IMMANANTS AND THE DUAL CANONICAL BASIS OF THE QUANTUM POLYNOMIAL RING

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Outline

(1) The quantum polynomial ring
(2) Quantum immanants and the dual canonical basis
(3) An immanant formulation of the dual canonical basis of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$.
(4) Some quantum analogs.
(5) Total nonnegativity hierarchy

The quantum polynomial ring $\mathcal{A}(n;q)$

Let $\mathcal{A}(n;q) \cong \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]\langle x_{1,1}, \ldots, x_{n,n}\rangle$, modulo

$$
\begin{aligned}
x_{i,\ell}x_{j,k} &= x_{j,k}x_{i,\ell} && \text{if } i<j,\ k<\ell,\\
x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell} && \text{if } k<\ell,\\
x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k} && \text{if } i<j,\\
x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k} && \text{if } i<j,\ k<\ell.
\end{aligned}
$$

We have $\mathcal{O}_q(SL(n,\mathbb{C})) \cong \mathcal{A}(n;q)/(\det_q(x) - 1)$, where

$$\det_q(x) = \sum_{v\in\mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(v)} x_{1,v_1}\cdots x_{n,v_n} = \sum_{v\in\mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(v)} x^{e,v}.$$

$\mathrm{span}\{x^{e,v} \mid v \in \mathfrak{S}_n\}$ = (quantum) *immanant space.*

Multigrading of $\mathcal{A}(n;q)$ and immanants

$$\mathcal{A}(n;q) = \bigoplus_{r\geq 0} \bigoplus_{(L,M)} \mathcal{A}_{L,M}(n;q),$$

over r-element multisets L, M of $[n]$.

Ex: $x_{1,2}^2 x_{3,1} x_{3,2} - q^{\frac{1}{2}} x_{1,1} x_{1,2} x_{3,2}^2 \in \mathcal{A}_{1133,1222}(3;q)$.

By relations, immanant space is

$$\begin{aligned} \mathcal{A}_{[n],[n]}(n;q) &= \mathrm{span}\{x_{u_1,v_1} \cdots x_{u_n,v_n} \,|\, u, v \in \mathfrak{S}_n\} \\ &= \mathrm{span}\{x_{1,v_1} \cdots x_{n,v_n} \,|\, v \in \mathfrak{S}_n\}. \end{aligned}$$

Natural basis of $\mathcal{A}_{L,M}(n;q)$

Let $x_{L,M}$ be the L, M generalized submatrix of x.
Let generators I, J of $\mathfrak{S}_r$ stabilize $x_{L,M}$.

Ex: $$x_{1133,1222} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \end{bmatrix}, \qquad \begin{gathered} I = \{s_1, s_3\}, \\ J = \{s_2, s_3\}, \\ \mathfrak{S}_r = \mathfrak{S}_4. \end{gathered}$$

Natural basis of $\mathcal{A}_{L,M}(n;q)$ is $\{(x_{L,M})^{e,v} \mid v \in W_+^{I,J}\}$, where $W_+^{I,J} = \{v \in \mathfrak{S}_r \mid v \text{ maximal in } W_I v W_J\}$.

$$W_+^{I,J} = \{4132, 4321\},$$

Ex: $$(x_{1133,1222})^{1234,4132} = x_{1,2}x_{1,1}x_{3,2}x_{3,2} = q^{\frac{1}{2}}x_{1,1}x_{1,2}x_{3,2}^2,$$
$$(x_{1133,1222})^{1234,4321} = x_{1,2}x_{1,2}x_{3,2}x_{3,1} = q^{\frac{1}{2}}x_{1,2}^2x_{3,1}x_{3,2}.$$

Canonical bases

Modification $\dot{U}$ of $U_q(\mathfrak{sl}(n,\mathbb{C}))$ has *canonical basis*.
This aids in construction of $U_q(\mathfrak{sl}(n,\mathbb{C}))$-modules [L 90].

Modification $\mathcal{A}(n;q)$ of $\mathcal{O}_q(SL(n,\mathbb{C}))$ has *dual canonical basis*.
This aids in construction of $U_q(\mathfrak{sl}(n,\mathbb{C}))$-modules [T 91, D 92].

$U_q(\mathfrak{sl}(n,\mathbb{C}))$, $\mathcal{O}_q(SL(n,\mathbb{C}))$ are dual Hopf algebras.
$\dot{U}$, $\mathcal{A}(n;q)$ are not Hopf algebras.
Explicit duality of bases not published [D, G-L].
Some choices are involved.

Dual canonical basis of $\mathcal{A}_{L,M}(n;q)$

Define the *bar involution* on $\mathcal{A}(n;q)$ by $\overline{q} = q^{-1}$ and

$$\overline{x_{a_1,b_1} \cdots x_{a_r,b_r}} = (q^{\frac{1}{2}})^{\alpha(a)-\alpha(b)} x_{a_r,b_r} \cdots x_{a_1,b_1},$$

where $\alpha(a) = \#\{(i,j) \,|\, i < j, a_i = a_j\}$.

Theorem: (L) $\mathcal{A}_{L,M}(n;q)$ has a unique bar-invariant basis $\{B_w^{L,M}(x;q) \,|\, w \in W_+^{I,J}\}$ satisfying

$$B_v^{L,M}(x;q) \in (x_{L,M})^{e,v} + \sum_{w>v} q^{\frac{-1}{2}}\mathbb{Z}[q^{\frac{-1}{2}}](x_{L,M})^{e,w}.$$

Call this the *dual canonical basis*.

Specializations at $q = 1$ have important nonnegativity properties [L, H, R-S, S] and applications [L-P-P].

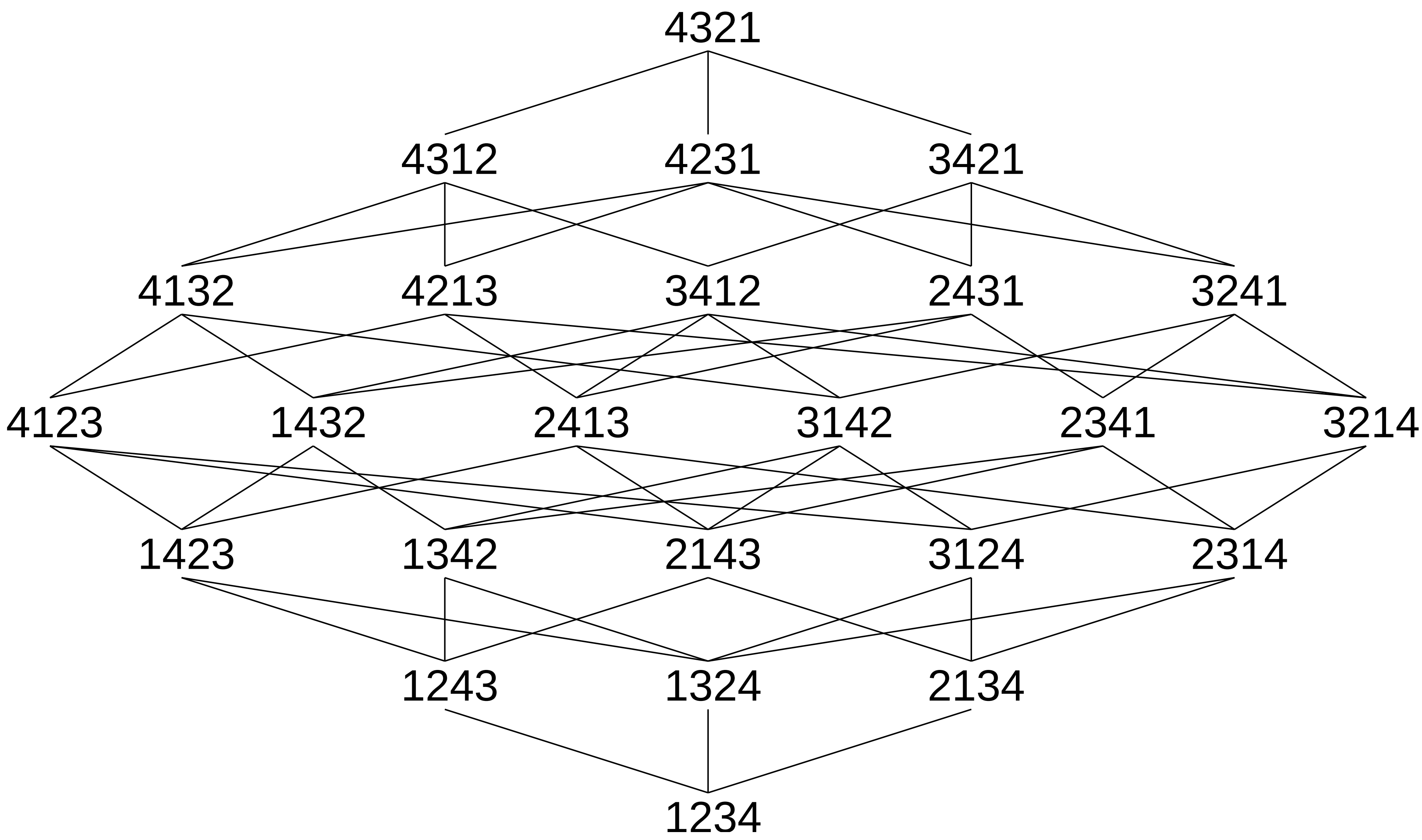
Dual canonical basis of $\mathcal{A}_{[n],[n]}(n;q)$

Immanants in DCB are $\{\mathrm{Imm}_v(x;q) \,|\, v \in \mathfrak{S}_n\}$, where

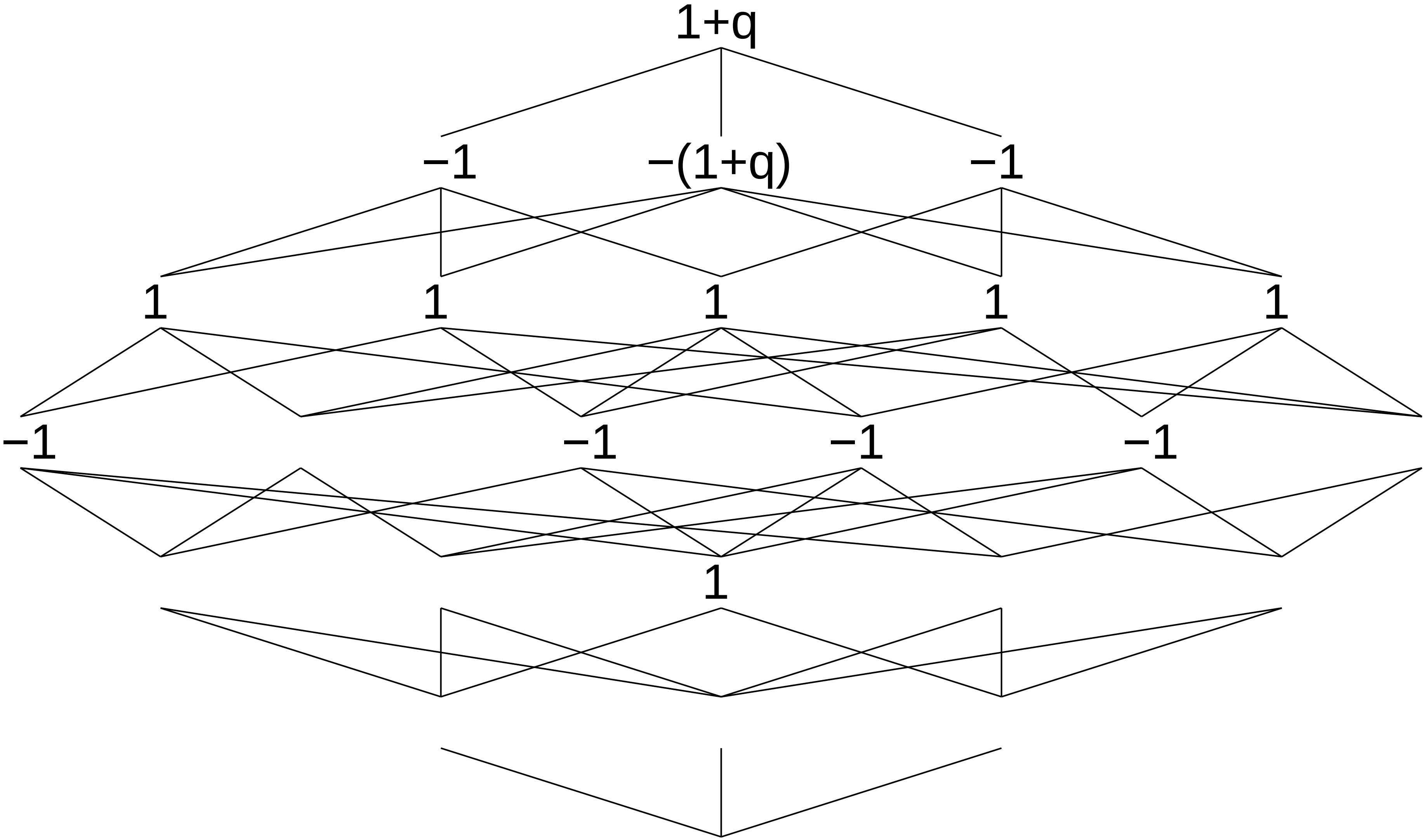
$$\begin{aligned}
\mathrm{Imm}_v(x;q) &= \sum_{w \geq v} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}(q) x_{1,w_1} \cdots x_{n,w_n}, \\
\epsilon_{v,w} &= (-1)^{\ell(w)-\ell(v)}, \\
q_{v,w} &= (q^{\frac{1}{2}})^{\ell(w)-\ell(v)}, \\
Q_{v,w}(q) &= P_{w_0 w, w_0 v}(q).
\end{aligned}$$

Nonquantum ($q = 1$) analogs in $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ are

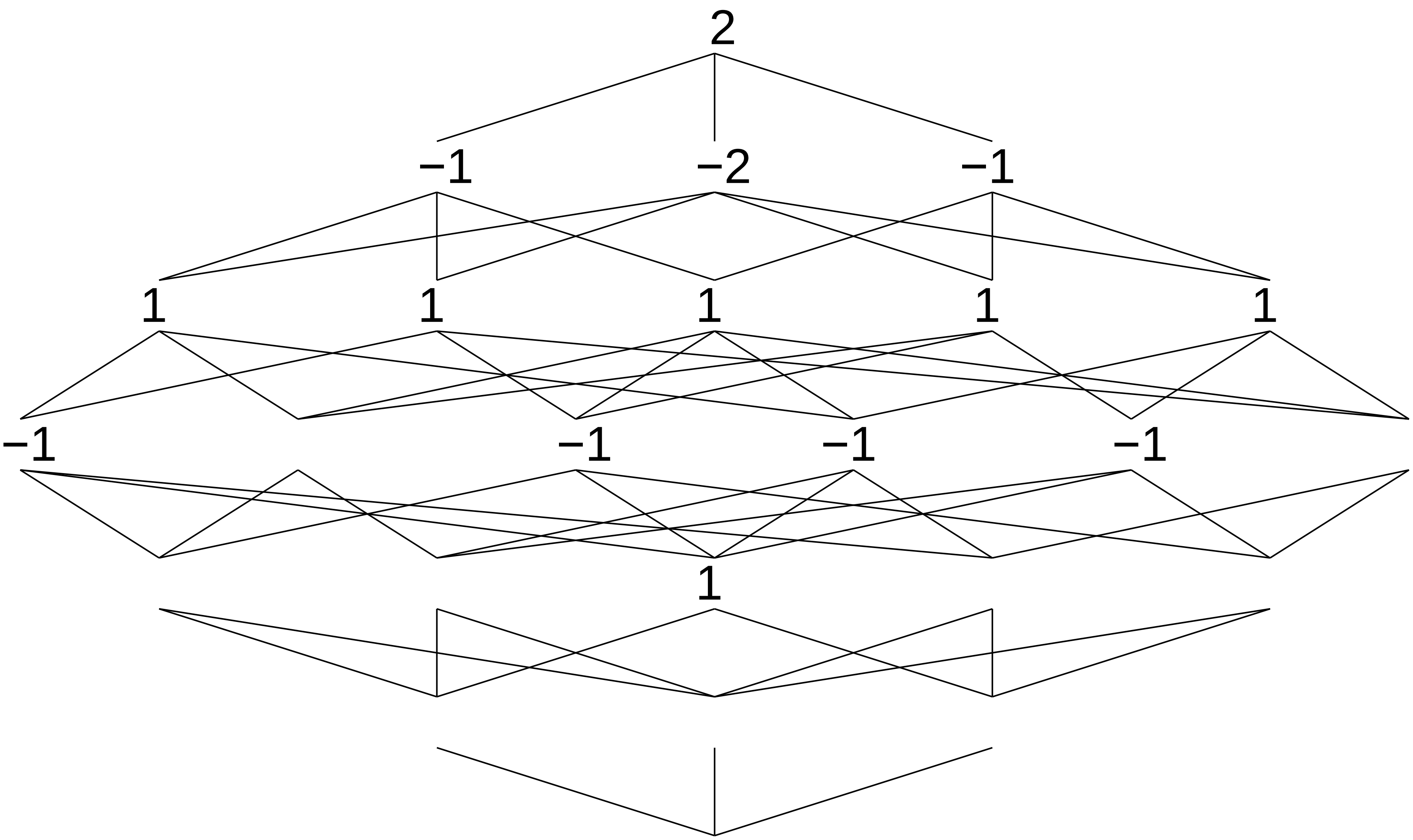
$$\mathrm{Imm}_v(x) = \sum_{w \geq v} \epsilon_{v,w} Q_{v,w}(1) x_{1,w_1} \cdots x_{n,w_n}.$$



The Bruhat order on $\mathfrak{S}_4$.



$$\mathrm{Imm}_{2143}(x;q) = x_{1,2}x_{2,1}x_{3,4}x_{4,3} - q^{\frac{-1}{2}}x_{1,4}x_{2,1}x_{3,2}x_{4,3} \pm \cdots$$



$$\mathrm{Imm}_{2143}(x) = x_{1,2}x_{2,1}x_{3,4}x_{4,3} - x_{1,4}x_{2,1}x_{3,2}x_{4,3} \pm \cdots$$

Immanant formulation of DCB of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$

Theorem: (S 04) DCB consists of all nonzero polynomials in

$$\bigcup_{r\geq 0} \{\mathrm{Imm}_w(x_{L,M}) \,|\, w \in \mathfrak{S}_r;\; L, M \text{ } r\text{-elt. multisets of } [n]\}.$$

Ex: Since $\mathrm{Imm}_{2143}(y) = y_{1,2}y_{2,1}y_{3,4}y_{4,3} - y_{1,4}y_{2,1}y_{3,2}y_{4,3} \pm \cdots$, we have

$$\mathrm{Imm}_{2143}(x_{1133,1222}) = \mathrm{Imm}_{2143} \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \end{bmatrix}$$

$$= x_{1,2}x_{1,1}x_{3,2}^2 - x_{1,2}x_{1,1}x_{3,2}^2 \pm \cdots .$$

Immanant formulation of DCB of $\mathcal{A}(n;q)$?

Q: Does DCB consist of all nonzero polynomials in

$$\bigcup_{r \geq 0} \{\mathrm{Imm}_w(x_{L,M};q) \,|\, w \in \mathfrak{S}_r;\ L, M \ r\text{-elt. multisets of } [n]\}?$$

Theorem: (L-S '10) We have single-parabolic identities

$$B_v^{L,M}(x;q) = \begin{cases} \mathrm{Imm}_v(x_{L,M};q) & \text{for } M \text{ mult.-free,} \\ \mathrm{Imm}_{v^{-1}}((x_{L,M})^\top;q) & \text{for } L \text{ mult.-free,} \end{cases}$$

and more generally,

$$B_v^{L,M}(x;q) = B_{v^{-1}}^{L,[r]}((x_{[r],M})^\top) = B_v^{[r],M}((x_{L,[r]})).$$

Since $\mathrm{Imm}_{2143}(y;q) = y_{1,2}y_{2,1}y_{3,4}y_{4,3} - q^{\frac{-1}{2}}y_{1,4}y_{2,1}y_{3,2}y_{4,3} \pm \cdots,$
we have

$$\mathrm{Imm}_{2143}(x_{1133,1234};q) = \mathrm{Imm}_{2143}\left(\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \end{bmatrix};q\right)$$

$$= x_{1,2}x_{1,1}x_{3,4}x_{3,3} - q^{\frac{-1}{2}}x_{1,4}x_{2,1}x_{3,2}x_{3,3} \pm \cdots ,$$

$$\mathrm{Imm}_{2143}(x_{1234,1222};q) = \mathrm{Imm}_{2143}((x_{1234,1222})^{\top};q)$$

$$= \mathrm{Imm}_{2143}\left(\begin{bmatrix} x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} \\ x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} \\ x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} \\ x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} \end{bmatrix};q\right)$$

$$= x_{2,1}x_{1,2}x_{4,2}x_{3,2} - q^{\frac{-1}{2}}x_{4,1}x_{1,2}x_{2,2}x_{3,2} \pm \cdots .$$

For $g_\lambda \in \Lambda$, define function $\gamma^\lambda : \mathfrak{S}_n \to \mathbb{C}$ by
$$g_\lambda = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \gamma^\lambda(w) p_{\rho(w)}.$$

For $f : \mathfrak{S}_n \to \mathbb{C}$, define generating function
$$\mathrm{Imm}_f(x) = \sum_{v \in \mathfrak{S}_n} f(v) x_{1,v_1} \cdots x_{n,v_n}.$$

Call a matrix *totally nonnegative* (TNN) if each of its minors is nonnegative.

Call an element $p(x) \in \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ *totally nonnegative* if $p(A) \geq 0$ for all TNN A.

Total nonnegativity table:

(quasi) symm. fn.	$f : \mathfrak{S}_n \to \mathbb{C}$	$\mathrm{Imm}_f(x)$	known TNN?	combin. interp?
h_λ	$\eta^\lambda = \mathrm{triv}\!\uparrow_\lambda^n$	$\mathrm{Imm}_{\eta^\lambda}(x)$	Y	Y
e_λ	$\epsilon^\lambda = \mathrm{sgn}\!\uparrow_\lambda^n$	$\mathrm{Imm}_{\epsilon^\lambda}(x)$	Y	Y
s_λ^k	?	?	Y	N
s_λ	χ^λ	$\mathrm{Imm}_\lambda(x)$	Y	N
p_λ	π^λ	$\mathrm{Imm}_{\pi^\lambda}(x)$	Y	Y
σ_λ^k	?	?	N	N
m_λ	ϕ^λ	$\mathrm{Imm}_{\phi^\lambda}(x)$	N	N
M_λ	?	?	N	N
G_λ	?	?	N	N
v	$w \mapsto \epsilon_{v,w} Q_{v,w}(1)$	$\mathrm{Imm}_v(x)$	Y	N