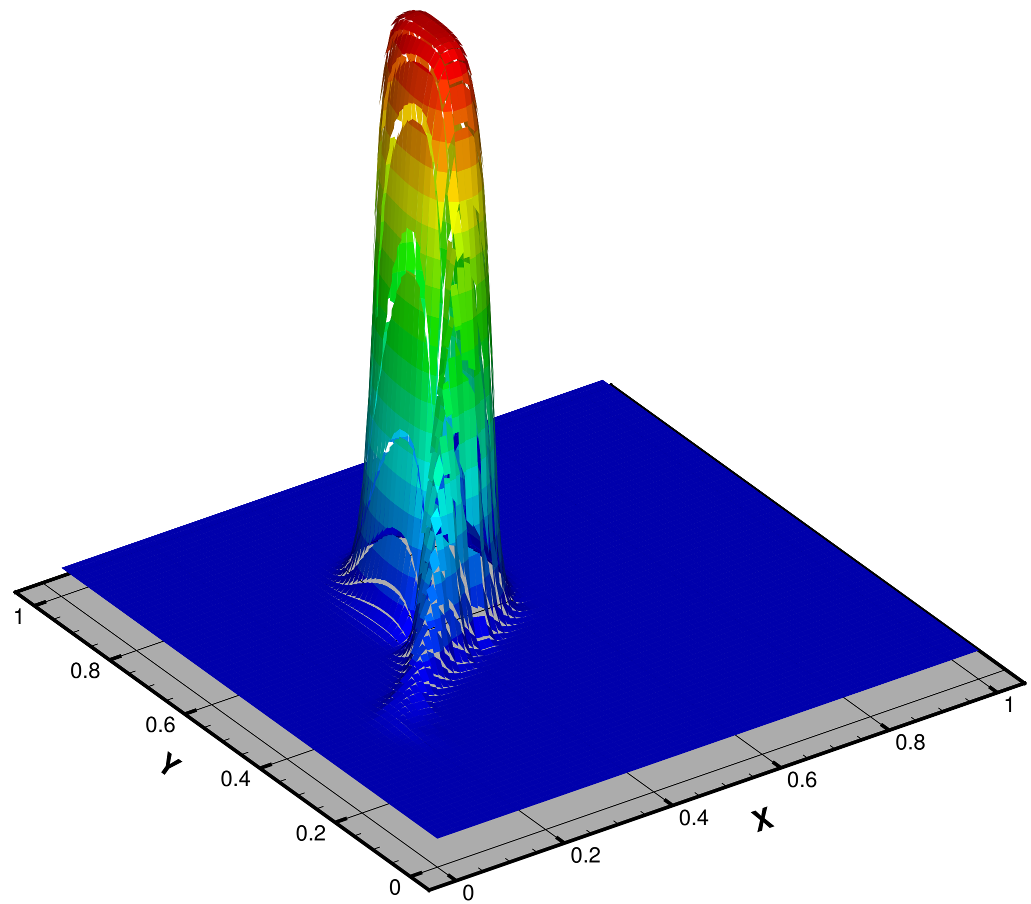


Time = 3

Transport: initial plume

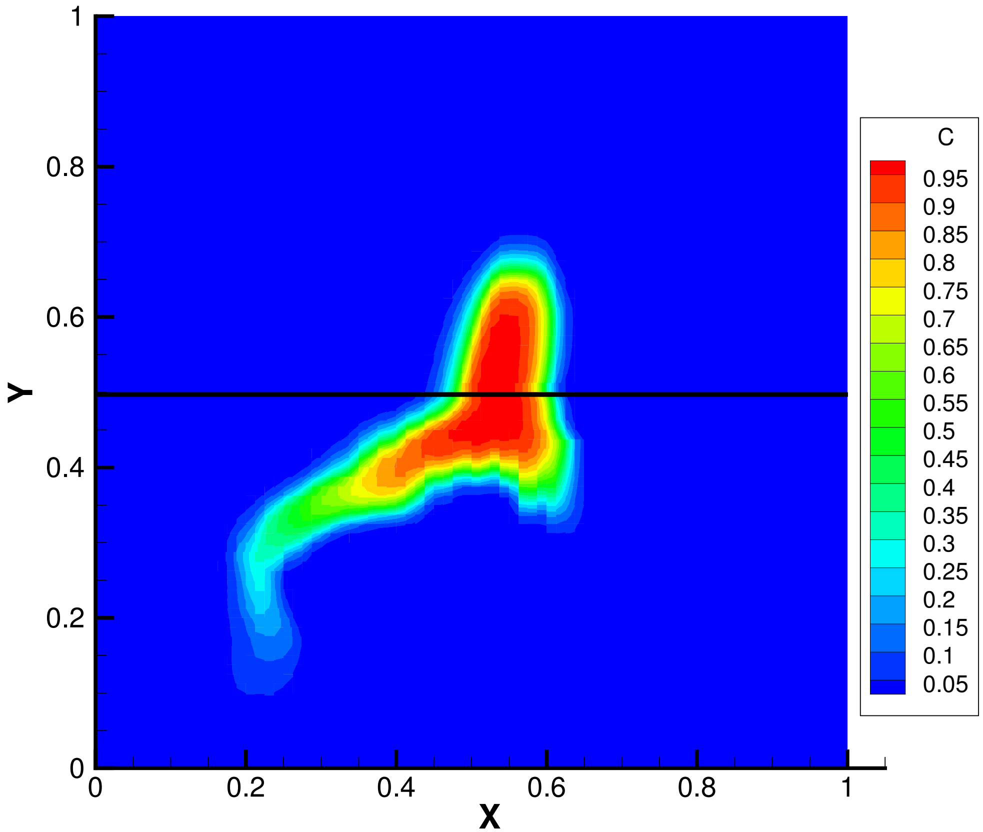


Time = 5

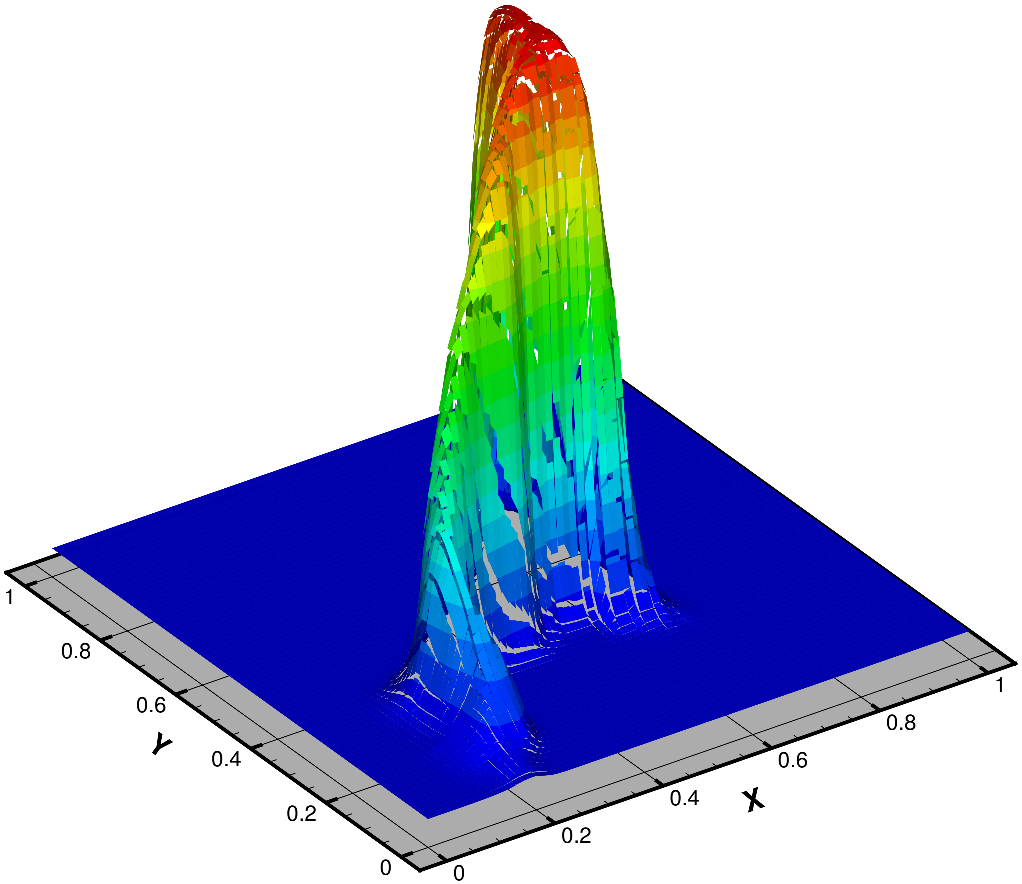


Time = 5

Transport: initial plume

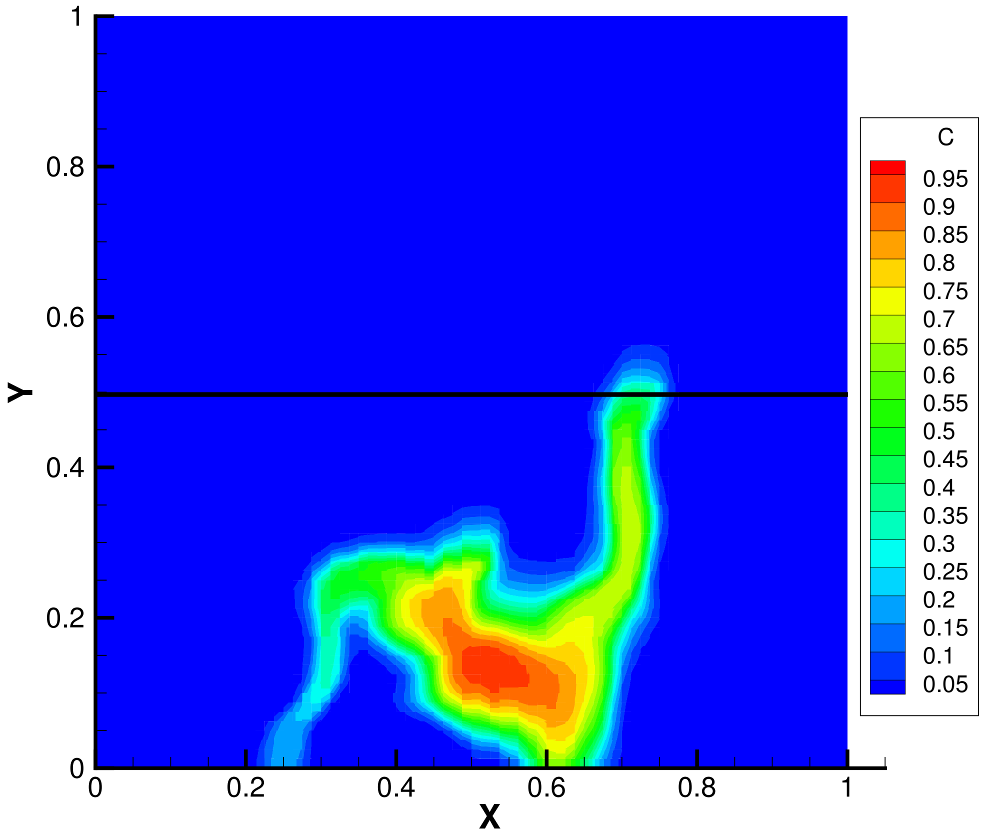


Time = 9

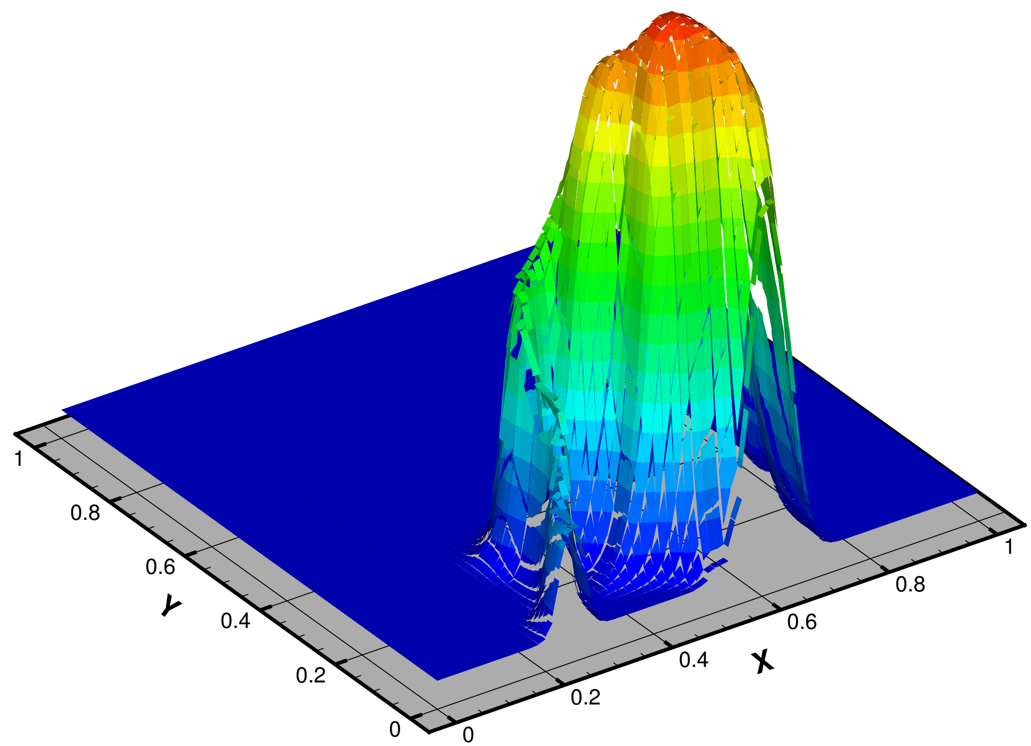


Time = 9

Transport: initial plume

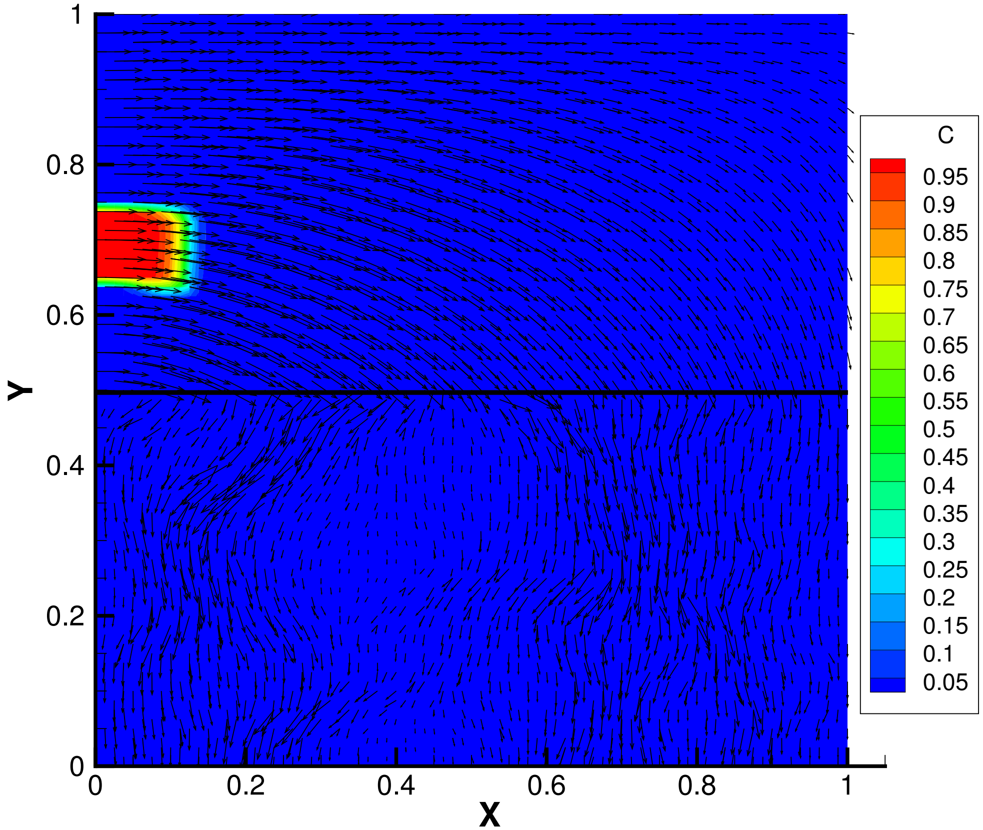


Time = 16

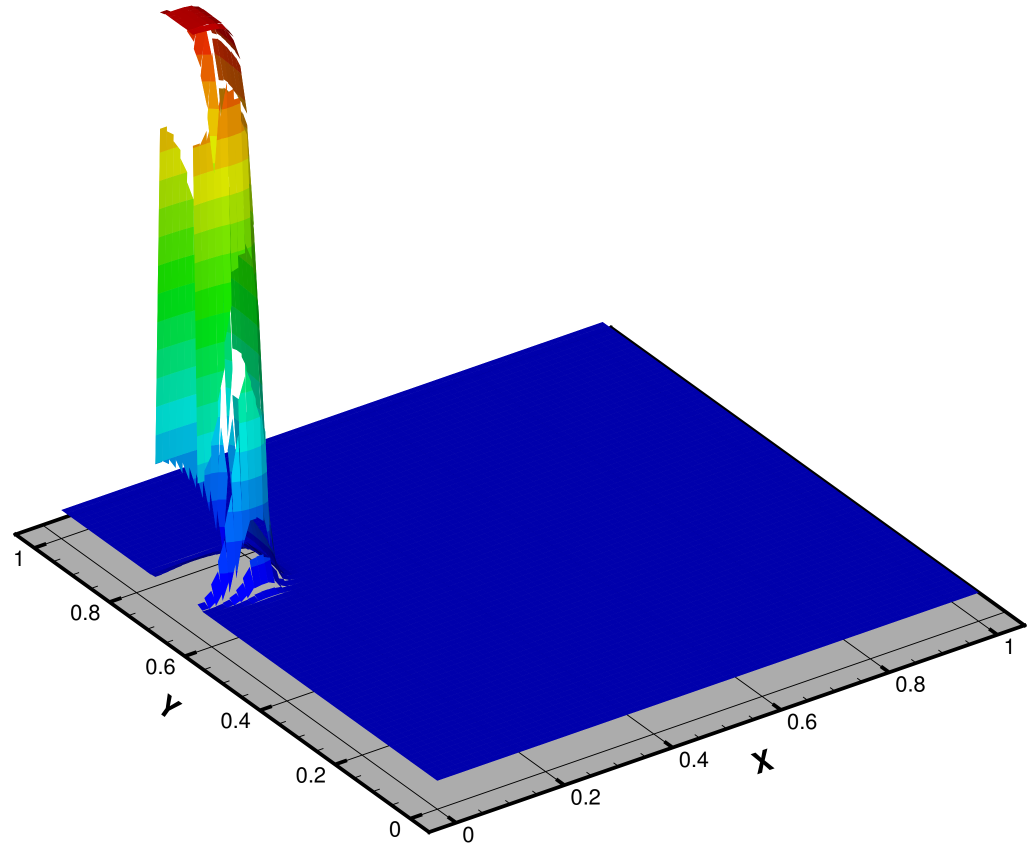


Time = 16

Transport: inflow

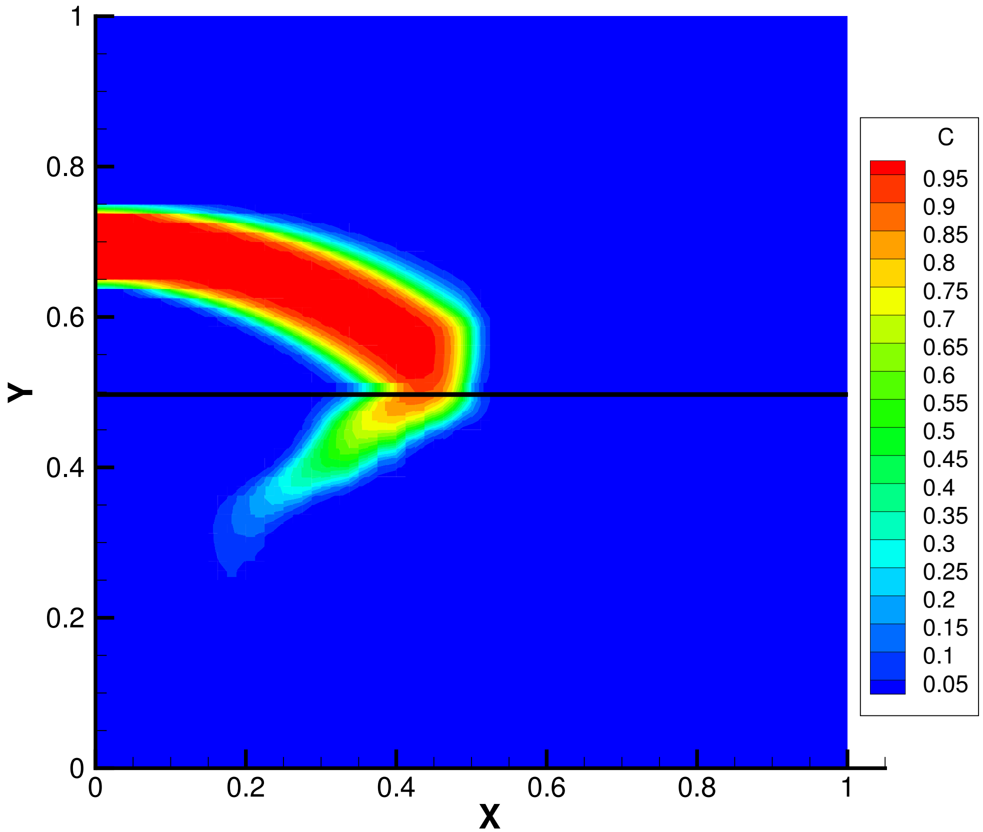


Time = 2

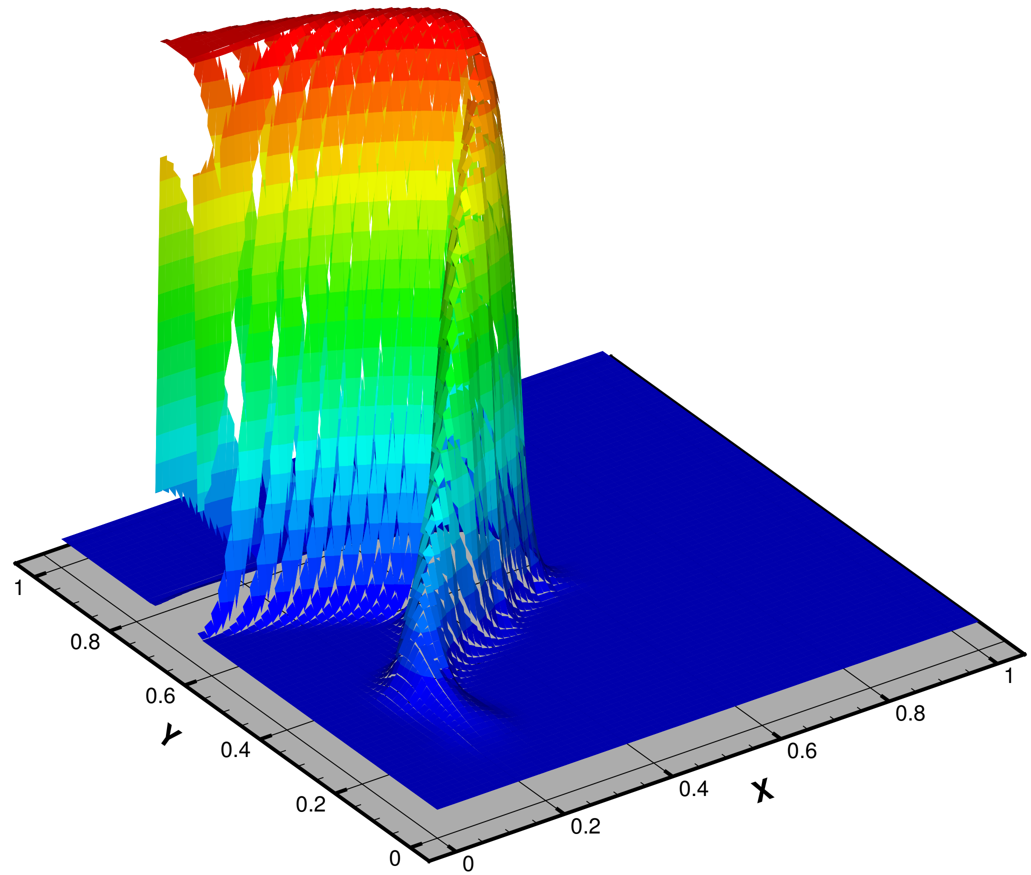


Time = 2

Transport: inflow

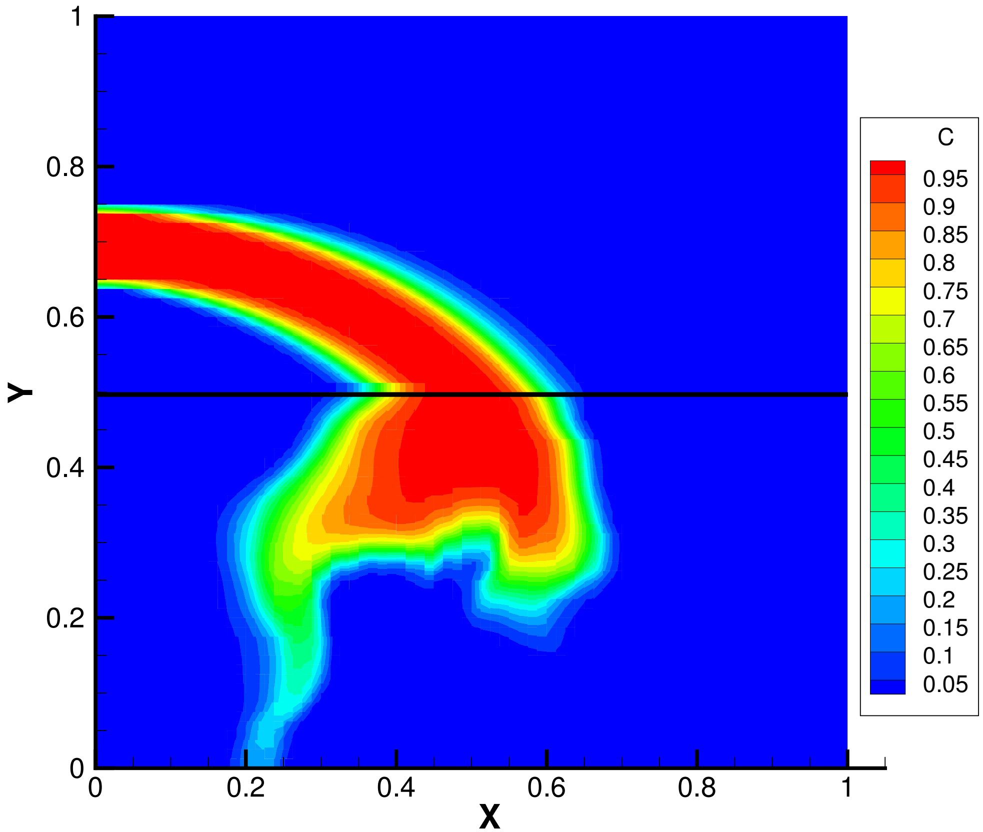


Time = 11

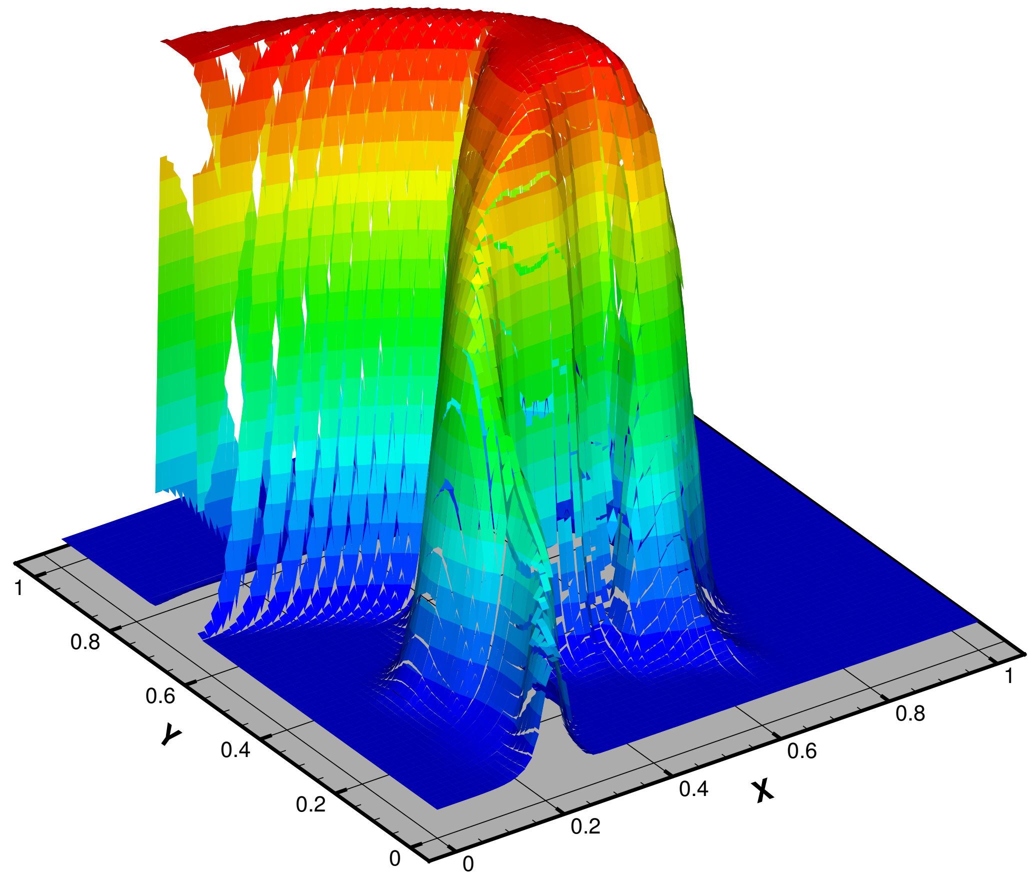


Time = 11

Transport: inflow



Time = 17



Time = 17

Summary

- Well-posed mathematical flow model based on continuity of flux and normal stress, and the Beavers-Joseph-Saffman condition

- Error analysis for mortar multiscale finite element approximations with very general mortar conditions

- Fine scale local resolution with coarse scale velocity matching

- Non-overlapping domain decomposition: s.p.d. interface problem; local Stokes and Darcy solves in parallel at every CG iteration

- Condition number of interface problem: $O(h^{-1})$ for two subdomains; $O(h^{-2})$ for multiple subdomains

- Multiscale flux basis provides efficient implementation

Current and future work

- Balancing preconditioner for the interface problem
- A posteriori error estimation and adaptive mesh refinement
- Extensions to Navier-Stokes and Brinkman