

The module of affine descents (over the descent algebra)

T. Kyle Petersen
DePaul University

joint with Marcelo Aguiar (Texas A&M),

Banff

November, 2010

The module of affine descents

Some background

Solomon's descent algebra

Affine structures

The affine descent module

Descents

$W = \langle s_\alpha : \alpha \in \Delta \rangle$ finite Coxeter group

$\Delta = \{\alpha_1, \dots, \alpha_n\}$ (nodes of Dynkin diagram)

$D(w) := \{1 \leq i \leq n : w\alpha_i < 0\}$

Descents

$W = \langle s_\alpha : \alpha \in \Delta \rangle$ finite Coxeter group

$\Delta = \{\alpha_1, \dots, \alpha_n\}$ (nodes of Dynkin diagram)

$D(w) := \{1 \leq i \leq n : w\alpha_i < 0\}$

$\Delta_0 = \Delta \cup \{\alpha_0 = -\tilde{\alpha}\}$ (nodes of extended Dynkin diagram)

$\tilde{D}(w) := \{0 \leq i \leq n : w\alpha_i < 0\}$

Descents

$W = \langle s_\alpha : \alpha \in \Delta \rangle$ finite Coxeter group

$\Delta = \{\alpha_1, \dots, \alpha_n\}$ (nodes of Dynkin diagram)

$D(w) := \{1 \leq i \leq n : w\alpha_i < 0\}$

$\Delta_0 = \Delta \cup \{\alpha_0 = -\tilde{\alpha}\}$ (nodes of extended Dynkin diagram)

$\tilde{D}(w) := \{0 \leq i \leq n : w\alpha_i < 0\}$

Type A is “cyclic” descents:

w	$D(w)$	$\tilde{D}(w)$
123	\emptyset	$\{0\}$
132	$\{2\}$	$\{0, 2\}$
213	$\{1\}$	$\{0, 1\}$
231	$\{2\}$	$\{2\}$
312	$\{1\}$	$\{1\}$
321	$\{1, 2\}$	$\{1, 2\}$

Descent algebras

$$X_J := \sum_{D(w) \subseteq J} w \quad (\text{in } \mathbb{Z}[W])$$

Solomon (1976): $\text{Sol}(W) := \text{Span}\{X_J : J \subseteq [n]\}$ is a subalgebra

Descent algebras

$$X_J := \sum_{D(w) \subseteq J} w \quad (\text{in } \mathbb{Z}[W])$$

Solomon (1976): $\text{Sol}(W) := \text{Span}\{X_J : J \subseteq [n]\}$ is a subalgebra

$$\tilde{X}_J := \sum_{\tilde{D}(w) \subseteq J} w \quad (\text{in } \mathbb{Z}[W])$$

$$\tilde{x}_k := \sum_{|J|=k} \tilde{X}_J \quad (\text{in } \mathbb{Z}[W])$$

Cellini (1995): $\text{Span}\{\tilde{x}_k : k \in [n]\}$ is a commutative subalgebra

Descent algebras

$$X_J := \sum_{D(w) \subseteq J} w \quad (\text{in } \mathbb{Z}[W])$$

Solomon (1976): $\text{Sol}(W) := \text{Span}\{X_J : J \subseteq [n]\}$ is a subalgebra

$$\tilde{X}_J := \sum_{\tilde{D}(w) \subseteq J} w \quad (\text{in } \mathbb{Z}[W])$$

$$\tilde{x}_k := \sum_{|J|=k} \tilde{X}_J \quad (\text{in } \mathbb{Z}[W])$$

Cellini (1995): $\text{Span}\{\tilde{x}_k : k \in [n]\}$ is a commutative subalgebra
 $\text{Span}\{\tilde{X}_J\}$ is NOT a subalgebra. However...

Main result

$$\widetilde{\text{Sol}}(W) := \text{Span}\{\widetilde{X}_J = \sum_{\widetilde{D}(w) \subseteq J} w : J \subseteq [0, n]\}$$

Theorem (Aguiar-P.)

$\widetilde{\text{Sol}}(W)$ is a left module over Solomon's descent algebra

- We call $\widetilde{\text{Sol}}(W)$ the “module of affine descents”

The geometric approach

Tits (1976), Bidigare (1997):

$$\text{Coxeter complex} \longleftrightarrow \text{Sol}(W)$$

The geometric approach

Tits (1976), Bidigare (1997):

$$\text{Coxeter complex} \longleftrightarrow \text{Sol}(W)$$

Our approach:

$$\text{Steinberg torus} \longleftrightarrow \widetilde{\text{Sol}}(W)$$

The module of affine descents

Some background

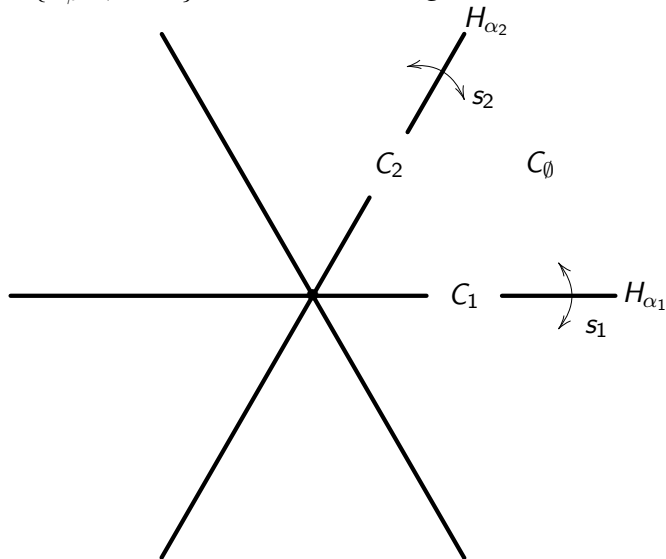
Solomon's descent algebra

Affine structures

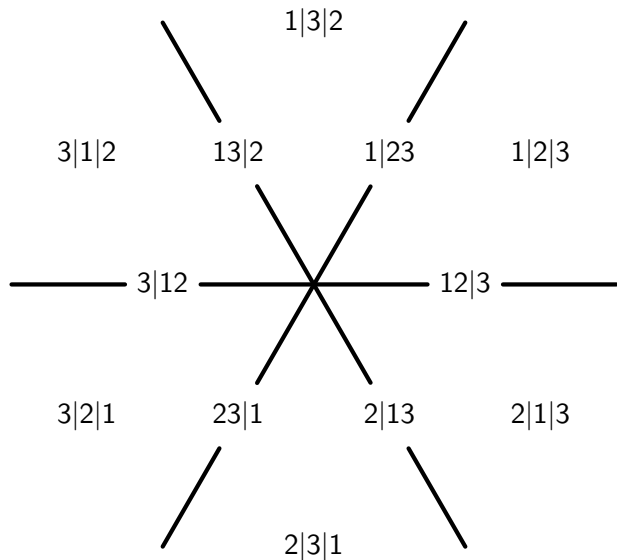
The affine descent module

The Coxeter arrangement

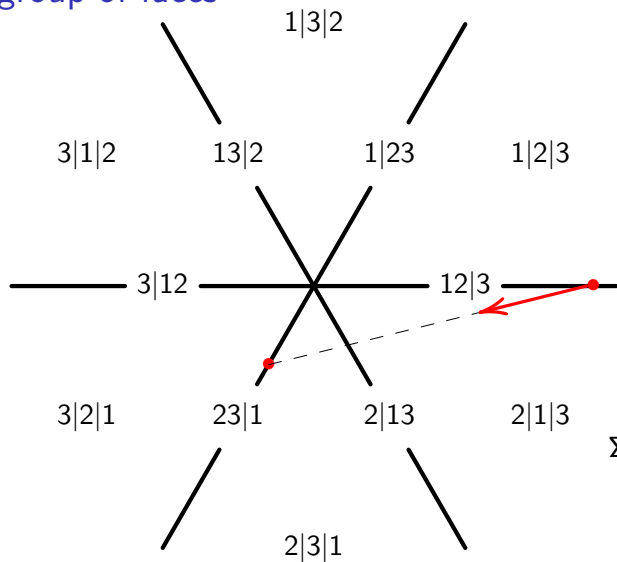
$\Sigma = \{H_\beta : \beta \in \Phi\}$, the Coxeter arrangement



Semigroup of faces

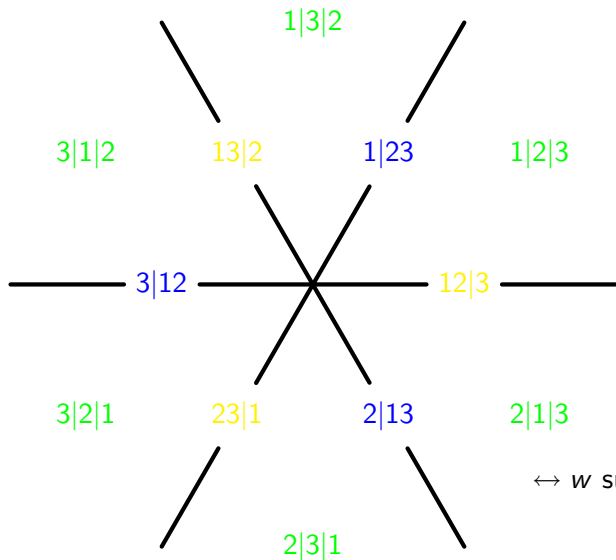


Semigroup of faces

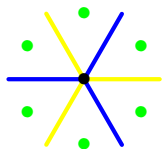


$$12|3 \cdot 23|1 = 2|1|3$$

W -orbits



The algebra Σ^W of invariants



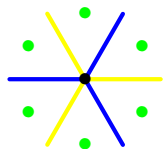
$$\sigma_{\emptyset} = 123 \text{ (origin)}$$

$$\sigma_1 = 1|23 + 2|13 + 3|12$$

$$\sigma_2 = 12|3 + 13|2 + 23|1$$

$$\sigma_{12} = 1|2|3 + 1|3|2 + 2|1|3 + 2|3|1 + 3|1|2 + 3|2|1$$

The algebra Σ^W of invariants



$$\sigma_{\emptyset} = 123 \text{ (origin)}$$

$$\sigma_1 = 1|23 + 2|13 + 3|12$$

$$\sigma_2 = 12|3 + 13|2 + 23|1$$

$$\sigma_{12} = 1|2|3 + 1|3|2 + 2|1|3 + 2|3|1 + 3|1|2 + 3|2|1$$

W -orbits closed under products of faces:

$$\begin{aligned} \sigma_1 \sigma_2 &= 1|2|3 + 1|3|2 + 1|23| + \\ &\quad 2|1|3 + 2|3|1 + 2|13| + \\ &\quad 3|1|2 + 3|2|1 + 3|12| \\ &= \sigma_{12} + \sigma_1 \end{aligned}$$

Solomon's descent algebra

Theorem (Bidigare, Tits, - see also Brown)

Σ^W is (anti-)isomorphic to Solomon's descent algebra via:

$$\sigma_J \leftrightarrow X_J := \sum_{D(w) \subseteq J} w \quad (J \subseteq [n])$$

$$\sigma_\emptyset = 123$$

$$\sigma_1 = 1|23 + 2|13 + 3|12$$

$$\sigma_2 = 12|3 + 13|2 + 23|1$$

$$\sigma_{12} = \begin{array}{l} 1|2|3 + 1|3|2 + 2|1|3 \\ + 2|3|1 + 3|1|2 + 3|2|1 \end{array}$$

$$X_\emptyset = 123$$

$$X_1 = 123 + 213 + 312$$

$$X_2 = 123 + 132 + 231$$

$$X_{12} = \begin{array}{l} 123 + 132 + 213 \\ + 231 + 312 + 321 \end{array}$$

Solomon's descent algebra

$$\sigma_J \sigma_K = \sum_L c_{J,K}^L \sigma_L$$
$$\sigma_1 \sigma_2 = \begin{array}{l} 1|2|3 + 1|3|2 + 1|23| + \\ 2|1|3 + 2|3|1 + 2|13| + \\ 3|1|2 + 3|2|1 + 3|12| \end{array}$$
$$= \sigma_{12} + \sigma_1$$

$$X_K X_J = \sum_L c_{J,K}^L X_L$$
$$X_2 X_1 = \begin{array}{l} 123 + 132 + 231 + \\ 213 + 312 + 321 + \\ 312 + 213 + 123 \end{array}$$
$$= X_{12} + X_1$$

The module of affine descents

Some background

Solomon's descent algebra

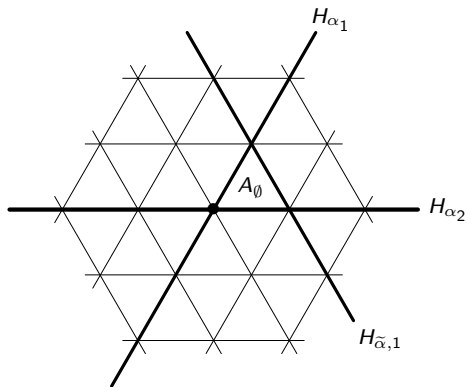
Affine structures

The affine descent module

The affine arrangement

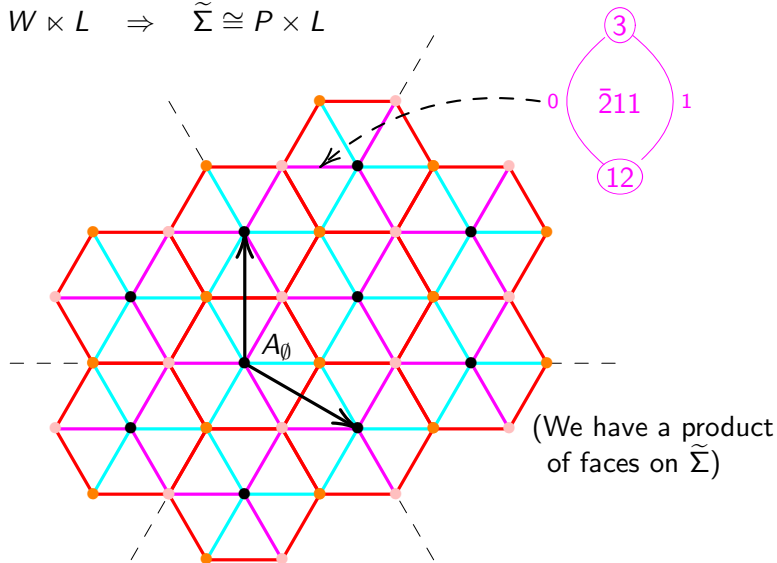
$$\widetilde{W} = \langle s_\alpha : \alpha \in \Delta_0 \rangle$$

$\widetilde{\Sigma}(A_2) :$



The affine arrangement

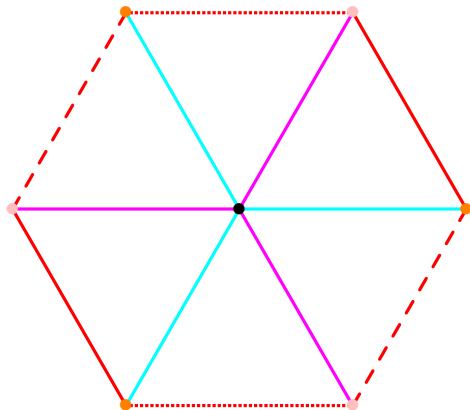
$$\widetilde{W} \cong W \ltimes L \Rightarrow \widetilde{\Sigma} \cong P \times L$$



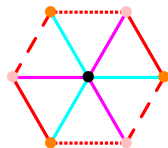
The Steinberg torus

$$\Sigma_T \cong \tilde{\Sigma}/L$$

(Product is ill-defined)



W-orbits in the Steinberg torus



$$\tilde{\sigma}_0 = \overset{0}{\circlearrowleft} (123), \quad \tilde{\sigma}_1 = \overset{1}{\circlearrowleft} (123), \quad \tilde{\sigma}_2 = \overset{2}{\circlearrowleft} (123)$$

$$\tilde{\sigma}_{01} = 0 \overset{1}{\circlearrowleft} (23) \overset{1}{\circlearrowright} + 0 \overset{2}{\circlearrowleft} (13) \overset{1}{\circlearrowright} + 0 \overset{3}{\circlearrowleft} (12) \overset{1}{\circlearrowright}$$

$$\tilde{\sigma}_{02} = 2 \overset{1}{\circlearrowleft} (23) \overset{0}{\circlearrowright} + 2 \overset{2}{\circlearrowleft} (13) \overset{0}{\circlearrowright} + 2 \overset{3}{\circlearrowleft} (12) \overset{0}{\circlearrowright}$$

$$\tilde{\sigma}_{12} = 1 \overset{1}{\circlearrowleft} (23) \overset{2}{\circlearrowright} + 1 \overset{2}{\circlearrowleft} (13) \overset{2}{\circlearrowright} + 1 \overset{3}{\circlearrowleft} (12) \overset{2}{\circlearrowright}$$

$$\tilde{\sigma}_{012} = \overset{0}{\circlearrowleft} \overset{1}{\circlearrowright} \overset{1}{\circlearrowright} (32) + \overset{0}{\circlearrowleft} \overset{1}{\circlearrowright} \overset{1}{\circlearrowright} (23) + \overset{0}{\circlearrowleft} \overset{2}{\circlearrowright} \overset{1}{\circlearrowright} (31) + \overset{0}{\circlearrowleft} \overset{2}{\circlearrowright} \overset{1}{\circlearrowright} (13) + \overset{0}{\circlearrowleft} \overset{3}{\circlearrowright} \overset{1}{\circlearrowright} (21) + \overset{0}{\circlearrowleft} \overset{3}{\circlearrowright} \overset{1}{\circlearrowright} (12)$$

The Steinberg torus

Theorem (Dilks-P.-Stembridge (2009))

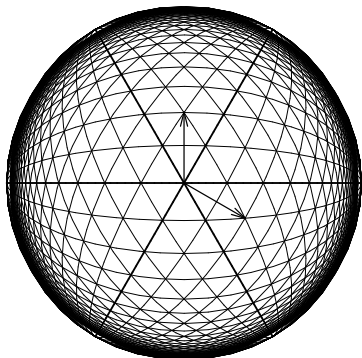
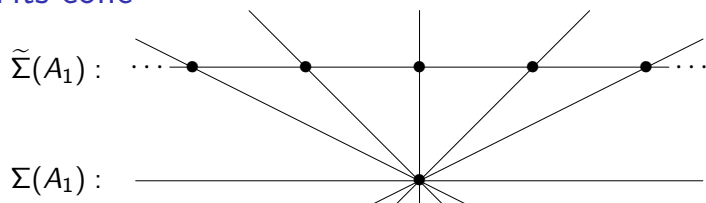
Affine descent sets characterize faces of the Steinberg torus:

$$\{\text{faces with color-set } J\} \leftrightarrow \{w \in W : \tilde{D}(w) \subseteq J\},$$

i.e.,

$$\tilde{\sigma}_J \leftrightarrow \tilde{X}_J$$

The Tits cone



The module of affine descents

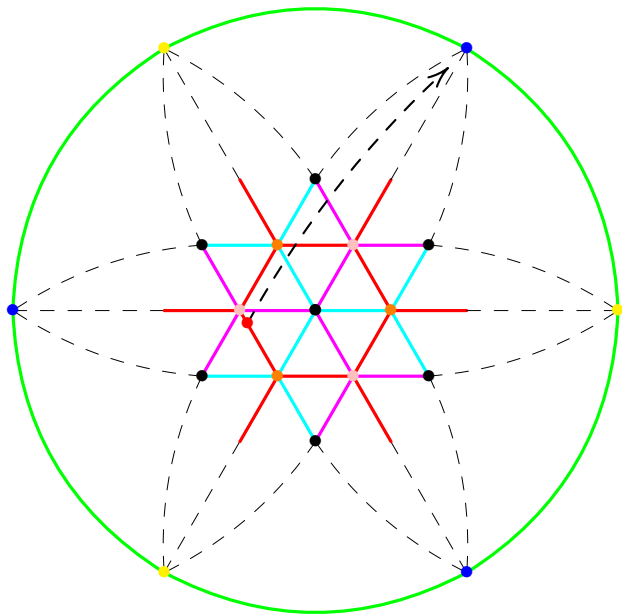
Some background

Solomon's descent algebra

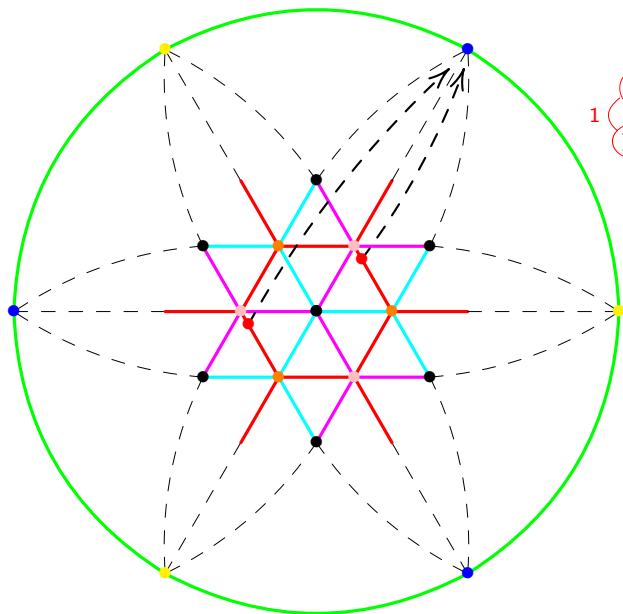
Affine structures

The affine descent module

The product $\tilde{\Sigma} \times \Sigma \rightarrow \tilde{\Sigma}$



The product $\Sigma_T \times \Sigma \rightarrow \Sigma_T$



$$1 \begin{array}{c} \textcircled{2} \\ \textcircled{13} \end{array} 2 \cdot 1|23$$

$$= \begin{array}{c} 0 \quad \textcircled{3} \quad 1 \\ \textcircled{1} \quad \quad \textcircled{2} \\ \quad \quad \quad 2 \end{array}$$

The semigroup Σ
acts on Σ_T

W -invariants

$$\Sigma^W \cong \text{Sol}(W)$$

W -invariants

$$\Sigma^W \cong \text{Sol}(W)$$

acts on

$$(\Sigma_T)^W \cong \widetilde{\text{Sol}}(W)$$

$$\tilde{\sigma}_J \sigma_K = \sum_L c_{J,K}^L \tilde{\sigma}_L \leftrightarrow X_K \tilde{X}_J = \sum_L c_{J,K}^L \tilde{X}_L$$

W -invariants

$$\Sigma^W \cong \text{Sol}(W)$$

acts on

$$(\Sigma_T)^W \cong \widetilde{\text{Sol}}(W)$$

$$\tilde{\sigma}_J \sigma_K = \sum_L c_{J,K}^L \tilde{\sigma}_L \leftrightarrow X_K \tilde{X}_J = \sum_L c_{J,K}^L \tilde{X}_L$$

$$2 \begin{array}{c} \textcircled{1} \\ \textcircled{23} \end{array} 0 + 2 \begin{array}{c} \textcircled{2} \\ \textcircled{13} \end{array} 0 + 2 \begin{array}{c} \textcircled{3} \\ \textcircled{12} \end{array} 0 \leftrightarrow 231 + 132 + 123$$

Main result

Theorem (Aguar-P.)

For any irreducible (finite) Weyl group

1. $\widetilde{\Sigma}(W)$ is a right module over $\Sigma(W)$
2. $\Sigma_T(W)$ is a right module over $\Sigma(W)$
3. $\widetilde{\text{Sol}}(W)$ is a left module over $\text{Sol}(W)$