Quasisymmetric expansions of symmetric functions

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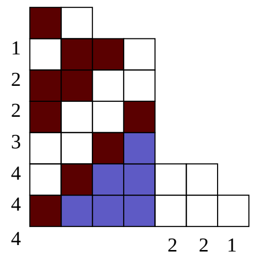

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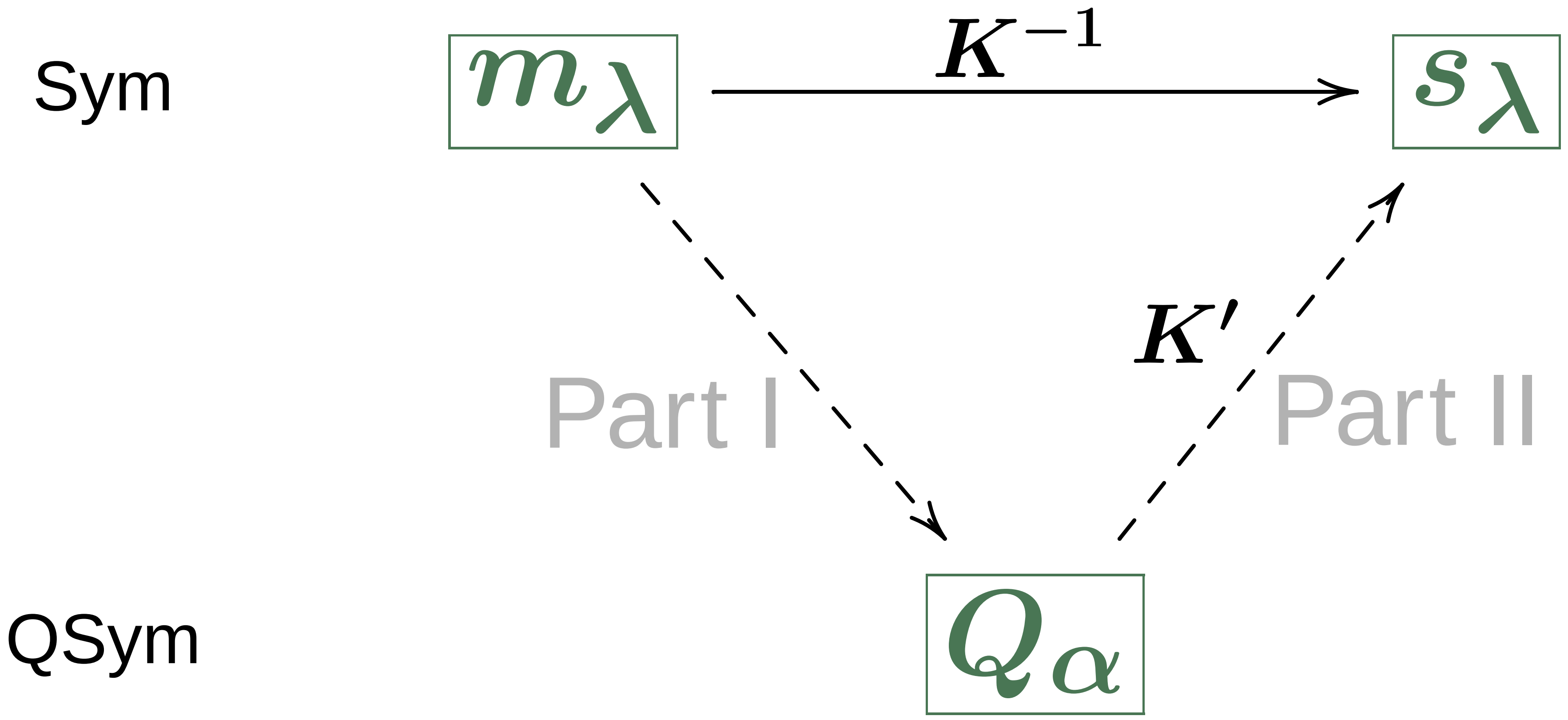
BIRS: Quasisymmetric Functions

November 15, 2010

Expansions

m_λ	s_λ
Have	Want

Expansions



Part I

Q-expansion of the plethysm $s_\mu[s_\lambda]$.

Schur functions

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$$

1	1	3
3	6	6
6		

$$\longrightarrow \quad x_1^2 x_3^2 x_6^3$$

Schur functions

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$$

1	1	3
3	6	6
6		

$\longrightarrow x_1^2 x_3^2 x_6^3$

Define $M_{223} = \sum_{i<j<k} x_i^2 x_j^2 x_k^3$

1	1
2	

1	1
3	

2	2
3	

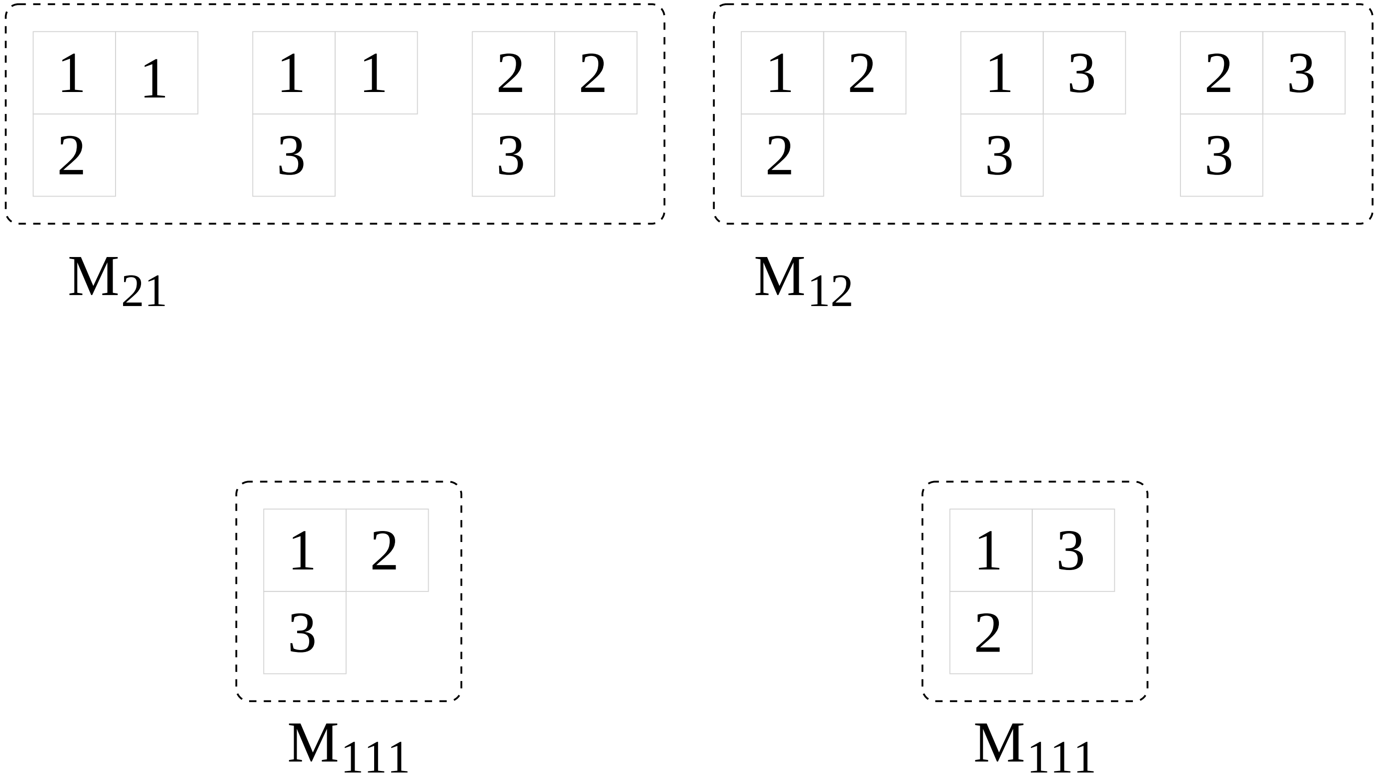
1	2
2	

1	3
3	

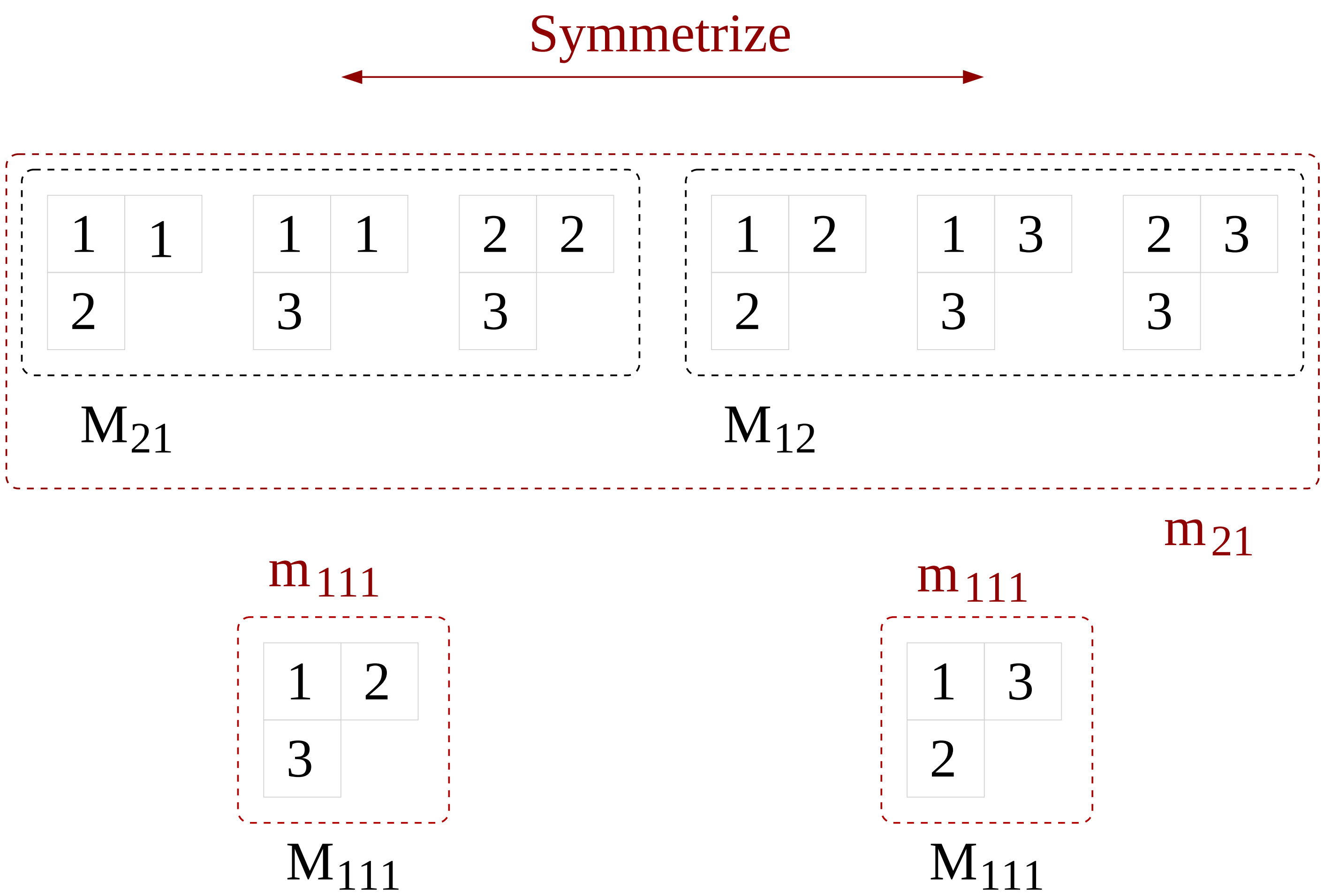
2	3
3	

1	2
3	

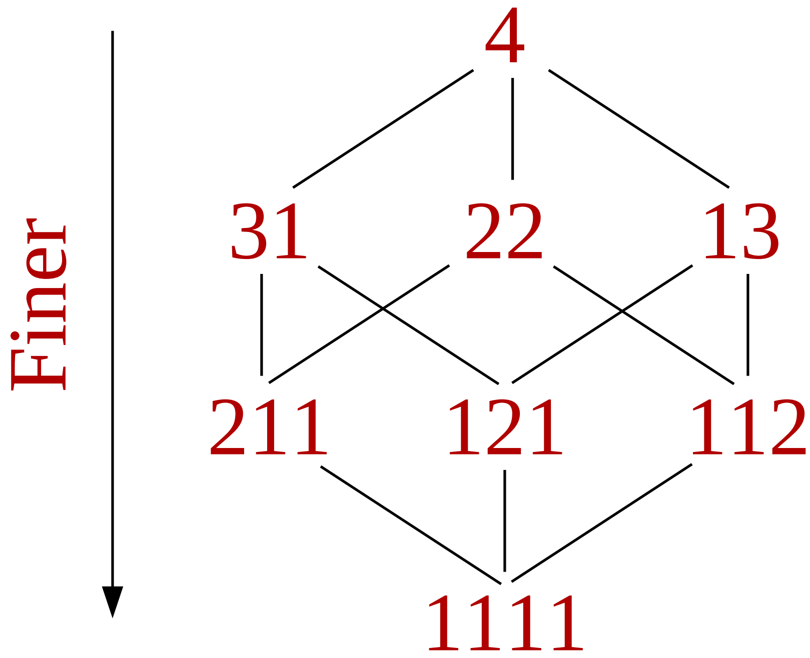
1	3
2	



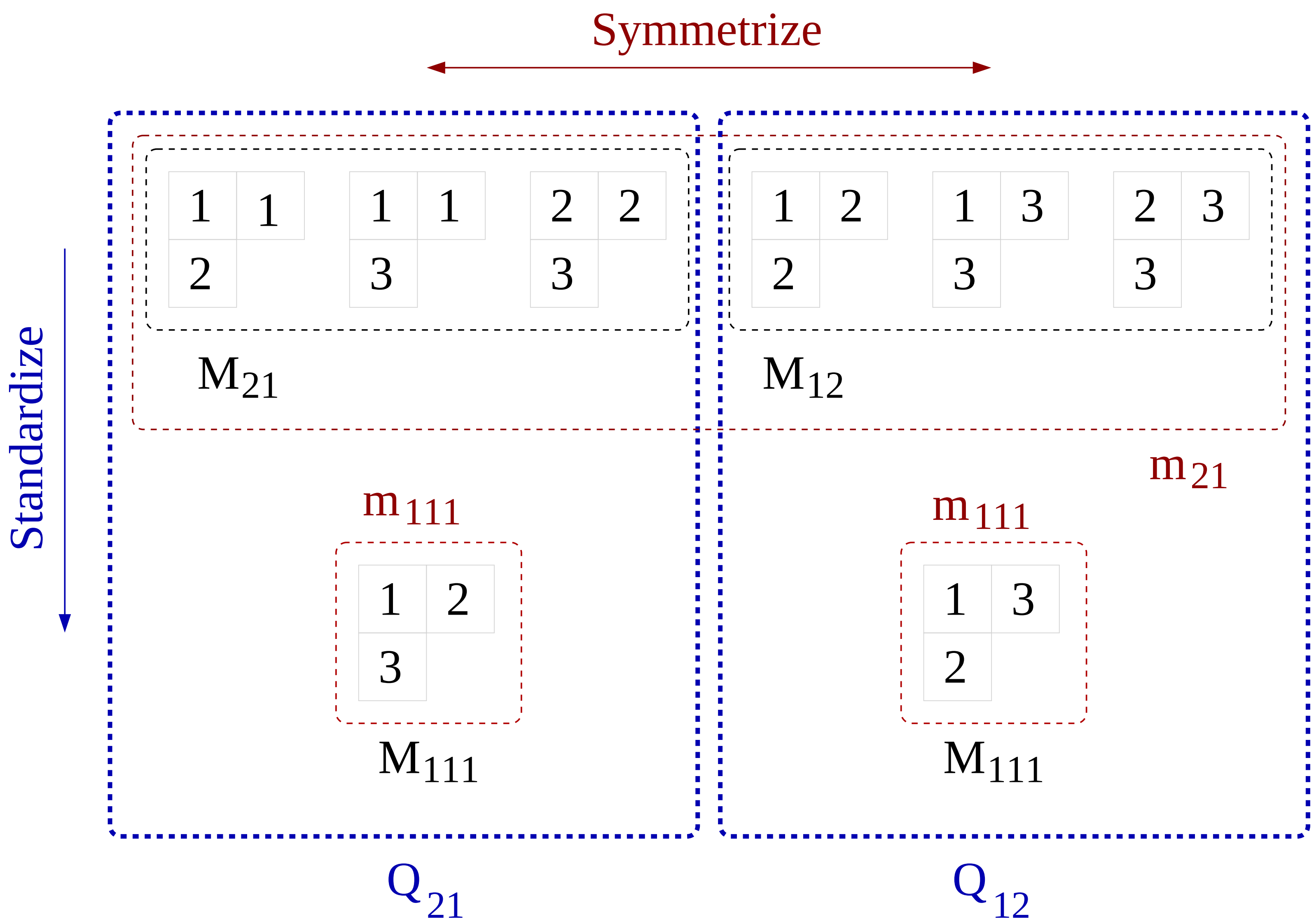
$$s_{21} = M_{21} + 2M_{111} + M_{12}$$



$$s_{21} = m_{21} + 2m_{111}$$



Define $\boldsymbol{Q_\alpha} = \sum_{\beta \text{ finer than } \boldsymbol{\alpha}} \boldsymbol{M_\beta}$


Symmetrize
Standardize
1 1
2
1 1
3
2 2
3
1 2
2
1 3
3
2 3
3
M21
M12
m21
m111
1 2
3
M111
m111
1 3
2
M111
Q21
Q12

$$s_{21} = Q_{21} + Q_{12}$$

Standardization

1	1	2	2
2	2	3	
3	3		

1	2	5	6
3	4	9	
7	8		

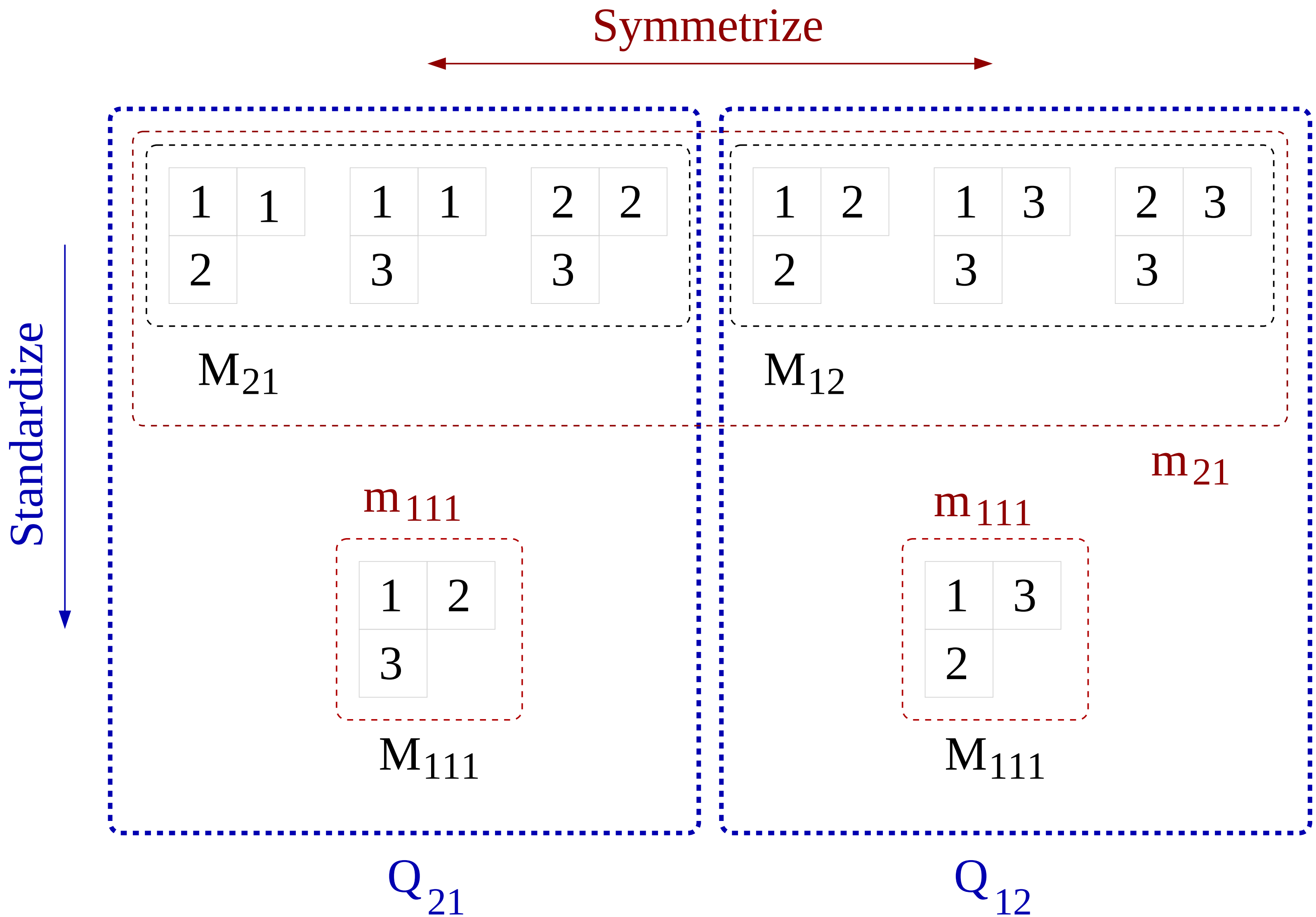
3 3 2 2 3 1 1 2 2

7 8 3 4 9 1 2 5 6

So $\mathbf{IDes} = \{\mathbf{2}, \mathbf{6}\} \rightarrow \mathbf{Asc} = \mathbf{243} \models \mathbf{9}$

Q_{221} living in s_{32}

T	rw	sort(rw)	β
$\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array}$	23112	$11 \mid 22 \mid 3$	221
$\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 4 & \end{array}$	34123	$1.2 \mid 33 \mid 4$	1121
$\begin{array}{ccc} 1 & 1 & 3 \\ 2 & 4 & \end{array}$	24113	$11 \mid 2.3 \mid 4$	2111
$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array}$	35124	$1.2 \mid 3.4 \mid 5$	11111



$$\boldsymbol{s_{21} = Q_{21} + Q_{12} = \sum_{T \in SYT(21)} Q_{\mathrm{Asc}(T)}}$$

Plethysm

Consider

- $x_1, \ldots, x_t$ variables
- $m_1, \ldots, m_s$ monic monomials in x_i
- $g(x_1, \ldots, x_t) = m_1 + \cdots + m_s$
- $f(x_1, \ldots, x_s)$ symmetric.

Define the plethysm

$$f[g] = f(m_1, m_2, \ldots, m_s).$$

Plethysm example

Let

- $g(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$
- Set $m_1 = x_1^2$, $m_2 = m_3 = x_1x_2$ and $m_3 = x_2^2$.
- $f(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} x_i^2$.

Then $f[g] = x_1^4 + 2x_1^2x_2^2 + x_2^4$.

Part I: Main result

Theorem[Loehr-W '10]

Let $\mu \vdash a$ and $\lambda \vdash b$. Then

$$s_\mu[s_\lambda] = \sum_{A \in \mathrm{Std}(\mu,\lambda)} Q_{\mathrm{Asc}(A)}.$$

cf. Malvenuto-Reutenauer '98

Schur plethysms

Sample term in $h_3[h_5] = s_{(3)}[s_{(5)}]$

1 1 3 3 5	1 2 2 2 4	2 2 3 3 3

↓

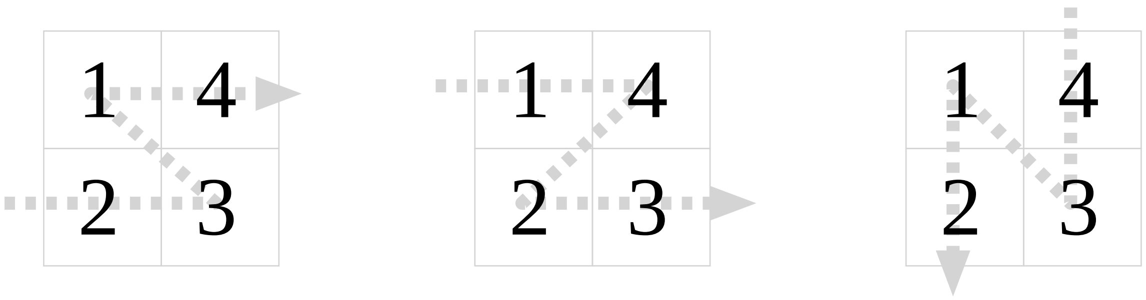
1	1	3	3	5
1	2	2	2	4
2	2	3	3	3

Q What is reading order?

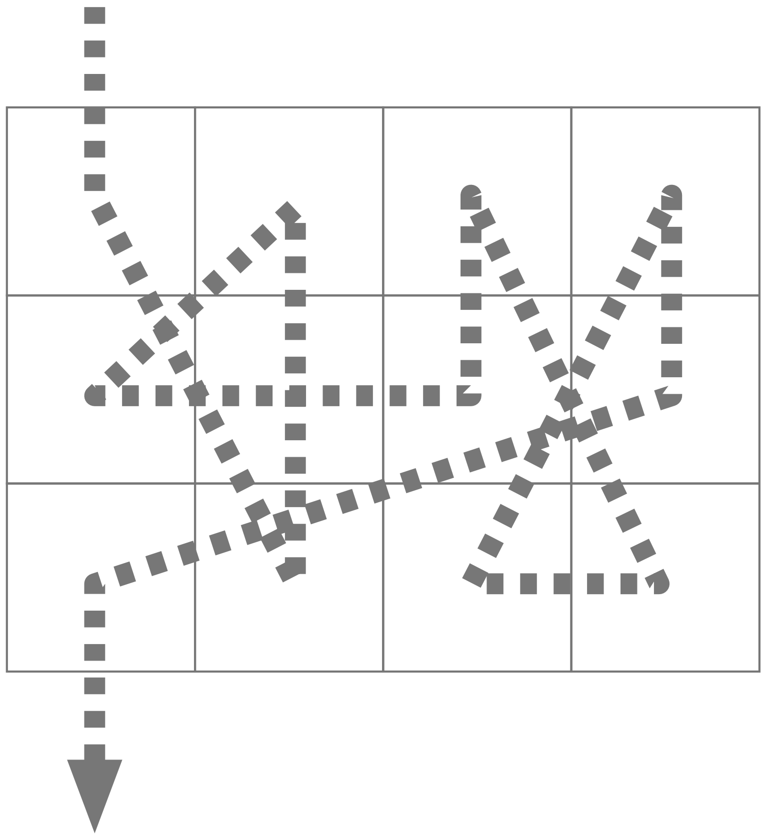
Naive approach

$$h_2[h_2] = Q_4 + Q_{22} + Q_{121}$$

Order



α	13	31	211

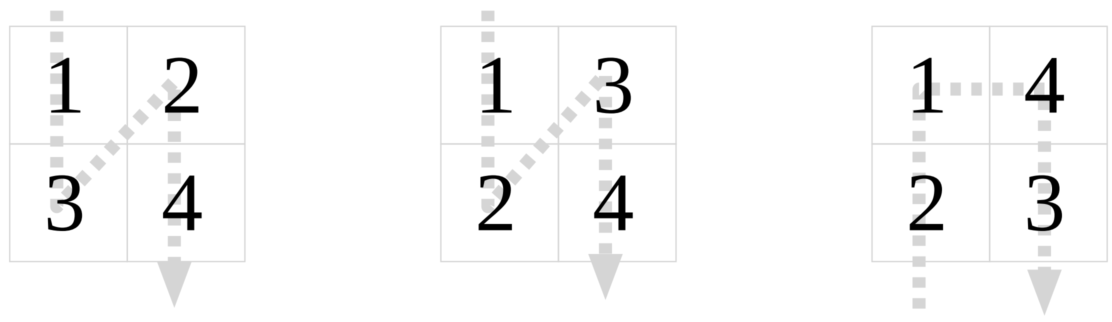


Correct approach

$$h_2[h_2] = Q_4 + Q_{22} + Q_{121}$$

The reading order is dynamic

Order



α	22	4	121

Reading order for $h_a[h_b]$

1. Read columns from left to right.

2. Visit cells in i-th column according to order of cells in $(i+1)$-st column.

Which Q_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

Which Q_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

2 3 1

Which Q_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

2 3 1 6 4 5

Which Q_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

$\mathbf{2\ 3\ 1\ 6\ 4\ 5\ 7\ 10\ 9}$

Which Q_α do we live in?

A=

1	6	7	8
2	4	9	12
3	5	10	11

2 3 1 6 4 5 7 10 9 8 12 11

A contributes to Q_{143121}

Partie II

Schur expansion from Q-expansion

cf. Assaf '10

The Kostka matrix

$$s_\lambda = \sum_{\mu \vdash n} K_n(\lambda, \mu) m_\mu$$

$$m_\lambda = \sum_{\mu \vdash n} K_n^{-1}(\lambda, \mu) s_\mu$$

The Kostka matrix

$$s_\lambda = \sum_{\mu \vdash n} K_n(\lambda, \mu) m_\mu$$

$$m_\lambda = \sum_{\mu \vdash n} K_n^{-1}(\lambda, \mu) s_\mu$$

$$f = \sum_{\mu \vdash n} x_\mu m_\mu = \sum_{\lambda \vdash n} y_\lambda s_\lambda$$

$$y_\lambda = \sum_{\mu \vdash n} x_\mu K_n^{-1}(\mu, \lambda)$$

K_4 and K_4^{-1}

$$K_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

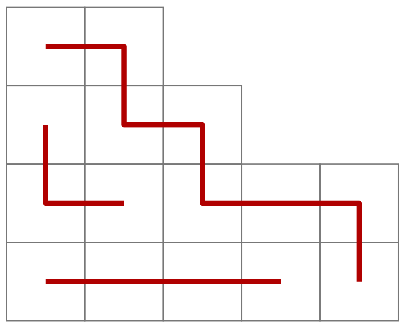
$$K_4^{-1} = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$\boldsymbol{K}$

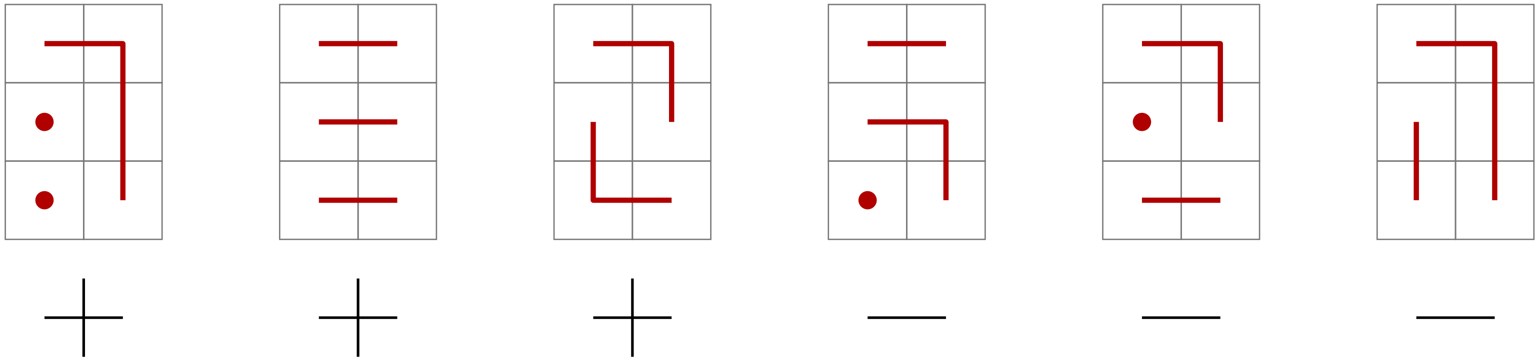
Theorem[Eğecioğlu-Remmel ’90]

$$\boldsymbol{K}_n^{-1}(\boldsymbol{\mu},\boldsymbol{\lambda}) = \sum_{\boldsymbol{S}\in \mathbf{srht}(\boldsymbol{\lambda},\boldsymbol{\mu})} \mathbf{sgn}(\boldsymbol{S})$$

Example $+\mathbf{1}$ to $\boldsymbol{K}_{\mathbf{15}}^{-\mathbf{1}}(\mathbf{843}, \mathbf{5532})$



y_{222}



Example

$$y_{222} = x_{411} + x_{222} + x_{33} - 2x_{321} - x_{42}$$

Q_α coefficients

$$f = \sum_{\alpha \models n} z_\alpha Q_\alpha = \sum_{\lambda \vdash n} y_\lambda s_\lambda$$

Goal Find a combinatorial formula

$$y_\lambda = \sum_{\alpha \models n} z_\alpha K_n^*(\alpha, \lambda)$$

Rectangular Kostka

$$s_{\lambda} = \sum_{\alpha \models n} \hat{K}_n(\lambda, \alpha) M_{\alpha}$$

$\hat{K}_4 =$

	4	31	22	211	13	121	112	1111
4	1	1	1	1	1	1	1	1
31	0	1	1	2	1	2	2	3
22	0	0	1	1	0	1	1	2
211	0	0	0	1	0	1	1	3
1111	0	0	0	0	0	0	0	1

Right inverse $\hat{K}'$

$$\hat{K}'_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\hat{K}'_n(\alpha, \lambda) = \begin{cases} \mathrm{sgn}(S), & S \in \mathrm{srht}(\lambda, \alpha), \\ 0, & S \text{ doesn't exist.} \end{cases}$$

For $\alpha, \beta \models n$, define A_n by

$$A_n(\alpha, \beta) = [\beta \text{ finer than } \alpha]$$

Then $s_\lambda = \sum_\alpha [\hat{K}_n A_n^{-1}](\lambda, \alpha) Q_\alpha$

Define $K_n^* = A_n \hat{K}_n' = \sum_{\beta \text{ finer than } \alpha} K_n'(\beta, \lambda)$

So $y_\lambda = \sum_{\alpha \models n} z_\alpha K_n^*(\alpha, \lambda)$

$$
\boldsymbol{K}_4^* = \begin{array}{c} \\ \mathbf{4} \\ \mathbf{31} \\ \mathbf{22} \\ \mathbf{211} \\ \mathbf{13} \\ \mathbf{121} \\ \mathbf{112} \\ \mathbf{1111} \end{array}
\begin{array}{c} \begin{array}{ccccc} \mathbf{4} & \mathbf{31} & \mathbf{22} & \mathbf{211} & \mathbf{1111} \end{array} \\
\begin{pmatrix}
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}
\end{pmatrix} \end{array}
$$

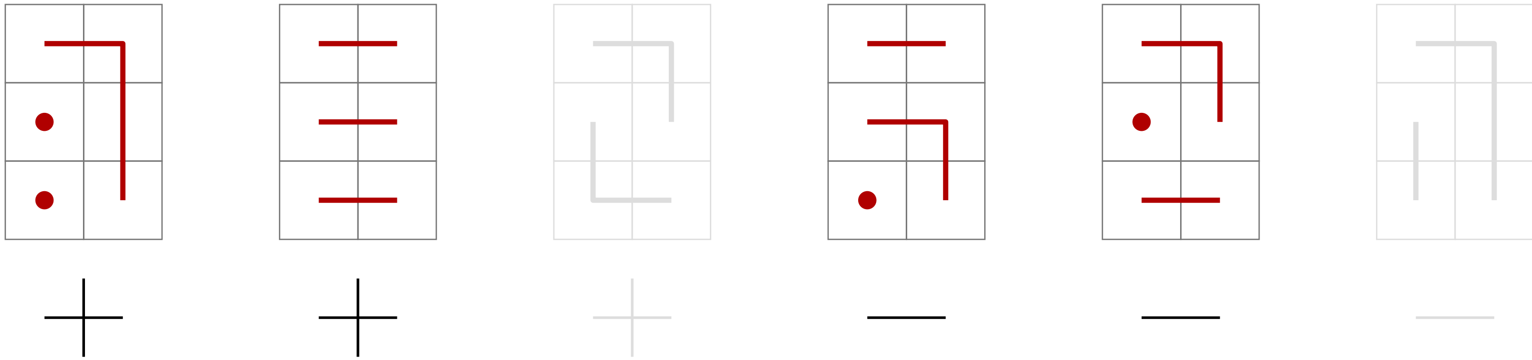
Partie II: Main result

Theorem[Egge-Loehr-W '10]

For $\boldsymbol{\alpha} \models \boldsymbol{n}$ and $\boldsymbol{\lambda} \vdash \boldsymbol{n}$,

$$\boldsymbol{K}_n^*(\boldsymbol{\alpha}, \boldsymbol{\lambda}) = \begin{cases} \mathbf{sgn}(\boldsymbol{S}), & \boldsymbol{S} \in \underline{\mathbf{srht}}(\boldsymbol{\lambda}, \boldsymbol{\alpha}), \\ \mathbf{0}, & \boldsymbol{S} \text{ doesn't exist.} \end{cases}$$

An example



Example

$$y_{222} = z_{114} + z_{222} - z_{132} - z_{213}$$

Part III (Bonus!)

Theorem [Thrall '42, . . . , L-W '10]

$$\boldsymbol{h_2[h_n] = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} s_{(2n-k,k)}}$$

Proof Sign-reversing involution on pairs $(\boldsymbol{A}, \boldsymbol{S})$ where

- $\boldsymbol{A \in S_{2,n}((2),(n))}$
- $\boldsymbol{S \in \underline{\mathrm{srht}}(\lambda, \mathrm{Asc}(A))}$

Enumerative results

For $\boldsymbol{\lambda} \vdash \boldsymbol{n}$, the number of flat special rim-hook tableaux of shape $\boldsymbol{\lambda}$ is $\prod_{i=1}^{k} \mathbf{DF}(\boldsymbol{\lambda}^i)$.

$\boldsymbol{n}$	1	2	3	4	5	6	7	8	9
$\underline{\mathbf{srht}}(\boldsymbol{n})$	1	2	3	6	9	18	27	50	79
$\mathbf{srht}(\boldsymbol{n})$	1	3	7	17	37	85	181	399	841

$$A(q) = \sum_{n} \sum_{\lambda \vdash n} \underline{\mathrm{srht}}(\lambda) q^n$$

$$= \sum_{k=0}^{\infty} k! q^{k^2} \prod_{j=1}^{k} \frac{1}{1 - jq^j} \prod_{j=1}^{k} \frac{1}{1 - q^j}$$