Eulerian quasisymmetric functions for the type B Coxeter group and other wreath product groups

Matthew Hyatt

University of Miami

November 17, 2010

Eulerian polynomials

Let S_n denote the symmetric group. Given $\sigma \in S_n$, let

$$\mathrm{exc}(\sigma) := |\{i \in [n-1] : \sigma(i) > i\}|$$

and

$$\mathrm{des}(\sigma) := |\{i \in [n-1] : \sigma(i) > \sigma(i+1)\}|$$

Let $A_n(t)$ denote the Eulerian polynomial

$$A_n(t) = \sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in S_n} t^{\mathrm{exc}(\sigma)}$$

where the second equality is due to MacMahon. Prior to this combinatorial interpretation of the Eulerian polynomials, Euler had proved the following formula

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{e^z(1-t)}{e^{tz} - te^z}$$

The group of colored permutations, $C_N \wr S_n$

Let C_N be the cyclic group of order N, and let S_n be the symmetric group on $[n]$. The wreath product $C_N \wr S_n$ is the group of colored permutations. Its elements are words in S_n where each letter has a color in $\{0, 1, 2, ..., N-1\}$ assigned to it. We denote this color with a superscript of the letter. For example

$$\begin{bmatrix} 5^2 & 3^0 & 1^1 & 4^0 & 2^1 & 6^2 \end{bmatrix} \in C_3 \wr S_6$$

and

$$\begin{bmatrix} 1^0 & 2^0 & 3^0 & 4^0 & 5^0 & 6^0 \end{bmatrix} = \mathrm{Id} \in C_3 \wr S_6$$

Note that $C_1 \wr S_n = S_n$, and $C_2 \wr S_n = B_n$ the type B Coxeter group.

Statistics on $C_N \wr S_n$

Totally order the letters that may appear in elements of $C_N \wr S_n$

$$\mathcal{E} := \left\{ 1^{N-1} < \dots < n^{N-1} < 1^{N-2} < \dots < n^{N-2} < \dots < 1^0 < \dots < n^0 \right\}$$

Let $\pi = \begin{bmatrix} \pi_1^{\epsilon_1} & \pi_2^{\epsilon_2} & \dots & \pi_n^{\epsilon_n} \end{bmatrix}$ and define the following statistics

$$\mathrm{des}^*(\pi) := |\mathrm{DES}(\pi)| + \chi(\epsilon_1 > 0)$$

$$\mathrm{maj}(\pi) := \sum_{i \in \mathrm{DES}(\pi)} i$$

$$\mathrm{exc}(\pi) := |\{ i \in [n-1] : \pi_i > i \text{ and } \epsilon_i = 0 \}|$$

$$\mathrm{fix}_m(\pi) := |\{ i \in [n] : \pi_i = i \text{ and } \epsilon_i = m \}|$$

$$\mathrm{col}_m(\pi) := |\{ i \in [n] : \epsilon_i = m \}|$$

where m is an integer in $[0, N-1]$.

Example

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2^2 & 3^2 & 6^0 & 4^1 & 8^0 & 1^1 & 7^0 & 9^2 & 5^2 \end{bmatrix} \in C_3 \wr S_9$$

$$\mathrm{des}^*(\pi) = 5$$

$$\mathrm{maj}(\pi) = 3 + 5 + 7 + 8 = 23$$

$$\mathrm{exc}(\pi) = 2$$

$$\mathrm{fix}_0(\pi) = 1, \quad \mathrm{fix}_1(\pi) = 1, \quad \mathrm{fix}_2(\pi) = 0, \quad \vec{\mathrm{fix}}(\pi) = (1, 1, 0)$$

$$\mathrm{col}_1(\pi) = 2, \quad \mathrm{col}_2(\pi) = 4, \quad \vec{\mathrm{col}}(\pi) = (2, 4)$$

Example

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2^2 & 3^2 & 6^0 & 4^1 & 8^0 & 1^1 & 7^0 & 9^2 & 5^2 \end{bmatrix} \in C_3 \wr S_9$$

$\mathrm{des}^*(\pi) = 5$

$\mathrm{maj}(\pi) = 3 + 5 + 7 + 8 = 23$

$\mathrm{exc}(\pi) = 2$

$\mathrm{fix}_0(\pi) = 1, \quad \mathrm{fix}_1(\pi) = 1, \quad \mathrm{fix}_2(\pi) = 0, \quad \vec{\mathrm{fix}}(\pi) = (1, 1, 0)$

$\mathrm{col}_1(\pi) = 2, \quad \mathrm{col}_2(\pi) = 4, \quad \vec{\mathrm{col}}(\pi) = (2, 4)$

Example

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2^2 & 3^2 & 6^0 & 4^1 & 8^0 & 1^1 & 7^0 & 9^2 & 5^2 \end{bmatrix} \in C_3 \wr S_9$$

$\mathrm{des}^*(\pi) = 5$

$\mathrm{maj}(\pi) = 3 + 5 + 7 + 8 = 23$

$\mathrm{exc}(\pi) = 2$

$\mathrm{fix}_0(\pi) = 1, \quad \mathrm{fix}_1(\pi) = 1, \quad \mathrm{fix}_2(\pi) = 0, \quad \vec{\mathrm{fix}}(\pi) = (1, 1, 0)$

$\mathrm{col}_1(\pi) = 2, \quad \mathrm{col}_2(\pi) = 4, \quad \vec{\mathrm{col}}(\pi) = (2, 4)$

Example

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2^2 & 3^2 & 6^0 & 4^1 & 8^0 & 1^1 & 7^0 & 9^2 & 5^2 \end{bmatrix} \in C_3 \wr S_9$$

$\mathrm{des}^*(\pi) = 5$

$\mathrm{maj}(\pi) = 3 + 5 + 7 + 8 = 23$

$\mathrm{exc}(\pi) = 2$

$\mathrm{fix}_0(\pi) = 1, \quad \mathrm{fix}_1(\pi) = 1, \quad \mathrm{fix}_2(\pi) = 0, \quad \vec{\mathrm{fix}}(\pi) = (1, 1, 0)$

$\mathrm{col}_1(\pi) = 2, \quad \mathrm{col}_2(\pi) = 4, \quad \vec{\mathrm{col}}(\pi) = (2, 4)$

Example

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2^2 & 3^2 & 6^0 & 4^1 & 8^0 & 1^1 & 7^0 & 9^2 & 5^2 \end{bmatrix} \in C_3 \wr S_9$$

$\mathrm{des}^*(\pi) = 5$

$\mathrm{maj}(\pi) = 3 + 5 + 7 + 8 = 23$

$\mathrm{exc}(\pi) = 2$

$\mathrm{fix}_0(\pi) = 1, \quad \mathrm{fix}_1(\pi) = 1, \quad \mathrm{fix}_2(\pi) = 0, \quad \vec{\mathrm{fix}}(\pi) = (1, 1, 0)$

$\mathrm{col}_1(\pi) = 2, \quad \mathrm{col}_2(\pi) = 4, \quad \vec{\mathrm{col}}(\pi) = (2, 4)$

Example

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2^2 & 3^2 & 6^0 & 4^1 & 8^0 & 1^1 & 7^0 & 9^2 & 5^2 \end{bmatrix} \in C_3 \wr S_9$$

$\mathrm{des}^*(\pi) = 5$

$\mathrm{maj}(\pi) = 3 + 5 + 7 + 8 = 23$

$\mathrm{exc}(\pi) = 2$

$\mathrm{fix}_0(\pi) = 1, \quad \mathrm{fix}_1(\pi) = 1, \quad \mathrm{fix}_2(\pi) = 0, \quad \vec{\mathrm{fix}}(\pi) = (1, 1, 0)$

$\mathrm{col}_1(\pi) = 2, \quad \mathrm{col}_2(\pi) = 4, \quad \vec{\mathrm{col}}(\pi) = (2, 4)$

Example

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2^2 & 3^2 & 6^0 & 4^1 & 8^0 & 1^1 & 7^0 & 9^2 & 5^2 \end{bmatrix} \in C_3 \wr S_9$$

$\mathrm{des}^*(\pi) = 5$

$\mathrm{maj}(\pi) = 3 + 5 + 7 + 8 = 23$

$\mathrm{exc}(\pi) = 2$

$\mathrm{fix}_0(\pi) = 1, \quad \mathrm{fix}_1(\pi) = 1, \quad \mathrm{fix}_2(\pi) = 0, \quad \vec{\mathrm{fix}}(\pi) = (1, 1, 0)$

$\mathrm{col}_1(\pi) = 2, \quad \mathrm{col}_2(\pi) = 4, \quad \vec{\mathrm{col}}(\pi) = (2, 4)$

Connections to the type B Coxeter group ($N = 2$)

The type B descent number is equidistributed with des^*.

The flag major index, denoted fmaj, is equidistributed with Coxeter length [Adin, Roichman], and is defined by

$$\mathsf{fmaj}(\pi) := 2\mathrm{maj}(\pi) + \mathrm{col}_1(\pi)$$

Remark: The word flag is used because of the connection between fmaj and a flag of parabolic subgroups $1 < G_1 < G_2 < \cdots < G_n$, where $G_i \simeq C_N \wr S_i$.

The flag descent number is a weighted count of Coxeter descents. It is equidistributed with the flag excedance number [Foata,Han], denoted fexc, and defined by

$$\mathsf{fexc}(\pi) := 2\mathrm{exc}(\pi) + \mathrm{col}_1(\pi)$$

Quasisymmetric functions and two specializations

Given a subset $T \subseteq [n-1]$, the fundamental quasisymmetric function of degree n, denoted $F_{T,n}$, is defined by

$$F_{T,n}(\mathbf{x}) := \sum_{\substack{i_1 \geq i_2 \geq \ldots \geq i_n \geq 1 \\ i_j > i_{j+1} \text{ if } j \in T}} x_{i_1} x_{i_2} \ldots x_{i_n}$$

The stable principal specialization $\mathbf{ps} : \mathrm{QSym} \to \mathbb{Q}[q]$ is a homomorphism defined by $x_i \mapsto q^{i-1}$ for all i.

The nonstable principal specialization $\mathbf{ps}_k : \mathrm{QSym} \to \mathbb{Q}[q]$ is a homomorphism defined by $x_i \mapsto q^{i-1}$ for $1 \leq i \leq k$ and $x_i \mapsto 0$ otherwise.

A useful lemma

Lemma (Gessel, Reutenauer)

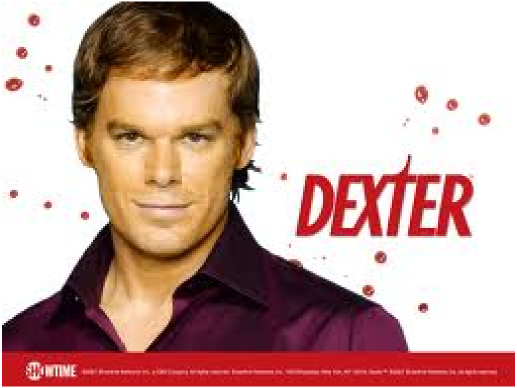
$$\mathbf{ps}(F_{T,n}) = \frac{q^{\sum_{i \in T} i}}{(q;q)_n}$$

$$\sum_{k \geq 0} p^k \mathbf{ps}_k(F_{T,n}) = \frac{p^{|T|+1} q^{\sum_{i \in T} i}}{(p;q)_{n+1}}$$

where

$$(p;q)_n := (1-p)(1-pq)\ldots(1-pq^{n-1}) \quad \textit{if } n \geq 1$$

and $(p;q)_0 := 1$.



No, not this guy.

Extend the definition of $\mathrm{DEX}(\pi)$

For $\pi \in C_N \wr S_n$, we want to define $\mathrm{DEX}(\pi) \subseteq [n-1]$, such that this definition coincides with the definition given by Shareshian and Wachs when $N = 1$.

Extend the totally ordered alphabet $\mathcal{E}$ to

$$\mathcal{A} := \left\{ \widetilde{1^0} < \widetilde{2^0} < \dots < \widetilde{n^0} \right\} < \mathcal{E}$$

Given π, obtain a word $\widetilde{\pi}$ over $\mathcal{A}$ by replacing $\pi_i^{\epsilon_i}$ by $\widetilde{\pi_i^{\epsilon_i}}$ if $\epsilon_i = 0$ and $\pi_i > i$ (i.e. i is an excedance position). Then define $\mathrm{DEX}(\pi) := \mathrm{DES}(\widetilde{\pi})$.

For example,
$\pi = \left[5^0 \quad 3^0 \quad 4^1 \quad 6^0 \quad 2^1 \quad 1^2\right]$

Extend the definition of $\mathrm{DEX}(\pi)$

For $\pi \in C_N \wr S_n$, we want to define $\mathrm{DEX}(\pi) \subseteq [n-1]$, such that this definition coincides with the definition given by Shareshian and Wachs when $N = 1$.

Extend the totally ordered alphabet $\mathcal{E}$ to

$$\mathcal{A} := \left\{ \widetilde{1^0} < \widetilde{2^0} < \ldots < \widetilde{n^0} \right\} < \mathcal{E}$$

Given π, obtain a word $\widetilde{\pi}$ over $\mathcal{A}$ by replacing $\pi_i^{\epsilon_i}$ by $\widetilde{\pi_i^{\epsilon_i}}$ if $\epsilon_i = 0$ and $\pi_i > i$ (i.e. i is an excedance position). Then define $\mathrm{DEX}(\pi) := \mathrm{DES}(\widetilde{\pi})$.

For example,

$$\pi = \begin{bmatrix} 5^0 & 3^0 & 4^1 & 6^0 & 2^1 & 1^2 \end{bmatrix}$$

$$\widetilde{\pi} = \begin{bmatrix} \widetilde{5^0} & \widetilde{3^0} & 4^1 & \widetilde{6^0} & 2^1 & 1^2 \end{bmatrix}$$

Extend the definition of $\mathrm{DEX}(\pi)$

For $\pi \in C_N \wr S_n$, we want to define $\mathrm{DEX}(\pi) \subseteq [n-1]$, such that this definition coincides with the definition given by Shareshian and Wachs when $N = 1$.

Extend the totally ordered alphabet $\mathcal{E}$ to

$$\mathcal{A} := \left\{ \widetilde{1^0} < \widetilde{2^0} < \ldots < \widetilde{n^0} \right\} < \mathcal{E}$$

Given π, obtain a word $\widetilde{\pi}$ over $\mathcal{A}$ by replacing $\pi_i^{\epsilon_i}$ by $\widetilde{\pi_i^{\epsilon_i}}$ if $\epsilon_i = 0$ and $\pi_i > i$ (i.e. i is an excedance position). Then define $\mathrm{DEX}(\pi) := \mathrm{DES}(\widetilde{\pi})$.

For example,

$$\pi = \begin{bmatrix} 5^0 & 3^0 & 4^1 & 6^0 & 2^1 & 1^2 \end{bmatrix}$$

$$\widetilde{\pi} = \begin{bmatrix} \widetilde{5^0} & \widetilde{3^0} & 4^1 & \widetilde{6^0} & 2^1 & 1^2 \end{bmatrix}$$

$$\mathrm{DEX}(\pi) = \{1, 3, 5\}$$

Properties of $\mathrm{DEX}(\pi)$

Lemma (H)

For every $\pi \in C_N \wr S_n$ we have

$$|\mathrm{DEX}(\pi)| = \begin{cases} \mathrm{des}^*(\pi) - 1 & \text{if } \pi_1^{\epsilon_1} \neq 1^0 \\ \mathrm{des}^*(\pi) & \text{if } \pi_1^{\epsilon_1} = 1^0 \end{cases}$$

and

$$\sum_{i \in \mathrm{DEX}(\pi)} i = \mathrm{maj}(\pi) - \mathrm{exc}(\pi)$$

A family of quasisymmetric functions

Given $\pi \in C_N \wr S_n$, let $F_\pi := F_{\mathrm{DEX}(\pi),n}(\mathbf{x})$.

For each $n, j \in \mathbb{N}, \vec{\alpha} \in \mathbb{N}^N, \vec{\beta} \in \mathbb{N}^{N-1}$, define the fixed point colored Eulerian quasisymmetric functions by

$$\bar{Q}(n, j, \vec{\alpha}, \vec{\beta}) := \sum_{\substack{\pi \in C_N \wr S_n \\ \mathrm{exc}(\pi) = j \\ \vec{\mathrm{fix}}(\pi) = \vec{\alpha} \\ \vec{\mathrm{col}}(\pi) = \vec{\beta}}} F_\pi$$

Some examples with $N = 2$

$$\bar{Q}(3, 1, \vec{0}, 1)$$

$$= F_{\left[2^0 \quad 3^1 \quad 1^0\right]} + F_{\left[3^0 \quad 1^1 \quad 2^0\right]} + F_{\left[3^0 \quad 1^0 \quad 2^1\right]} + F_{\left[2^1 \quad 3^0 \quad 1^0\right]}$$
$$= F_{\phi,3} + F_{\phi,3} + F_{\{2\},3} + F_{\{1\},3} = h_3 + h_3 + (m_{2,1} + e_3) = h_3 + h_{2,1}$$

$$\bar{Q}(3, 0, \vec{0}, 3) = h_{2,1} - h_3 = s_{2,1}$$

A generating function formula

Theorem (H)

$$\sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} \bar{Q}(n,j,\vec{\alpha},\vec{\beta}) z^n t^j r_0^{\alpha_0} \left(\prod_{m=1}^{N-1} (r_m)^{\alpha_m} (s_m)^{\beta_m}\right)$$

$$= \frac{H(r_0 z)(1-t)\left(\prod_{m=1}^{N-1} E(-s_m z) H(r_m s_m z)\right)}{\left(1+\sum_{m=1}^{N-1} s_m\right) H(tz) - \left(t + \sum_{m=1}^{N-1} s_m\right) H(z)}$$

where $E(z) := \sum_{i\geq 0} e_i z^i$, and $H(z) := \sum_{i\geq 0} h_i z^i$. If $N=1$ this reduces to a formula of Shareshian and Wachs.

A generating function formula

Theorem (H)

$$\sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} \bar{Q}(n,j,\vec{\alpha},\vec{\beta})z^n t^j r_0^{\alpha_0}\left(\prod_{m=1}^{N-1}(r_m)^{\alpha_m}(s_m)^{\beta_m}\right)$$

$$= \frac{H(r_0 z)(1-t)\left(\prod_{m=1}^{N-1}E(-s_m z)H(r_m s_m z)\right)}{\left(1+\sum_{m=1}^{N-1}s_m\right)H(tz)-\left(t+\sum_{m=1}^{N-1}s_m\right)H(z)}$$

where $E(z) := \sum_{i\geq 0} e_i z^i$, and $H(z) := \sum_{i\geq 0} h_i z^i$. If $N=1$ this reduces to a formula of Shareshian and Wachs.

A generating function formula

Theorem (H)

$$\sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} \bar{Q}(n,j,\vec{\alpha},\vec{\beta})z^n t^j r_0^{\alpha_0}\left(\prod_{m=1}^{N-1}(s_m)^{\beta_m}\right)$$

$$= \frac{H(r_0 z)(1-t)}{\left(1+\sum_{m=1}^{N-1} s_m\right) H(tz) - \left(t+\sum_{m=1}^{N-1} s_m\right) H(z)}$$

where $E(z) := \sum_{i\geq 0} e_i z^i$, and $H(z) := \sum_{i\geq 0} h_i z^i$. If $N=1$ this reduces to a formula of Shareshian and Wachs.

A generating function formula

Theorem (H)

$$\sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} \bar{Q}(n,j,\vec{\alpha},\vec{\beta}) z^n t^j r_0^{\alpha_0} \left(\prod_{m=1}^{N-1} (s_m)^{\beta_m}\right)$$

$$= \frac{H(r_0 z)(1-t)}{\left(1+\sum_{m=1}^{N-1} s_m\right) H(tz) - \left(t + \sum_{m=1}^{N-1} s_m\right) H(z)}$$

where $E(z) := \sum_{i\geq 0} e_i z^i$, and $H(z) := \sum_{i\geq 0} h_i z^i$. If $N=1$ this reduces to a formula of Shareshian and Wachs.

A generating function formula

Theorem (H)

$$\sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} \bar{Q}(n,j,\vec{\alpha},\vec{\beta})z^n t^j r_0^{\alpha_0}$$

$$= \frac{H(r_0 z)(1-t)}{NH(tz)-(t+N-1)\,H(z)}$$

where $E(z) := \sum_{i\geq 0} e_i z^i$, and $H(z) := \sum_{i\geq 0} h_i z^i$. If $N=1$ this reduces to a formula of Shareshian and Wachs.

Another family of quasisymmetric functions

Let $\check{\lambda}(\pi) = ((\lambda_1, \vec{\beta}^{(1)}), (\lambda_2, \vec{\beta}^{(2)}), \ldots, (\lambda_k, \vec{\beta}^{(k)}))$ denote the cycle type of a colored permutation. For example if

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3^2 & 4^1 & 1^1 & 2^2 & 8^0 & 9^2 & 5^1 & 7^1 & 6^0 \end{bmatrix}$$

$$= (1^1, 3^2)(2^2, 4^1)(6^0, 9^2)(5^1, 8^0, 7^1)$$

then

$$\check{\lambda}(\pi) = \{(3, (2, 0)), (2, (1, 1)), (2, (1, 1)), (2, (0, 1))\}$$

The cycle type colored Eulerian quasisymmetric functions are

$$\bar{Q}(\check{\lambda}, j) := \sum_{\substack{\pi \in C_N \wr S_n \\ \text{exc}(\pi) = j \\ \check{\lambda}(\pi) = \check{\lambda}}} F_\pi$$

Another family of quasisymmetric functions

Let $\check{\lambda}(\pi) = ((\lambda_1, \vec{\beta}^{(1)}), (\lambda_2, \vec{\beta}^{(2)}), \ldots, (\lambda_k, \vec{\beta}^{(k)}))$ denote the cycle type of a colored permutation. For example if

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3^2 & 4^1 & 1^1 & 2^2 & 8^0 & 9^2 & 5^1 & 7^1 & 6^0 \end{bmatrix}$$

$$= (1^1, 3^2)(2^2, 4^1)(6^0, 9^2)(5^1, 8^0, 7^1)$$

then

$$\check{\lambda}(\pi) = \{(3, (2, 0)), (2, (1, 1)), (2, (1, 1)), (2, (0, 1))\}$$

The cycle type colored Eulerian quasisymmetric functions are

$$\bar{Q}(\check{\lambda}, j) := \sum_{\substack{\pi \in C_N \wr S_n \\ \mathrm{exc}(\pi) = j \\ \check{\lambda}(\pi) = \check{\lambda}}} F_\pi$$

Another family of quasisymmetric functions

Let $\check{\lambda}(\pi) = ((\lambda_1, \vec{\beta}^{(1)}), (\lambda_2, \vec{\beta}^{(2)}), \ldots, (\lambda_k, \vec{\beta}^{(k)}))$ denote the cycle type of a colored permutation. For example if

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3^2 & 4^1 & 1^1 & 2^2 & 8^0 & 9^2 & 5^1 & 7^1 & 6^0 \end{bmatrix}$$

$$= (1^1, 3^2)(2^2, 4^1)(6^0, 9^2)(5^1, 8^0, 7^1)$$

then

$$\check{\lambda}(\pi) = \{(3, (2, 0)), (2, (1, 1)), (2, (1, 1)), (2, (0, 1))\}$$

The cycle type colored Eulerian quasisymmetric functions are

$$\bar{Q}(\check{\lambda}, j) := \sum_{\substack{\pi \in C_N \wr S_n \\ \text{exc}(\pi) = j \\ \check{\lambda}(\pi) = \check{\lambda}}} F_\pi$$

Another family of quasisymmetric functions

Let $\check{\lambda}(\pi) = ((\lambda_1, \vec{\beta}^{(1)}), (\lambda_2, \vec{\beta}^{(2)}), \ldots, (\lambda_k, \vec{\beta}^{(k)}))$ denote the cycle type of a colored permutation. For example if

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3^2 & 4^1 & 1^1 & 2^2 & 8^0 & 9^2 & 5^1 & 7^1 & 6^0 \end{bmatrix}$$

$$= (1^1, 3^2)(2^2, 4^1)(6^0, 9^2)(5^1, 8^0, 7^1)$$

then

$$\check{\lambda}(\pi) = \{(3, (2, 0)), (2, (1, 1)), (2, (1, 1)), (2, (0, 1))\}$$

The cycle type colored Eulerian quasisymmetric functions are

$$\bar{Q}(\check{\lambda}, j) := \sum_{\substack{\pi \in C_N \wr S_n \\ \mathrm{exc}(\pi) = j \\ \check{\lambda}(\pi) = \check{\lambda}}} F_\pi$$

Another family of quasisymmetric functions

Let $\check{\lambda}(\pi) = ((\lambda_1, \vec{\beta}^{(1)}), (\lambda_2, \vec{\beta}^{(2)}), \ldots, (\lambda_k, \vec{\beta}^{(k)}))$ denote the cycle type of a colored permutation. For example if

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3^2 & 4^1 & 1^1 & 2^2 & 8^0 & 9^2 & 5^1 & 7^1 & 6^0 \end{bmatrix}$$

$$= (1^1, 3^2)(2^2, 4^1)(6^0, 9^2)(5^1, 8^0, 7^1)$$

then

$$\check{\lambda}(\pi) = \{(3, (2, 0)), (2, (1, 1)), (2, (1, 1)), (2, (0, 1))\}$$

The cycle type colored Eulerian quasisymmetric functions are

$$\bar{Q}(\check{\lambda}, j) := \sum_{\substack{\pi \in C_N \wr S_n \\ \mathrm{exc}(\pi) = j \\ \check{\lambda}(\pi) = \check{\lambda}}} F_\pi$$

Outline of the proof

View $\bar{Q}(\check{\lambda}, j)$ as a multiset of monomials.

Construct a bijection from $\bar{Q}(\check{\lambda}, j)$ to a certain set of colored ornaments. A colored ornament is (roughly) a multiset of primitive circular words over a certain alphabet. This bijection is a nontrivial extension of a bijection due to Shareshian and Wachs, which is in turn a nontrivial extension of a bijection due to Gessel and Reutenauer.

Use the ornament description of the monomials appearing in $\bar{Q}(n, j, \vec{\alpha}, \vec{\beta})$ to obtain a recurrence relation, which is equivalent to the theorem. The proof of this recurrence uses the increasing factorization of Désarménien and Wachs, which is a refinement of Lyndon factorization.

A little bit on colored ornaments

Map a pair (π, s) into a multiset R of necklaces on the ordered alphabet $1^0 < 1^1 < 1^2 < \bar{1}^0 < 2^0 < 2^1 < 2^2 < \bar{2}^0 < \ldots$

$$\begin{array}{lcllllllll} \mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \pi & = & 3^0 & 4^0 & 8^1 & 1^0 & 7^0 & 6^1 & 5^2 & 2^1 \\ & & & & & & & & & \\ s & = & x_7 & x_7 & x_7 & x_7 & x_5 & x_5 & x_3 & x_3 \end{array}$$

A little bit on colored ornaments

Map a pair (π, s) into a multiset R of necklaces on the ordered alphabet $1^0 < 1^1 < 1^2 < \overline{1^0} < 2^0 < 2^1 < 2^2 < \overline{2^0} < \ldots$

$$
\begin{array}{cccccccccc}
\mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\tilde{\pi} & = & \tilde{3^0} & \tilde{4^0} & 8^1 & 1^0 & \tilde{7^0} & 6^1 & 5^2 & 2^1 \\
 & & & & & & & & & \\
s & = & x_7 & x_7 & x_7 & x_7 & x_5 & x_5 & x_3 & x_3
\end{array}
$$

A little bit on colored ornaments

Map a pair (π, s) into a multiset R of necklaces on the ordered alphabet $1^0 < 1^1 < 1^2 < \bar{1}^0 < 2^0 < 2^1 < 2^2 < \bar{2}^0 < \ldots$

$$
\begin{array}{lccccccccc}
\mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\tilde{\pi} & = & \tilde{3}^0 & \tilde{4}^0 & 8^1 & 1^0 & \tilde{7}^0 & 6^1 & 5^2 & 2^1 \\
 & & & & & & & & & \\
s & = & x_7 & x_7 & x_7 & x_7 & x_5 & x_5 & x_3 & x_3 \\
\alpha & = & \bar{7}^0 & \bar{7}^0 & 7^1 & 7^0 & \bar{5}^0 & 5^1 & 3^2 & 3^1
\end{array}
$$

A little bit on colored ornaments

Map a pair (π, s) into a multiset R of necklaces on the ordered alphabet $1^0 < 1^1 < 1^2 < \bar{1}^0 < 2^0 < 2^1 < 2^2 < \bar{2}^0 < \ldots$

$$
\begin{array}{lcllllllll}
\mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\tilde{\pi} & = & \tilde{3}^0 & \tilde{4}^0 & 8^1 & 1^0 & \tilde{7}^0 & 6^1 & 5^2 & 2^1 \\
 & & & & & & & & & \\
s & = & x_7 & x_7 & x_7 & x_7 & x_5 & x_5 & x_3 & x_3 \\
\alpha & = & \bar{7}^0 & \bar{7}^0 & 7^1 & 7^0 & \bar{5}^0 & 5^1 & 3^2 & 3^1 \\
 & & & & & & & & & \\
\sigma & = & (1, & 3, & 8, & 2, & 4) & (6) & (5, & 7)
\end{array}
$$

A little bit on colored ornaments

Map a pair (π, s) into a multiset R of necklaces on the ordered alphabet $1^0 < 1^1 < 1^2 < \bar{1}^0 < 2^0 < 2^1 < 2^2 < \bar{2}^0 < \dots$

$$
\begin{array}{lcllllllll}
\mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\tilde{\pi} & = & \tilde{3}^0 & \tilde{4}^0 & 8^1 & 1^0 & \tilde{7}^0 & 6^1 & 5^2 & 2^1 \\
\\
s & = & x_7 & x_7 & x_7 & x_7 & x_5 & x_5 & x_3 & x_3 \\
\alpha & = & \bar{7}^0 & \bar{7}^0 & 7^1 & 7^0 & \bar{5}^0 & 5^1 & 3^2 & 3^1 \\
\\
\sigma & = & (1, & 3, & 8, & 2, & 4) & (6) & (5, & 7) \\
R & = & (\bar{7}^0, & 7^1, & 3^1, & \bar{7}^0, & 7^0) & (5^1) & (\bar{5}^0, & 3^2)
\end{array}
$$

A little bit on colored ornaments

Map a pair (π, s) into a multiset R of necklaces on the ordered alphabet $1^0 < 1^1 < 1^2 < \bar{1}^0 < 2^0 < 2^1 < 2^2 < \bar{2}^0 < \ldots$

$$\begin{array}{lcllllllll}
\mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\tilde{\pi} & = & \tilde{3}^0 & \tilde{4}^0 & 8^1 & 1^0 & \tilde{7}^0 & 6^1 & 5^2 & 2^1 \\
 & & & & & & & & & \\
s & = & x_7 & x_7 & x_7 & x_7 & x_5 & x_5 & x_3 & x_3 \\
\alpha & = & \bar{7}^0 & \bar{7}^0 & 7^1 & 7^0 & \bar{5}^0 & 5^1 & 3^2 & 3^1 \\
 & & & & & & & & & \\
\sigma & = & (1, & 3, & 8, & 2, & 4) & (6) & (5, & 7) \\
R & = & (\bar{7}^0, & 7^1, & 3^1, & \bar{7}^0, & 7^0) & (5^1) & (\bar{5}^0, & 3^2)
\end{array}$$

To recover σ we rank each position of R, and α is the weakly decreasing rearrangement of the letters in R.

A recurrence

$$\bar{Q}(n,j,\vec{0},\vec{\beta}) = \sum_{\substack{0\le i\le n-2 \\ j-n+i<k<j}} \bar{Q}(i,k,\vec{0},\vec{\beta})h_{n-i}$$

$$+\sum_{m=1}^{N-1}\left(\sum_{\substack{0\le i\le n-1 \\ j-n+i<k\le j}} \bar{Q}(i,k,\vec{0},\vec{\beta}(\hat{m}))h_{n-i}\right)$$

$$+\chi\left(j=0\right)\chi\left(\left|\vec{\beta}\right|=n\right)(-1)^n\prod_{m=1}^{N-1} e_{\beta_m}$$

where if $\vec{\beta}=(\beta_1,\beta_2,...,\beta_{N-1})$ then

$$\vec{\beta}(\hat{m}) := (\beta_1,...,\beta_{m-1},\beta_m-1,\beta_{m+1},...,\beta_{N-1})$$

Apply the stable principal specialization

Theorem (H)

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{[n]_q!} q^{\mathrm{maj}(\pi)} t^{\mathrm{exc}(\pi)} r_0^{\mathrm{fix}_0(\pi)} \left(\prod_{m=1}^{N-1} (r_m)^{\mathrm{fix}_m(\pi)} (s_m)^{\mathrm{col}_m(\pi)}\right)$$

$$= \frac{\exp_q(r_0 z)(1-tq)\left(\prod_{m=1}^{N-1} \mathrm{Exp}_q(-s_m z)\exp_q(r_m s_m z)\right)}{\left(1+\sum_{m=1}^{N-1} s_m\right)\exp_q(tqz) - \left(tq+\sum_{m=1}^{N-1} s_m\right)\exp_q(z)}$$

If $N=1$ this reduces to the q-analog of Euler's formula due to Shareshian and Wachs.

Apply the stable principal specialization

Theorem (H)

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{[n]_q!} q^{\mathrm{maj}(\pi)} t^{\mathrm{exc}(\pi)} r_0^{\mathrm{fix}_0(\pi)} \left(\prod_{m=1}^{N-1} (r_m)^{\mathrm{fix}_m(\pi)} (s_m)^{\mathrm{col}_m(\pi)}\right)$$

$$= \frac{\exp_q(r_0 z)(1-tq)\left(\prod_{m=1}^{N-1} \mathrm{Exp}_q(-s_m z)\exp_q(r_m s_m z)\right)}{\left(1+\sum_{m=1}^{N-1} s_m\right)\exp_q(tqz) - \left(tq + \sum_{m=1}^{N-1} s_m\right)\exp_q(z)}$$

If $N=1$ this reduces to the q-analog of Euler's formula due to Shareshian and Wachs.

Apply the stable principal specialization

Theorem (H)

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{[n]_q!} q^{\mathrm{maj}(\pi)} t^{\mathrm{exc}(\pi)} r_0^{\mathrm{fix}_0(\pi)}$$

$$= \frac{\exp_q(r_0 z)(1-tq)}{N\exp_q(tqz) - (tq+N-1)\exp_q(z)}$$

If $N=1$ this reduces to the q-analog of Euler's formula due to Shareshian and Wachs.

Apply the nonstable principal specialization (and then do some more work)

Theorem (H)

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{(p;q)_{n+1}} q^{\mathrm{maj}(\pi)} t^{\mathrm{exc}(\pi)} p^{\mathrm{des}^*(\pi)} \left(\prod_{m=0}^{N-1} (r_m)^{\mathrm{fix}_m(\pi)}\right) \left(\prod_{m=1}^{N-1} (s_m)^{\mathrm{col}_m(\pi)}\right)$$

$$= \sum_{k\geq 0} \frac{p^k (1-tq)(z;q)_k (tqz;q)_k \left(\prod_{m=1}^{N-1} (s_m z;q)_k\right)}{\left(\prod_{m=1}^{N-1} (r_m s_m z;q)_k\right) (r_0 z;q)_{k+1}}$$

$$\times \frac{1}{\left[\left(1+\sum_{m=1}^{N-1} s_m\right)(z;q)_k - \left(tq + \sum_{m=1}^{N-1} s_m\right)(tqz;q)_k\right]}$$

Recall that $(p;q)_n := \prod_{i=1}^{n} (1-pq^{i-1})$.
If $N=1$ this reduces to a formula of Foata and Han.

A special case: the type B Coxeter group

Corollary (H)

$$\sum_{\substack{n\geq 0\\ \pi\in B_n}} \frac{z^n}{(p;q^2)_{n+1}} q^{\mathrm{fmaj}(\pi)} t^{\mathrm{fexc}(\pi)} p^{\mathrm{des}_\mathrm{B}(\pi)} r^{\mathrm{fix}^+(\pi)} s^{\mathrm{neg}(\pi)}$$

$$= \sum_{k\geq 0} \frac{p^k(1-t^2q^2)(z;q^2)_k(t^2q^2z;q^2)_k}{(rz;q^2)_{k+1}\left[(1+sqt)(z;q^2)_k-(t^2q^2+sqt)(t^2q^2z;q^2)_k\right]}$$

With a little work, this reduces to

Corollary (Chow, Gessel)

$$\sum_{\pi\in B_n} q^{\mathrm{fmaj}(\pi)} p^{\mathrm{des}_\mathrm{B}(\pi)} = (p;q^2)_{n+1} \sum_{k\geq 0} p^k [2k+1]_q^n$$

A cool formula

Let $N \geq 2$, and let ω be a primitive N^{th} root of unity. Set $r_m = 1$, set $s_m = \omega^m$, and extract the coefficient of z^n to obtain

Corollary (H)

$$\sum_{\pi \in C_N \wr S_n} t^{\mathrm{exc}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}^*(\pi)} \left(\prod_{m=1}^{N-1} \omega^{m \cdot \mathrm{col}_m(\pi)} \right) = (p; q)_n$$

An example, $C_3 \wr S_2$

π	monomial	π	monomial	π	monomial	π	monomial
$1^0 2^0$	1	$1^1 2^2$	$p^2 q$	$2^0 1^0$	tpq	$2^1 1^2$	$p^2 q$
$1^0 2^1$	ωpq	$1^2 2^0$	$\omega^2 p$	$2^0 1^1$	ωtpq	$2^2 1^0$	$\omega^2 p$
$1^0 2^2$	$\omega^2 pq$	$1^2 2^1$	p	$2^0 1^2$	$\omega^2 tpq$	$2^2 1^1$	p
$1^1 2^0$	ωp	$1^2 2^2$	ωp	$2^1 1^0$	ωp	$2^2 1^2$	$\omega p^2 q$
$1^1 2^1$	$\omega^2 p$			$2^1 1^1$	$\omega^2 p^2 q$		

$$\sum_{\pi \in C_3 \wr S_2} t^{\mathrm{exc}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}(\pi)} \left(\prod_{m=1}^{2} \omega^{m \cdot \mathrm{col}_m(\pi)} \right)$$

$$= 1 - p - pq + p^2 q = (1-p)(1-pq) = (p;q)_2$$

Thank you