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Stabilized finite elements for Darcy flow and application to hydrothermal flows

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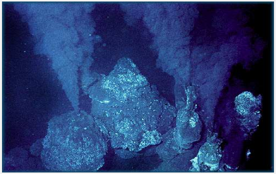
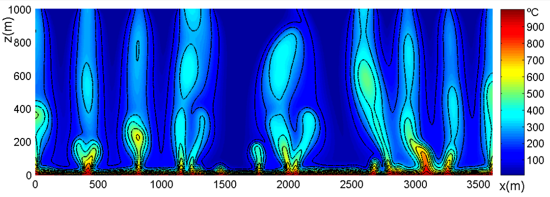
Outline

1. Introduction
2. Local projection stabilization (LPS) for Darcy-Brinkman
3. Stability and a priori error estimates
4. Model problem
5. Application to black smokers
   - Effect of numerical scheme to statistical quantities

1. Introduction

Black smokers





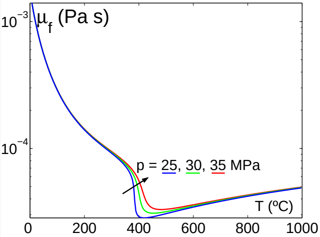
- Undersea volcanoes are interesting due to biochemical mineral formation
- Numerical simulation may help to understand their formation:
  - dynamical processes in the ocean's crust (porous media)
  - PDEs with nonlinear mathematical models for the coefficients
- Numerical schemes should be improved when model complexity increases

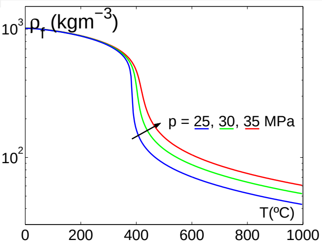
Governing equations

Darcy + temperature eq:

$$\begin{aligned}
\phi\frac{\partial \rho_f}{\partial t} + \nabla\cdot(\rho_f v) &= 0 \\
\frac{\mu_f}{k} v + \nabla p &= \rho_f g \\
(\phi\rho_f c_{p_f} + (1-\phi)\rho_r c_{p_r})\,\frac{\partial T}{\partial t} + (\rho_f c_{p_f} v\cdot\nabla)T - \nabla\cdot(D\nabla T) &= 0
\end{aligned}$$

- **Model I:** μ =const, $\rho_f = \rho_0 + \alpha(T - T_0) + \beta(p - p_0)$
- **Model II:** variable μ_f, c_p's and α, β:





2. Darcy-Brinkman-problem

$$\begin{aligned} \sigma v - \nu \Delta v + \nabla p &= f \quad \text{in } \Omega\,, \\ \operatorname{div} v &= g \quad \text{in } \Omega\,, \\ v \cdot n &= 0 \quad \text{on } \partial\Omega\,, \\ \nu v \cdot t &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Compatibility conditions $\int_\Omega g\,dx = 0$, $\int_\Omega p\,dx = 0$.

Brinkman 1947: consistent bc for Darcy

Different regimes:

- $\sigma = 0$, $g = 0$, $\nu > 0$: standard stationary Stokes system
- $\sigma \sim (t_n - t_{n-1})^{-1}$, $g = 0$: one time step of non-stationary Stokes
- $\nu = 0$: Darcy case

Variational formulation

Variational spaces:

$$
\begin{aligned}
V &:= \{v \in H^1(\Omega)^d \,:\, v = 0 \text{ on } \partial\Omega\} && \text{for } \nu > 0\,,\\
V &:= \{v \in L^2(\Omega)^d \,:\, \operatorname{div} v \in L^2(\Omega)\,, v \cdot n = 0 \text{ on } \partial\Omega\} && \text{for } \nu = 0\,,\\
X &:= V \times L^2_0(\Omega)
\end{aligned}
$$

Bilinear form for $u = (v, p), \varphi = (\phi, \xi) \in X$:

$$
A(u, \varphi) \;:=\; (\sigma v, \phi) + (\nu \nabla v, \nabla \phi) - (p, \operatorname{div} \phi) + (\operatorname{div} v, \xi)
$$

Uniformly stable and accurate fem with respect to σ and ν ?

- Stable families of finite elements for Stokes and Darcy are usually not the same.
- $H(div)$-conforming FE for Darcy are not suitable for Stokes (Mardal et al. 2002)
- Stokes MINI-element not suitable for Darcy

Finite elements for Darcy-Brinkmann

Hetereogeneous approaches:

- Layton, Schieweck, Yotov 2003:
  RT or Brezzi-Douglas-Marini for Darcy, Taylor-Hood or Mini for Stokes

Unified approaches:

- Karper, Mardal, Winther 2006: **non-conforming FE** of Crouzeix-Raviart type
- Burman, Hansbo 2007: P_1/P_0+edge stab.
- Badia, Codina 2009: ASGS and OSS

Aim:

- Discretization with equal-order elements
- stable and accurate for $\nu \in [0, \infty)$, in particular for $\nu = 0$
- stable and accurate for $\sigma \in [0, \infty)$
- uniform approach

Local projection stabilization (LPS)

- Replace X by discrete space $X_h = V_h \times Q_h$:

$$u_h \in X_h : \quad A(u_h, \varphi) + S_h(u_h, \varphi) \quad = \quad F(\varphi) \quad \forall \varphi \in X_h$$

- Symmetric stabilization term

$$S_h(u, z) \quad := \quad (\delta \kappa_h(\operatorname{div} v), \kappa_h(\operatorname{div} w)) + (\alpha \kappa_h(\nabla p), \kappa_h(\nabla q))$$

  Fluctuation operator $\kappa_h := I - \pi_h$ and local L^2-projection π_h:

$$\pi_h : L^2(\Omega) \to D_h$$

  D_h: Inf-sup condition, locally L^2-stable, interpolation property.

- Two-level method: $D_h := Q^{dc}_{2h,r-1}$
  Several one-level methods for Oseen (Matthies, Skrzypacz, Tobiska)
- Appropriate scaling of stabilization parameters in dependence of h, σ and ν.

3. Stability and a priori error estimates

- We show an inf-sup condition with respect to the triplenorm ($\theta > 0$):

$$\|\!|u|\!\|^2 \quad := \quad \underbrace{\nu|v|_1^2 + \sigma\|v\|_0^2 + S_h(y,y)}_{=|\cdot|_a^2} + \underbrace{\theta^2\|p\|_0^2}_{=|\cdot|_b^2} .$$

  θ will be fixed later, but h independent.

- Stability with respect to $|\cdot|_a$:

$$A_h(u_h, u_h) \; := \; A(u_h, u_h) + S_h(u_h, u_h) \quad = \quad |u_h|_a^2 \qquad \forall u_h \in X_h$$

- How do we obtain stability with respect to $|\cdot|_b$?

Abstract stability result

Let $(H, \|\cdot\|_H)$ be a Hilbert space where $\|\cdot\|_H$ is defined by means of two semi-norms $|\cdot|_a$ and $|\cdot|_b$

$$\|y\|_H^2 \quad := \quad |y|_a^2 + |y|_b^2$$

Lemma (Br., Schieweck 2010)

Let $B : H \times H \to \mathbb{R}$ bilinear with

$$\begin{aligned} \forall y \in H : \qquad & B(y,y) \ \geq \ c_0 |y|_a^2 \\ \forall y \in H \ \exists x \in H : \qquad & B(y,x) \ \geq \ c_2 |y|_b^2 - c_1 |y|_a^2 \quad \textit{and} \quad \|x\|_H \leq \|y\|_H \end{aligned}$$

with $c_0, c_2 > 0$ and $c_1 \geq 0$. Then there is $\gamma > 0$ s.t.

$$\forall y \in H \ \exists z \in H \setminus \{0\} : \qquad B(y,z) \ \geq \ \gamma \|y\|_H \|z\|_H$$

$$\gamma \quad = \quad \min\left\{ \frac{c_2}{1+\varrho}, \frac{c_2}{1+(c_1+c_2)/c_0} \right\} > 0,$$

where $\varrho > 1$ is arbitrary.

Stability of Darcy-Brinkman-LPS

Proposition

For patch-wise constants

$$0 < \alpha_0 h_K^2 \leq \alpha_K \quad \textit{and} \qquad 0 \leq \delta_K \leq \overline{\delta} \qquad \forall\, K \in \mathcal{T}_{2h}\,.$$

the stabilized Darcy-Brinkman bilinear form satisfies the inf-sup condition

$$\forall\, y_h \in X_h \;\exists\, z_h \in X_h \setminus \{0\} : \qquad A_h(y_h, z_h) \geq \frac{1}{2} \|\!| y_h |\!\|\, \|\!| z_h |\!\|$$

with $\theta > 0$, and for the the corrsponding discrete problem there exists always a unique solution.

$$\theta = \frac{4\gamma_c}{5 c_i \alpha_0^{-2} + 9 c_s (\sigma + \nu + c_\kappa^2 \overline{\delta})^{1/2}} > 0\,,$$

c_i, c_s, c_κ depend on κ_h but not on h.

Proof. Using previous Lemma and continuous inf-sup condition.

Proposition (A priori estimate)

Under the assumption $v \in H^{r+1}(\Omega)^d$ *and* $p \in H^{r+1}(\Omega)$*, the choice of stabilization constants*

$$\alpha \;=\; (\sigma + h^{-2}\nu)^{-1} \qquad \text{and} \qquad \delta \;=\; \sigma h^2\,,$$

leads to the a priori bound

$$|||u - u_h||| \;\lesssim\; \varrho_h h^r |v|_{r+1} + \varrho_h^{-1} h^{r+1} |p|_{r+1}\,,$$

where $\varrho_h := \sigma^{1/2} h + \nu^{1/2}$.

Proof.

- Standard techniques on bases of the discrete inf-sup condition for arbitrary coefficients.
- "Optimization" of the parameters.

Extreme cases:

- **Stokes:** $(\alpha = h^2\nu^{-1},\ \delta = 0)$

$$|||u|||^2 = \nu|v|_1^2 + \theta^2\|p\|_0^2 + S_h(u,u)$$

A priori est.:

$$|||u - u_h||| \lesssim \nu^{1/2}h^r|v|_{r+1} + \nu^{-1/2}h^r|p|_r$$

optimal

- **Darcy:** $(\alpha = \sigma^{-1},\ \delta = \sigma h^2)$

$$|||u|||^2 = \sigma\|v\|_0^2 + \theta^2\|p\|_0^2 + S_h(u,u)$$

A priori est.:

$$|||u - u_h||| \lesssim \sigma^{1/2}h^r|v|_r + \sigma^{-1/2}h^r|p|_{r+1}$$

Not yet optimal for L^2-errors $\rightarrow$ duality arguments needed

Duality argument (1)

(R1) For $\varphi \in \{\mathbf{0}\} \times L_0^2(\Omega)$, the solution $z \in Y$ of

$$A(w,z) \quad = \quad \langle \varphi, w \rangle \qquad \forall w \in Y$$

is in $H^1(\Omega)^d \times H^2(\Omega)$ and it holds the estimate

$$\|z_v\|_1 + \sigma^{-1}\|z_p\|_2 \lesssim \|\varphi_p\| .$$

- Example: $\nu = 0$, $\sigma > 0$, $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a open convex polyhedron or has C^2-boundary $\Longrightarrow$ (R1).

Proposition

Let $\nu = 0$, $\sigma > 0$ *and assume that the regularity property (R1) is satisfied. Then*

$$\|p - p_h\| \quad \lesssim \quad \sigma h^{r+1}|v|_r + h^{r+1}|p|_{r+1} .$$

Duality argument (2)

(R2) For $\varphi \in L^2(\Omega)^d \times \{0\}$, the solution $z \in Y$ of

$$A(w,z) \quad = \quad \langle \varphi, w \rangle \qquad \forall w \in Y$$

is in $H^2(\Omega)^d \times H^1(\Omega)$ and it holds the estimate

$$\nu \|z_v\|_2 + \|z_p\|_1 \lesssim \|\varphi_v\| .$$

- Example: $\nu > 0$, $\sigma \geq 0$, $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a open convex polyhedron or has C^2-boundary $\Longrightarrow$ (R2).

Proposition

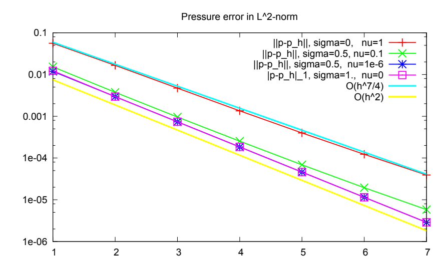
Let $\nu > 0$, $\sigma \geq 0$ and assume that the regularity property (R2) is satisfied. Then

$$\|v - v_h\| \quad \lesssim \quad h^{r+1} \left((1 + \frac{\sigma}{\nu} h^2) |v|_{r+1} + \frac{h}{\nu} |p|_{r+1} \right) .$$

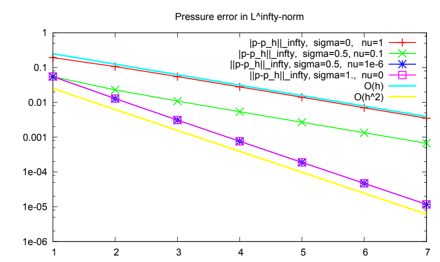
4. Numerical test

- Smooth (not divergence-free) exact solution.
- Q_1/Q_1 FEM with varying ν, σ and varying h.

Error in pressure



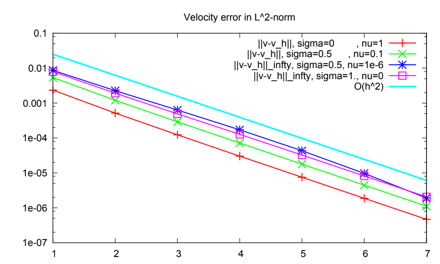
$\|p - p_h\|$



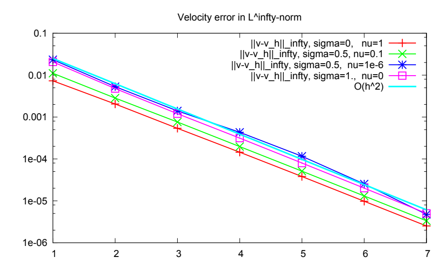
$\|p - p_h\|_\infty$

- $\nu = 0$ (Darcy): $\mathcal{O}(h^2)$ in both norms (as theory).
- $\nu = 10^{-6}$: $\mathcal{O}(h^2)$ in both norms, optimal order
- $\nu = 1$ and $\sigma = 0$ (Stokes): $\|p - p_h\| = \mathcal{O}(h^{7/4})$, $\|p - p_h\|_\infty = \mathcal{O}(h)$

Error in velocity



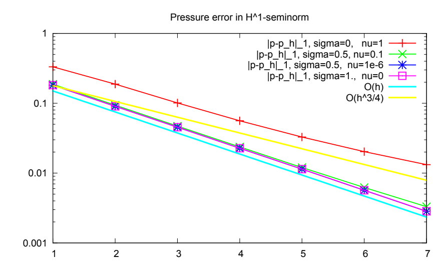
$\|v - v_h\|$



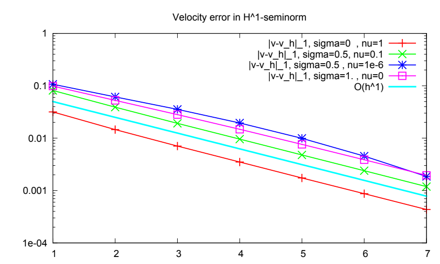
$\|v - v_h\|_\infty$

- $\mathcal{O}(h^2)$ in both norms completely independent of the parameters σ and ν.
- Optimal order of convergence as in theory ($\nu > 0$) and even for $\nu = 0$.

Errors in H^1-seminorms



$\|\nabla(p - p_h)\|$



$\|\nabla(v - v_h)\|$

- $\|\nabla(v - v_h)\| = \mathcal{O}(h)$ independent of the parameters σ and ν.
- $\|\nabla(p - p_h)\| = \mathcal{O}(h)$ for $\sigma > 0$ and $\|\nabla(p - p_h)\| = \mathcal{O}(h^{3/4})$ for $\sigma = 0$

Numerical observations:

σ	ν	$\lVert p - p_h \rVert$	$\lVert p - p_h \rVert_\infty$	$\lvert p - p_h \rvert_{H^1}$	$\lVert v - v_h \rVert$	$\lvert v - v_h \rvert_{H^1}$
0	1	$h^{7/4}$	h	$h^{3/4}$	h^2	h
0.5	0.1	$h^{7/4}$	h	h	h^2	h
0.5	10^{-6}	h^2	h^2	h	h^2	h
1	0	h^2	h^2	h	h^2	h

Numerical observations:

σ	ν	$\Vert p-p_h\Vert$	$\Vert p-p_h\Vert_\infty$	$\vert p-p_h\vert_{H^1}$	$\Vert v-v_h\Vert$	$\vert v-v_h\vert_{H^1}$
0	1	$h^{7/4}$ h	h	$h^{3/4}$	h^2 h^2	h h
0.5	0.1	$h^{7/4}$ h	h	h	h^2 h^2	h h
0.5	10^{-6}	h^2 h	h^2	h	h^2 h^2	h h
1	0	h^2 h^2	h^2	h	h^2 h	h

theory

5. Numerical schemes for hydrothermal flows

BDF-2 in time

2 alternatives in space for T eq:

- **SUPG:**

$$
\begin{aligned}
& (\gamma^n T_h^n + (\beta^n \cdot \nabla) T_h^n, \varphi_h) + (D \nabla T_h^n, \nabla \varphi) + \\
\sum_{K \in \mathcal{T}_h} \delta_K (\gamma^n T_h^n + (\beta^n \cdot \nabla) T_h^n - \nabla \cdot (D \nabla T_h^n) - f^n, (\beta^n \cdot \nabla) \varphi_h)_K & = (f^n, \varphi) , \\
& f^n = \tfrac{4}{3} \gamma^{n-1} T_h^{n-1} - \tfrac{1}{3} \gamma^{n-2} T_h^{n-2}
\end{aligned}
$$

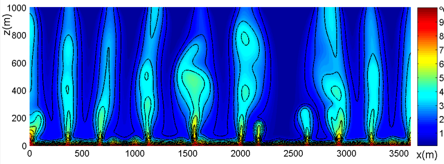
- **SLM: Semi-Lagrangian method**

$$
(\gamma^n T_h^n, \varphi_h) + (D \nabla T_h^n, \nabla \varphi_h) = (\tfrac{4}{3} \gamma^{n-1} \overline{T}_h^{n-1} - \tfrac{1}{3} \gamma^{n-2} \overline{T}_h^{n-2}, \varphi)
$$

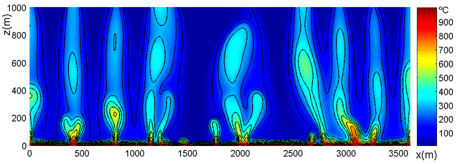
$$
\overline{T}_h^{n-l}(x) = T_h^{n-l}(X(x, t_n; t_{n-l})) , \quad l = 1, 2 .
$$

Discrete solutions at the same time instant:

SUPG



SLM



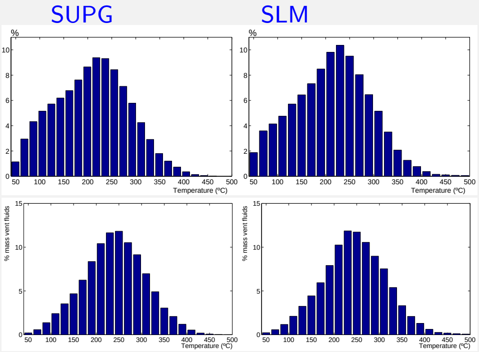
- Configuration of Coumou, Driesner, Geiger et al. (2006)
- Rel. fine mesh with 50,032 quadratic elements. Total time 4,000 a.
- System is unstable, numerical errors trigger instabilities.
- Direct comparison of the discrete solutions are not meaningful.
- One should compare statistical quantities.

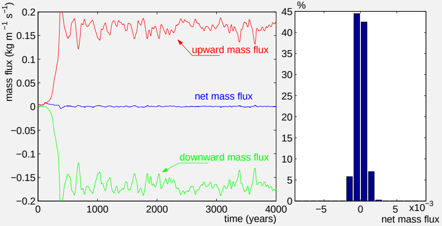
Statistical quantities for constant coefficients (Model I)

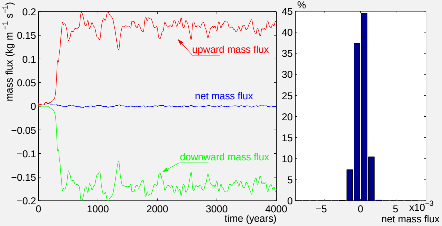
We concentrate on three different quantities:

- R_1: T at upper bd averaged over total time and all mesh points
- R_2 : momentum $\rho_f \mathbf{v}$ at top boundary as function of T
- R_3: up- and downward mass flux:

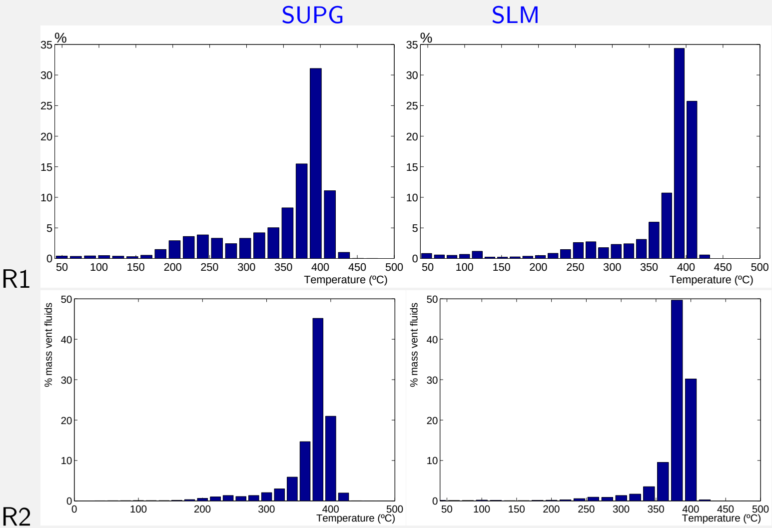
$$\dot{m}^{+/-} = \int_{\Gamma_{top}} (\rho_f \mathbf{v}_z)^{+/-} \, ds \,,$$







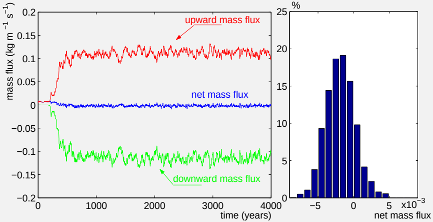
Variabel coefficients (Model II)

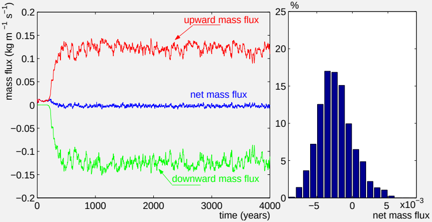


- Qualitatively similar structure
- Quantitatively: Substantial differences compared to Model I (const coef).

SUPG SLM

R3:





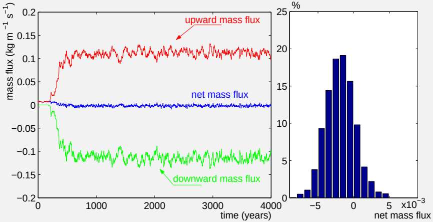
Differences quantitatively:

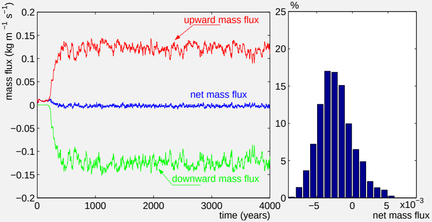
$$E_i = \sum_{k=0}^{n} |R_{k,SFEM} - R_{k,SLM}|$$

i	Model I	Model II
1	10.8 %	123.1 %
2	4.3 %	28.1 %

SUPG SLM

R3:





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Considerable impact of numerical scheme even on statistical quantites.

SUPG SLM

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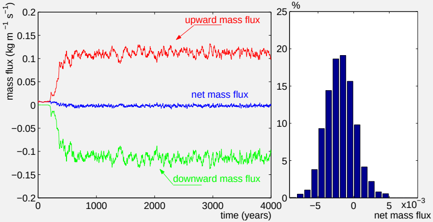

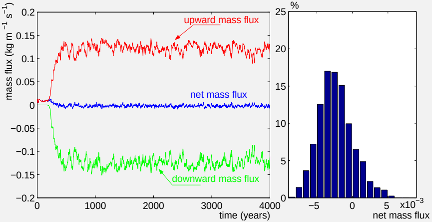

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Considerable impact of numerical scheme even on statistical quantites.

There is still a demand for improved discretizations for complex nonlinear models.

Conclusion

1. Uniform approach with LPS for Darcy-Stokes, i.e. Darcy-Brinkman model,
2. Robustness with respect to model parameters, even in extremal cases
   - for Darcy and Stokes: optimal order of convergence
3. FEM for hydrothermal flows:
   - **large** effect of the numerical scheme
   - **small** effect of the numerical scheme on statistical quantities for eq with constant coefficients
   - **considerable** effect of the numerical scheme on statistical quantities for eq with variable coefficients.
4. Improvement of numerical schemes (accuacy) has to go hand in hand with increasing model complexity.

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Thanks a lot !