Combinatorics of symmetric functions in non-commutative settings

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Objects of interest

- Schur functions
- Stanley symmetric functions
- Stable Grothendieck functions
- Affine Stanley symmetric functions / Dual k-Schur functions
- Affine Grothendieck functions

These all have weight-generating functions $F = \sum_{f \in \mathcal{F}} x^{ev(f)}$.

We want to understand symmetry of these and relationship to Hall-Littlewood and Macdonald polynomials.

Framework for standardization

$\mathcal{F}_n$: a set with

- (Evaluation) *ev*: $\mathcal{F}_n \to$ weak compositions of n
- (Descent set) *Des*: $\mathcal{F}_n \to$ strong compositions of n

$\mathcal{S}_n \subset \mathcal{F}_n$ the *standard* objects: $wt(s) = (1^n)$.

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A standardization map $\mathcal{F}_n \to \mathcal{S}_n$ satisfies:

- Identity on $\mathcal{S}_n$
- Preserves *Des*
- Fiber of $s \in \mathcal{S}_n$ consists of exactly one element of evaluation α for every α that refines $Des(s)$.

$$F_{\mathcal{F}_n}(x) := \sum_{f \in \mathcal{F}_n} x^{ev(f)} = \sum_{s \in \mathcal{S}_n} \mathcal{L}_{Des(s)}$$

Symmetry

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Symmetry:

- wt-preserving involutions s_i with

$$\begin{aligned} ev(f) &= (\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots) \\ ev(s_i(f)) &= (\alpha_1, \cdots, \alpha_{i+1}, \alpha_i, \cdots) \end{aligned}$$

The s_i operators act like permutations if

$$s_i s_j s_i = s_j s_i s_j$$

for $j = i + 1$ and $s_i s_j = s_j s_i$ for $|i - j| > 2$.

Introducing a t

If we have standardization and symmetrization, we have

$$F_{\mathcal{F}_n}(x) = \sum_{f \in \mathcal{F}_n} x^{ev(f)} = \sum_{s \in \mathcal{S}_n} \mathcal{L}_{Des(s)} = \sum_{\substack{f \in \mathcal{F}_n \\ ev(f) \in Par}} m_{ev(f)}$$

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We want

$$F_{\mathcal{F}_n}(x) = \sum_{\substack{f \in \mathcal{F}_n \\ ev(f) \in Par}} t^{ch(f)} P_{ev(f)}[X;t]$$

Hall-Littlewood polynomials

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$$F_{\mathcal{F}_n}(x) = \sum_{\substack{f \in \mathcal{F}_n \\ ev(f) \in Par}} t^{ch(f)} P_{ev(f)}[X; t]$$

This is in analogy to a theorem of Lascoux and Schützenberger which I take as a definition:

$$s_\lambda = \sum_{\substack{T \in SSYT(\lambda) \\ ev(T) \in Par}} t^{ch(T)} P_{ev(f)}[X; t]$$

Note that when $t = 1$, $P_\mu[X; 1] = m_\mu$.

Charge

Define cocharge instead of charge, and use:

$$ch(f) = n(\lambda(ev(f))) - cc(f)$$

where λ of a composition is the rearrangement to a partition shape, and

$$n(\alpha) = \sum (i-1)\alpha_i$$

Note: $n(\alpha) \neq n(\lambda(\alpha))$.

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Plan:

- Define cocharge on standard objects.
- Associate every object with a standard object.

Warning! Simple standardization *cannot* be this association.

Definition of cocharge

For standard objects, cocharge is a simple statistic on the descent set:

$$cc(s) = n(Des(s))$$

To associate an f with a standard object:

- Use symmetry operators to associate f with an element of partition evaluation
- Use simple standardization on the first part only.
- Repeat

Symmetrization operators on tableau

- Consider i's and $i+1$'s in the reading word of the tableau
- "Bracket" pairs of the form $(i+1, i)$
- Change unbracketed $i^a(i+1)^b$ to an $i^b(i+1)^a$

Example:

$$s_4\left(\begin{array}{cccccccccc} 4 & 4 & 5 & 5 & & & & & & \\ 3 & 3 & 4 & 4 & & & & & & \\ 2 & 2 & 2 & 3 & 3 & 4 & 5 & & & \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 & 5 \end{array}\right) = \begin{array}{cccccccccc} 4 & 5 & 5 & 5 & & & & & & \\ 3 & 3 & 4 & 4 & & & & & & \\ 2 & 2 & 2 & 3 & 3 & 5 & 5 & & & \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 5 \end{array}$$

Symmetrization operators on f-words

- Consider factor i and $i+1$
- Form brackets until smallest side is completely bracketed:
  - First bracket pairs $(i, i+1)$
  - Then pairs $(i, i+2)$
  - $\ldots$
  - Then pairs $(i, i-\infty)$
  - $\ldots$
  - Then pairs $(i, i-1)$
- Move unbracketed letters across the bracket by using whatever relations you have on words.

$$87653 \cdot 431 \to 873 \cdot 65431$$

$$4321 \cdot 432 \to 321 \cdot 4321$$

Recap

There are sets $\mathcal{F}_n$ with standardizations such that the polynomial

$$F_{\mathcal{F}_n} = \sum_{f \in \mathcal{F}_n} x^{ev(f)} = \sum_{s \in \mathcal{S}_n} \mathcal{L}_{Des(s)}$$

can be used to define the following symmetric functions:

Schur functions, Stanley symmetric functions, stable Grothendieck functions, affine Stanley symmetric functions, affine Grothendieck functions

We have operators that prove the symmetry of each of the above. Furthermore these operators provide a definition of charge so that, conjecturally, we have

$$F_{\mathcal{F}_n} = \sum_{\substack{f \in \mathcal{F}_n \\ ev(f) \in Par}} t^{ch(f)} P_{ev(f)}[X; t]$$

for the non-affine functions above (and hopefully for the affine version).