Combinatorial Integral Approximation for Mixed-Integer Nonlinear Optimal Control

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Banff


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Outline

MIOC introduction

Methods for Mixed-Integer Optimal Control

...based on HJB / Dynamic Programming

...based on Direct Methods

MINLP

MS MINTOC

Switching Costs and Combinatorial Constraints

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Applications I: Valves and ports

- Watergates and on/off pumps in water networks

  [Burgschweiger, Deuerlein, Gugat, Hante, Leugering, Steinbach, . . .]

Images: Wikipedia, BASF SE

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- Valves, (de)compressors in gas networks

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- Watergates and on/off pumps in water networks

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- Valves, (de)compressors in gas networks

  [Leugering, Martin, Schultz, Steinbach, S. Ulbrich, . . .]
- Valves in separation processes

  [Biegler, Engell, Findeisen, Grossmann, Kienle, Marquardt, Swartz, . . .]
  - Slop cut recycling in batch distillation

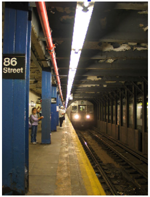
    [Bock, Diehl, S., . . .]
  - Evaporator operation [Sonntag, Engell, . . .]
  - Simulated Moving Bed superstructure

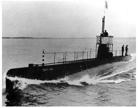
    [Biegler, Engell, Potschka, S., Weismantel, Westerlund, . . .]

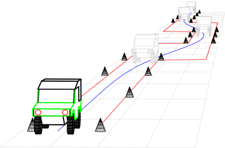
Images: Wikipedia, BASF SE

Applications II: optimal gear shifts in transport

- New York subway: energy optimal rides (?), (?)
- Submarine control (?)
- Look-ahead control in heavy duty trucks (?), (?), (?)
  - Feedback control, GPS data, $\geq$ 16 gears
- Elchtest benchmark: automobile testdriving (??), (?), [Borelli, ...]
- Periodic time-optimal automobile driving, (?)
- Formula 1 racing, (?)







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Images: Wikipedia, youtube (Hellström), Pesch, Gerdts in 'Mathematics — Key Technology for the Future'

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Applications III: Others

- Traffic lights in Traffic flow [Aw, Leugering, Rascle, . . .], (?), (?)
- Biology / Medicine: inhibition, treatment [Engelhart, Lebiedz, S., . . .]
- Economics: yes/no decisions (part time, 2nd job?) [Kübler, Kuhn]
- Robot swarm movement and communication (?)
- Optimum experimental design [Bock, Körkel, Hoffmann, Kostina, Lebiedz, Schittkowski, Schlöder, Seidel-Morgenstern, . . .]
  - Variables: is measurement done or not, $w(x, t) \in \{0, 1\}$
- . . .
- benchmark library: `http://mintoc.de`

Problem class

$$
\begin{aligned}
\min_{x,v,u}\ & \phi(x(t_f)) \\
\text{s.t.}\quad \dot{x}(t) &= f(x(t), v(t), u(t)), \\
0 &\leq c(x(t), u(t)), \\
0 &\leq r_{\text{in}}(x(t_0), \dots, x(t_f)), \\
0 &= r_{\text{eq}}(x(t_0), \dots, x(t_f)), \\
v(t) &\in \Omega := \{v^1, v^2, \dots, v^{n_\omega}\}, \quad t \in [t_0, t_f].
\end{aligned}
$$

- x differential states, u, v control functions
- All functions sufficiently smooth
- Generalizations: PDEs, algebraic variables, constant-in-time control values, multi-stage, ...
- Focus of talk: controls $v(t)$ from finite set $v^i \in \Omega \subseteq \mathbb{R}^{n_v}$.

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Why is this problem class difficult?

$$\begin{array}{llcl} \min\limits_{x,v,u} & \phi(x(t_f)) & & \\ \text{s.t.} & \dot{x}(t) & = & f(x(t), v(t), u(t)), \\ & 0 & \leq & c(x(t), u(t)), \\ & 0 & \leq & r_{\text{in}}(x(t_0), \ldots, x(t_{\text{f}})), \\ & 0 & = & r_{\text{eq}}(x(t_0), \ldots, x(t_{\text{f}})), \\ & v(t) & \in & \Omega := \{v^1, v^2, \ldots, v^{n_\omega}\}, \quad t \in [t_0, t_{\text{f}}]. \end{array}$$

- Nonlinear
- Differential equations
- Path and point constraints
- Integer decisions (in function space)

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Switching Costs and Combinatorial Constraints

Alternative Methods

- HJB / Dynamic Programming
  - based on Bellman's Principle of Optimality
  - tabulated enumeration
  - easy to extend to integer controls
  - but: curse of dimensionality

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Alternative Methods

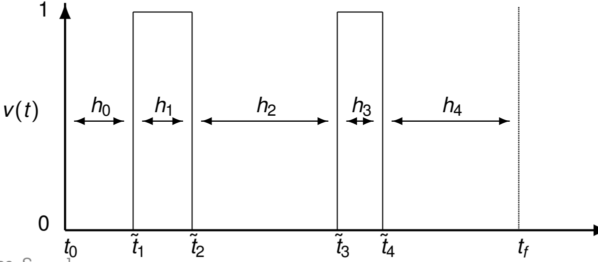

- HJB / Dynamic Programming
  - based on Bellman's Principle of Optimality
  - tabulated enumeration
  - easy to extend to integer controls
  - but: curse of dimensionality
- Pontryagin's Maximum Principle
  - use optimality conditions in function space
  - solve resulting boundary value problem
  - $u^*(t)$ is the pointwise maximum of the Hamiltonian

$$u^*(t) = \arg \max_{u \in U} H(x(t), u, \lambda(t))$$

  U may also be a discrete set!
  - practice: use switching functions to detect changes
  - difficult for singular and path-constrained arcs
  - analytical work, needs to be well initialized

Switching time optimization

Take given switching order with piecewise fixed controls $v(\cdot)$, optimize interval lengths (after standard time transformation)



[Gerdts, Kaya, Leineweber, Maurer, Noakes, S., . . .]

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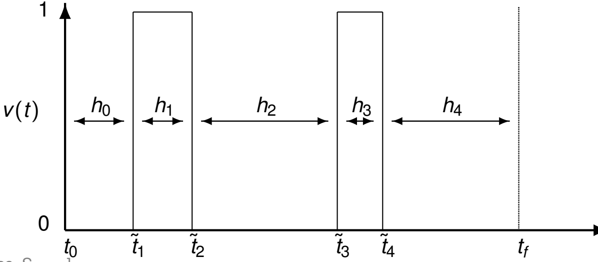

[Gerdts, Kaya, Leineweber, Maurer, Noakes, S., ...]

$$\min_{x,v,u} \phi(x(t_f))$$

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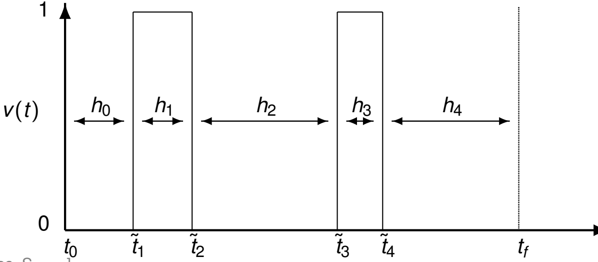

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$$\min_{x,v,u} \phi(x(t_f))$$
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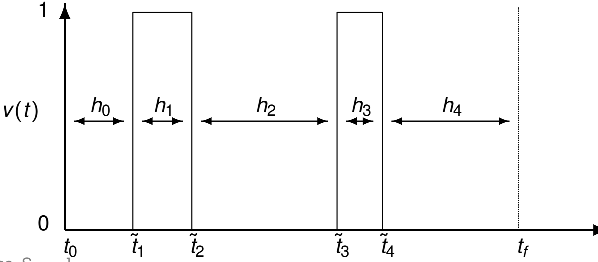
$$\min_{x,h,u} \phi(x(t_f))$$
$$\text{s.t.} \quad \dot{x}(t) = f(x(t), v^{i_j}, u(t)), \quad t \in [\tilde{t}_j, \tilde{t}_{j+1}]$$
$$\ldots$$

Switching time optimization

Take given switching order with piecewise fixed controls $v(\cdot)$, optimize interval lengths (after standard time transformation)



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$$\dots$$

- Main issue: initialization

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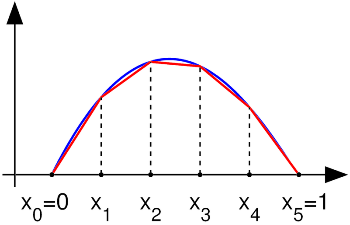
Linear approximations

- MILP way faster than MINLP solvers
- Idea: approximate nonlinearities and obtain MILP
  - A) Use linearizations in NMPC context [Jones, Morari, Borrelli, . . .]
  - B) Use piecewise linear approximation (?)



- Main issues: problem size, adaptivity, restarts

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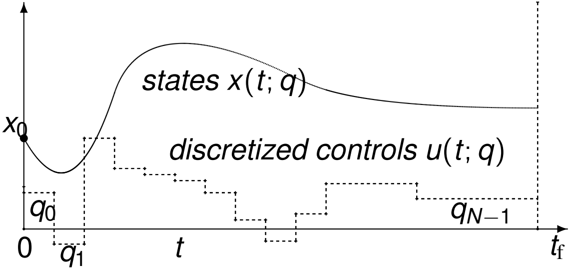
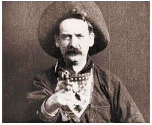
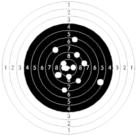
Image: Wikipedia

Direct Single Shooting [Hicks, Ray 1971; Sargent, Sullivan 1977]

Discretize controls $u(t)$ on fixed grid $0 = t_0 < t_1 < \ldots < t_N = t_\mathrm{f}$.

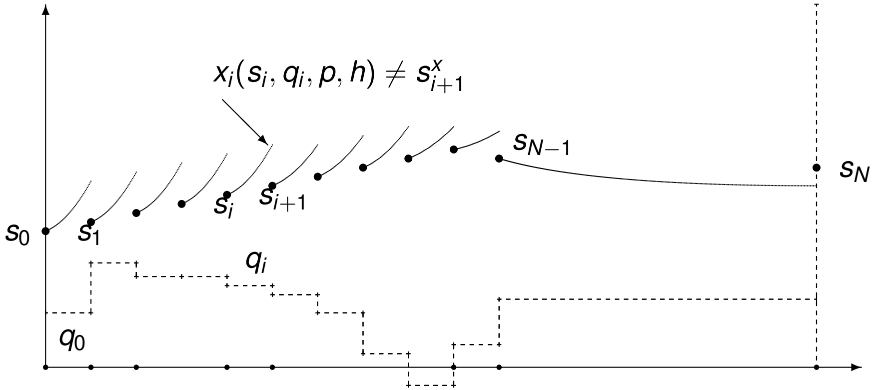






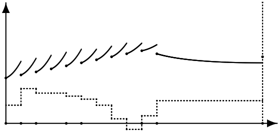
Regard states $x(t)$ on $[t_0, t_\mathrm{f}]$ as dependent variables.
Use numerical integration to obtain state as function $x(t; q, x_0)$ of finitely many control parameters $q = (q_0, q_1, \ldots, q_{N-1})$ and the initial value x_0.

Direct Multiple Shooting [Bock and Plitt, 1981, 1984, ...]



- Idea: Decouple intervals, add extra continuity constraints.
- Denote each interval's variables by $w_i := (s^x_i, s^z_i, q_i)$.
- Summarize all in large vector $w := (w_0, \ldots, w_N)$.

NLP in Direct Multiple Shooting [Bock and Plitt, 1981, 1984, . . .]



$$\min_{w} \sum_{i=0}^{N} \phi_i(w_i) \;\; \text{s.t.}$$

$$\left\{ \begin{array}{rcll} s^x_{i+1} - x_i(w_i) & = & 0 & \text{(continuity)} \\ g_i(w_i) & = & 0 & \text{(algebraic consistency)} \\ c_i(w_i) & \geq & 0 & \text{(path constraints)} \\ \sum_{i=0}^{N} r_{\mathrm{i}}(w_i) & \geq & 0 & \text{(multipoint inequality constraints)} \\ \sum_{i=0}^{N} r_{\mathrm{e}}(w_i) & \geq & 0 & \text{(multipoint equality constraints)} \end{array} \right.$$

Intermediate summary

- Large problem class with many applications
- Different approaches to solve control problems
  - Dynamic Programming
  - Maximum Principle
  - Direct Methods: discretize controls, solve NLP
- Direct Methods: different ways to parameterize states

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MINLP approach to solve MIOCPs

$$\begin{array}{llll} \min\limits_{x,v,u} \phi(x(t_f)) & & & \\ \text{s.t.} & \dot{x}(t) & = & f(t, x(t), v(t), u(t)), \quad t \in [t_0, t_f], \\ & 0 & \leq & c(t, x(t), v(t), u(t)), \\ & 0 & \leq & r_i(x(t_0), \dots, x(t_f)), \\ & 0 & = & r_e(x(t_0), \dots, x(t_f)), \\ & v(t) & \in & \Omega. \end{array}$$

- Consider the infinite dimensional problem
- Discretize

MINLP approach to solve MIOCPs

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0 &= r_e(x(t_0), \ldots, x(t_f)), \\
v(t) &\in \Omega.
\end{aligned}
$$

- Consider the infinite dimensional problem
- Discretize
- Let variables inherit integer constraint

$$
\begin{aligned}
\min_{\gamma,\beta} \quad & F(\gamma, \beta) \\
\text{s.t.} \quad 0 &= G(\gamma, \beta) \\
0 &\leq H(\gamma, \beta) \\
\beta_i &\in \Omega, i = 1..N
\end{aligned}
$$

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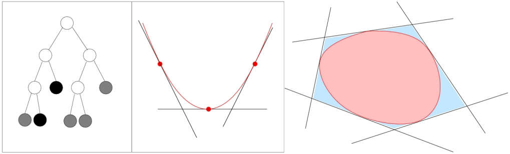
Mixed–Integer Nonlinear Programming

- Generic algorithms: Nonlinear Branch & Bound, Outer Approximation, Disjunctive Programming, ...



- Active research area
  - [Bonami, Wächter, ...] (Bonmin)
  - [Leyffer, Linderoth, ...] (FilMint)
  - [Biegler, Floudas, Grossmann, Lee, Marquardt, Oldenburg, Sahinidis, Weismantel, ...]
  - ...
- But: extremly expensive for control problems

New Approach

$$
\begin{aligned}
\min_{x,v,u} \ & \phi(x(t_f)) \\
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v(t) &\in \Omega := \{v^1, v^2, \dots, v^{n_\omega}\}, \quad t \in [t_0, t_f].
\end{aligned}
$$

- Important: lower bounds
- Relaxation $v \in \Omega \longrightarrow v \in \text{conv}\ \Omega$ is too weak!
- Is there a better relaxation?

(Partial) Outer convexification [Sager, 2005]

Easy to show:

$$\begin{aligned} v(\cdot) &\in \Omega := \{v^1, v^2, \dots, v^{n_\omega}\} \\ \dot{x}(t) &= f(x(t), v(t), u(t)) \end{aligned}$$

$$\Longleftrightarrow$$

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Easy to show:

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$$\iff$$

$$\begin{aligned} \omega(\cdot) &\in \{0,1\}^{n_\omega} \\ \dot{x}(t) &= \sum_{i=1}^{n_\omega} f(x(t), v^i, u(t))\, \omega_i(t) \\ \sum_{i=1}^{n_\omega} \omega_i(t) &= 1, \quad t \in [t_0, t_\mathrm{f}] \end{aligned}$$

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Problem class after Outer Convexification

$$
\begin{array}{llll}
\min\limits_{x,u,\omega} \phi(x(t_f)) & & & \\
\text{s.t.} & \dot{x}(t) & = & \sum_{i=1}^{n_\omega} f(x(t), v^i, u(t))\, \omega_i(t), \\
& 0 & \leq & c(x(t), u(t)), \\
& 0 & \leq & r_i(x(t_0), \ldots, x(t_f)), \\
& 0 & = & r_e(x(t_0), \ldots, x(t_f)), \\
& 1 & = & \sum_{i=1}^{n_\omega} \omega_i(t), \\
& \omega(t) & \in & \{0,1\}^{n_\omega}, \qquad t \in [t_0, t_f].
\end{array}
$$

- Important: lower bounds
- Is relaxation $\omega \in \{0,1\}^{n_\omega} \longrightarrow \alpha \in [0,1]^{n_\omega}$ good?

Approximating the state [S., Bock, Diehl, Mathematical Programming, to appear]

LEMMA. Let $x(\cdot)$ and $y(\cdot)$ be solutions of

$$\begin{aligned} \dot{x}(t) &= A(t, x(t)) \cdot \alpha(t), \quad x(0) = x_0, \\ \dot{y}(t) &= A(t, y(t)) \cdot \omega(t), \quad y(0) = y_0, \end{aligned}$$

with $t \in [0, t_\mathrm{f}]$, for given measurable functions $\alpha, \omega : [0, t_\mathrm{f}] \to [0, 1]^{n_\omega}$ and $A : \mathbb{R}^{n_x+1} \mapsto \mathbb{R}^{n_x \times n_\omega}$.

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$$\begin{aligned} \left\| \frac{\mathrm{d}}{\mathrm{d}t} A(t, x(t)) \right\| &\leq C, \\ \| A(t, y(t)) - A(t, x(t)) \| &\leq L \| y(t) - x(t) \|, \end{aligned}$$

for $t \in [0, t_\mathrm{f}]$ almost everywhere and $A(\cdot, x(\cdot))$ essentially bounded by M on $[0, t_\mathrm{f}]$,

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for $t \in [0, t_\mathrm{f}]$ almost everywhere and $A(\cdot, x(\cdot))$ essentially bounded by M on $[0, t_\mathrm{f}]$, and it exists $\epsilon \in \mathbb{R}^+$ such that

$$\left\| \int_0^t \alpha(\tau) - \omega(\tau) \, \mathrm{d}\tau \right\| \leq \epsilon$$

for all $t \in [0, t_\mathrm{f}]$,

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for all $t \in [0, t_f]$, then for all $t \in [0, t_f]$ it holds

$$\| y(t) - x(t) \| \leq (\| x_0 - y_0 \| + (M + C) t \epsilon) \, e^{Lt}$$

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Approximating the state [S., Bock, Diehl, Mathematical Programming, to appear]

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with $t \in [0, t_\mathrm{f}]$, for given measurable functions $\alpha, \omega : [0, t_\mathrm{f}] \to [0, 1]^{n_\omega}$ and $A : \mathbb{R}^{n_x+1} \mapsto \mathbb{R}^{n_x \times n_\omega}$. If $C, L, M \in \mathbb{R}^+$ exist such that

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Related results: [Gonzalez, Häckl, Pietrus, Tidball, Veliov]

Approximating the state [S., Bock, Diehl, Mathematical Programming, to appear]

In other words:

$$\| \, y(t) - x(t) \, \|$$

will be small for all t, if

$$\left\| \int_0^t \alpha(\tau) - \omega(\tau) \, \mathrm{d}\tau \right\|$$

is small for all times $t \in [0, t_\mathrm{f}]$.

How to get binary variables $\omega(\tau)$
from continuous variables $\alpha(\cdot)$?

Sum Up Rounding [Sager, 2005]

Assume continuous solution $q_{j,i} \in [0,1]$. Goal: $p_{j,i} \in \{0,1\}$.
For $\Delta t_i = t_{i+1} - t_i$ and all $j = 1 \dots n_\omega, i = 1 \dots n_{int}$ set

$$p_{j,i} = \begin{cases} 1 & \text{if } \sum_{k=0}^{i} q_{j,k} \Delta t_k - \sum_{k=0}^{i-1} p_{j,k} \Delta t_k \geq 0.5 \Delta t_i \\ 0 & \text{else} \end{cases} .$$

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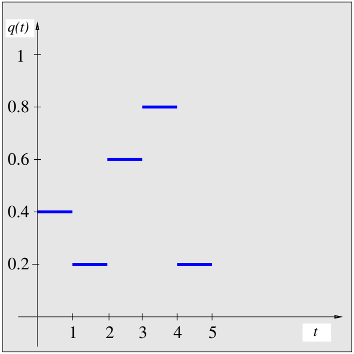

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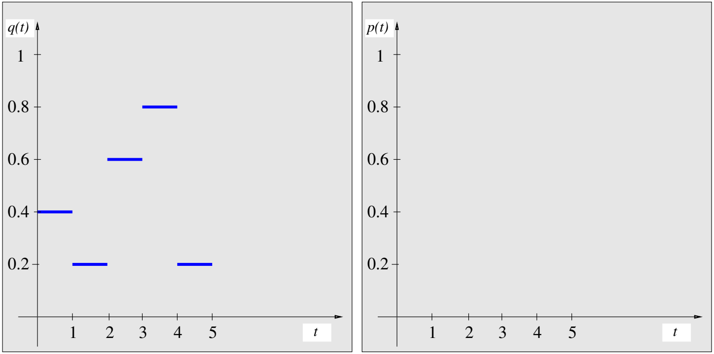

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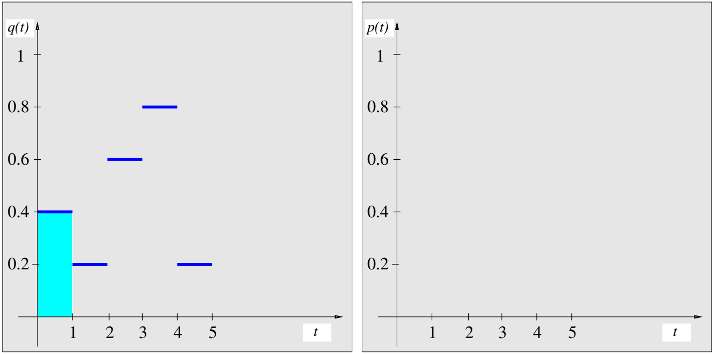

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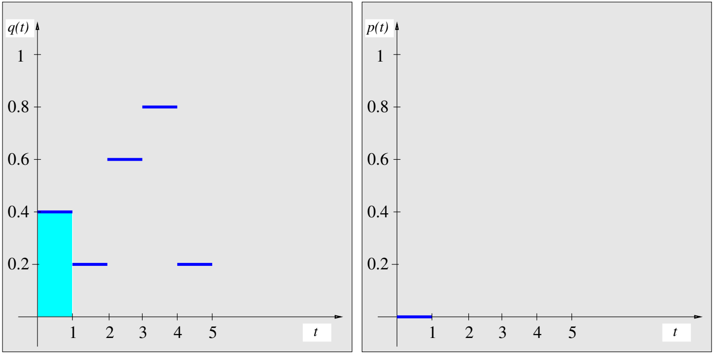

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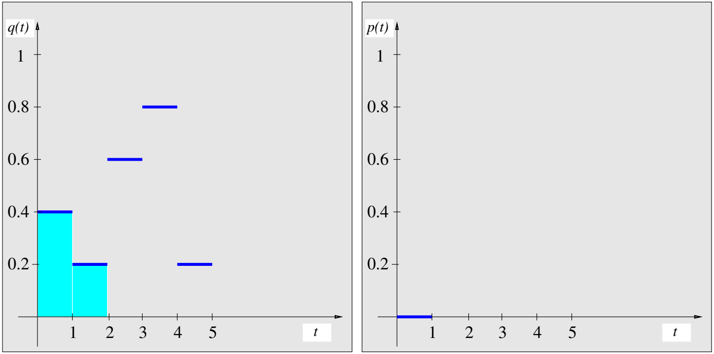

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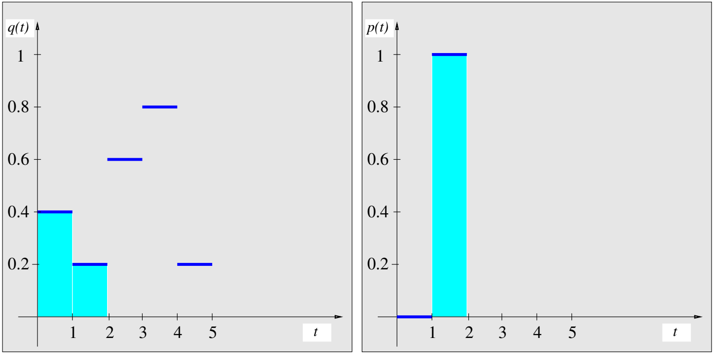

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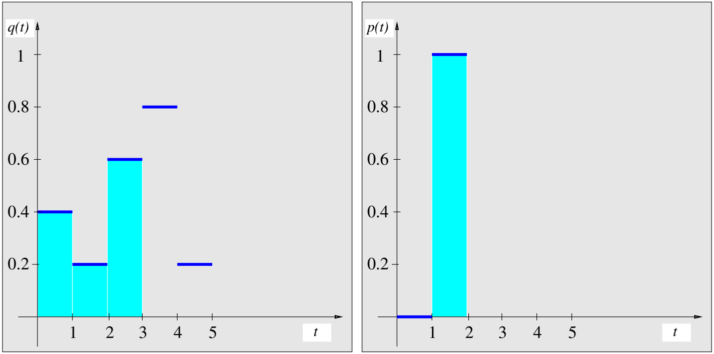

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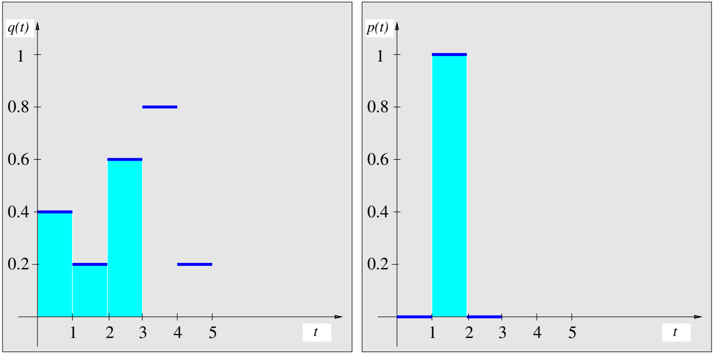

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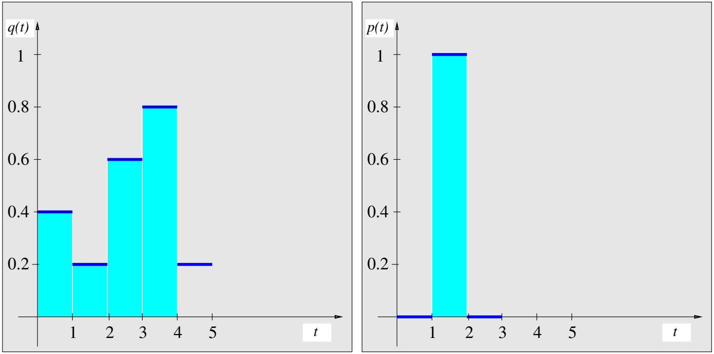

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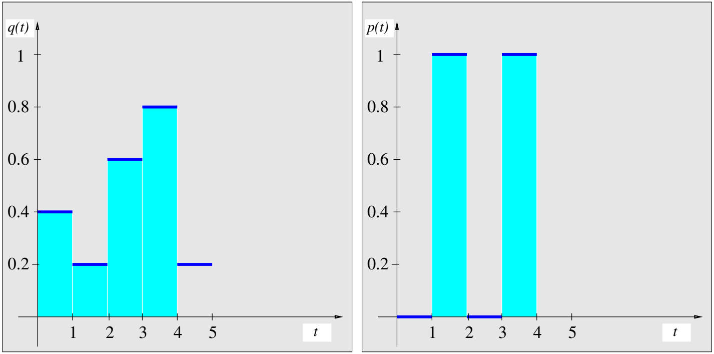

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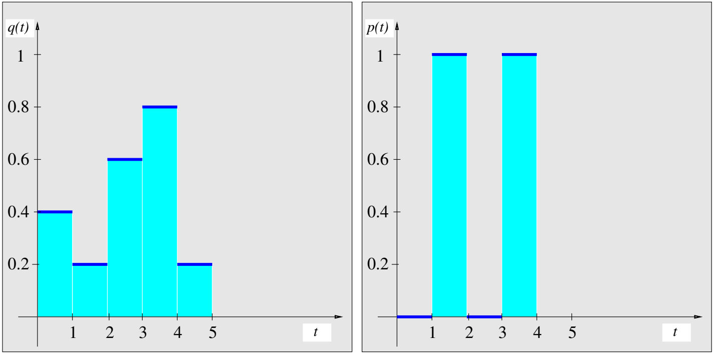

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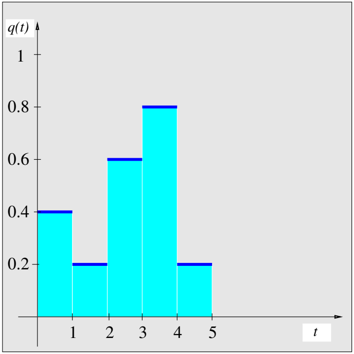

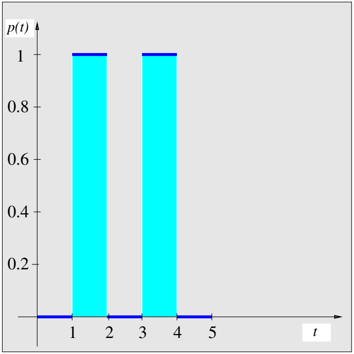

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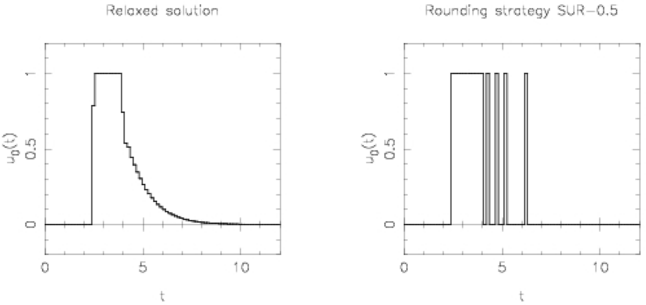

Approximating the control [S., Bock, Diehl, Mathematical Programming, to appear]

LEMMA. Let functions $\alpha : [0, t_\mathrm{f}] \mapsto [0, 1]^{n_\omega}$,

$$\alpha_j(t) = q_{j,i}, \quad t \in [t_i, t_{i+1}]$$

and $\omega : [0, t_\mathrm{f}] \mapsto \{0, 1\}^{n_\omega}$,

$$\begin{aligned} \omega_j(t) &= p_{j,i}, \quad t \in [t_i, t_{i+1}] \\ p_{j,i} &= \begin{cases} 1 & \text{if } \sum_{k=0}^{i} q_{j,k} \Delta t_k - \sum_{k=0}^{i-1} p_{j,k} \Delta t_k \geq 0.5 \Delta t_i \\ 0 & \text{else} \end{cases} . \end{aligned}$$

be given. Then it holds

$$\left\| \int_0^t \alpha(\tau) - \omega(\tau) \, \mathrm{d}\tau \right\| \leq 0.5 \max_i \Delta t_i$$

(Similar strategy and lemma allow to include SOS1 constraints for nonlinear case)

Implications

- 2 lemmata together: if SUR is used then

$$\| y(t) - x(t) \| \leq C\Delta t$$

  for any (relaxed) solution $x(\cdot)$, control grid size Δt
- Objective function is a continuous function, thus

$$|\Phi(x(t_\mathrm{f})) - \Phi(y(t_\mathrm{f}))| \leq \gamma\Delta t$$

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- Extension to continuous constraints straightforward
- We know the exact lower bound from $\alpha(\cdot)$ solution
- Sum Up Rounding constructive way to get integer solution

MS MINTOC algorithm [Sager 2005; Sager, J Proc Control, 2009]

1. $k = 0$. Input: control discretization grid $\mathcal{G}^0$, $TOL \in \mathbb{R}^+$.
2. If necessary, reformulate and convexify problem. Relax this problem to $\alpha(\cdot) \in [0,1]^{n_\omega}$.
3. REPEAT
   - 3.1 Solve relaxed problem on $\mathcal{G}^k$. Obtain $\mathcal{T}^k = (x^k(\cdot), u^k(\cdot), \alpha^k(\cdot))$ and the grid–dependent optimal value ϕ^{RC}.
   - 3.2 If $\mathcal{T}^k$ on $\mathcal{G}^k$ fulfills $\omega^k(\cdot) := \alpha^k(\cdot) \in \{0,1\}^{n_\omega}$ then STOP.
   - 3.3 Apply Sum Up Rounding to $\alpha^k(\cdot)$. Fix $u^k(\cdot)$. Obtain $y^k(\cdot)$ and upper bound ϕ^{STO} by simulation.
   - 3.4 If $\phi^{\text{STO}} < \phi^{\text{RC}} + TOL$ then STOP.
   - 3.5 Refine the control grid $\mathcal{G}^k$.
   - 3.6 $k = k + 1$.
4. Bijection to obtain solution $\Phi^* = \phi^{\text{STO}}$ for original problem.

Algorithm terminates in finite number of iterations [S., 2005; S. 2009]

Intermediate summary

- Main trick: binary controls propagated to continuous states
- Most applications: no explicit constraints on binary controls

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- Main trick: binary controls propagated to continuous states
- Most applications: no explicit constraints on binary controls
- At the price of chattering, relaxed solution can always be reached $\rightarrow$ no integer gap!!!
- However: for any fixed control grid there may be a gap
  $\rightarrow$ black box MINLP performs badly!
- Sum Up Rounding constructive way to get integer solution
- Needs to be combined with iterative refinement of grid

- Is Outer Convexification crucial?

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Example: MIOCP nonlinear in v

$$
\begin{aligned}
\min_{x,v} \quad & x_2(t_\mathrm{f}) \\
\text{s.t.} \quad & \dot{x}_0 = -\frac{x_0 \ \sin(v_1)}{\sin(1)} + (x_0 + x_1)\ {v_2}^2 + (x_0 - x_1)\ {v_3}^3 \\
& \dot{x}_1 = (x_0 + 2x_1)\ v_1 + (x_0 - 2x_1)\ v_2 + (x_0 + x_1)\ v_3 \\
& \qquad + (x_0 x_1 - x_2)\ {v_2}^2 - (x_0 x_1 - x_2)\ {v_2}^3, \qquad (1) \\
& \dot{x}_2 = x_0^2 + x_1^2, \\
& x(0) = (0.5, 0.5, 0)^T, \\
& x_1 \geq 0.4, \\
& v \in \{(1,0,0), (0,1,0), (0,0,1)\}
\end{aligned}
$$

with $t \in [t_0, t_\mathrm{f}] = [0, 1]$. This problem can be relaxed by requiring

$$
v_1, v_2, v_3 \in [0, 1], \quad \sum_{i=1}^{3} v_i = 1
$$

Example: partially outer convexified with ω

$$
\begin{aligned}
\min_{x,\omega} \quad & x_2(t_\mathrm{f}) \\
\text{s.t.} \quad & \dot{x}_0 = -x_0\,\omega_1 + (x_0 + x_1)\,\omega_2 + (x_0 - x_1)\,\omega_3, \\
& \dot{x}_1 = (x_0 + 2x_1)\,\omega_1 + (x_0 - 2x_1)\,\omega_2 + (x_0 + x_1)\,\omega_3, \\
& \dot{x}_2 = x_0^2 + x_1^2, \\
& x(0) = (0.5, 0.5, 0)^T, \\
& x_1 \geq 0.4, \\
& \omega_i \in \{0,1\}, \quad \sum_{i=1}^{3} \omega_i = 1
\end{aligned}
\tag{2}
$$

with $t \in [t_0, t_\mathrm{f}] = [0, 1]$. Relaxation: $\alpha_i \in [0, 1]$

Is (by construction) identical to the one in (?) and (?).

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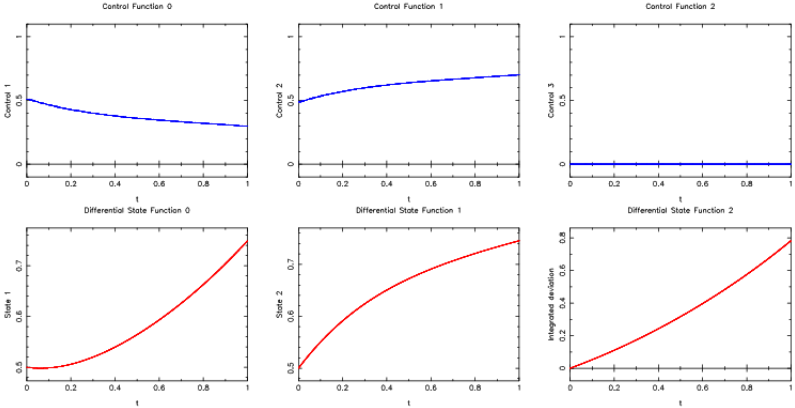
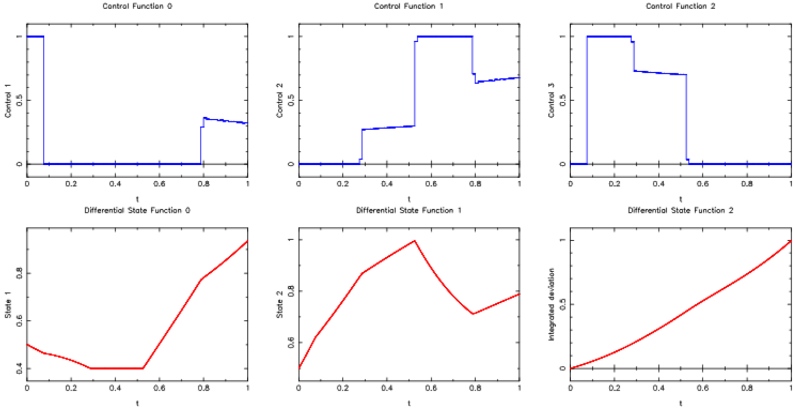
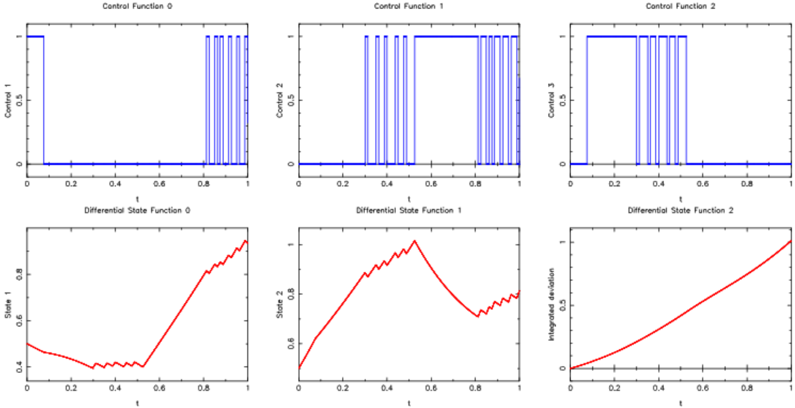
OCP (1) - nonlinear relaxed: $\Phi = 0.782$



OCP (2) - outer convexification relaxed: $\Phi = 0.996$



MIOCPs (1) and (2) - Sum Up Rounding: $\Phi = 1.012$



Elchtest benchmark problem [Gerdts, OCAM, 2005]

Elchtest in min. time with gear shifts, *N* discretization points.

N	t_f	CPU Time	t_f	CPU Time
20	6.779751	00:23:52	6.779035	00:00:24
40	6.786781	232:25:31	6.786730	00:00:46
80	–	–	6.789513	00:04:19

Left: Branch and Bound, Inner Convexification [Gerdts, OCAM, 2005]
Pentium III with 750 MHz

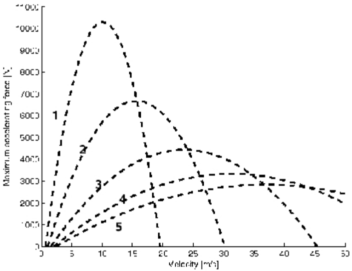
Right: MS MINTOC, Outer Convexific. [Kirches, Bock, Schlöder, S., OCAM, 2009]
AMD Athlon XP 3000+ with 2166 MHz

Similar speedup for free switching times [Gerdts OCAM 2006, Kirches et al. 2009]

Why is this formulation so beneficial?!?

$$\dot{v}(t) = \frac{1}{m}\Big(\big(F_{\mathrm{lr}}^{\mu} - F_{\mathrm{Ax}}\big)\ \cos\beta(t) + F_{\mathrm{lf}}\ \cos\big(\delta(t) + \beta(t)\big) \\ - \big(F_{\mathrm{sr}} - F_{\mathrm{Ay}}\big)\ \sin\beta(t) - F_{\mathrm{sf}}\ \sin\big(\delta(t) + \beta(t)\big)\Big).$$

F_{lr}^{μ} is a function of ϕ, F_{B}, v, and i_{g}^{μ}. Maximum acceleration $\to \phi := 1, F_{\mathrm{B}} := 0$.



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$$\dot{v}(t) = \frac{1}{m}\Big(\big(F_{\text{lr}}^{\mu} - F_{\text{Ax}}\big)\ \cos\beta(t) + F_{\text{lf}}\ \cos\big(\delta(t)+\beta(t)\big)$$
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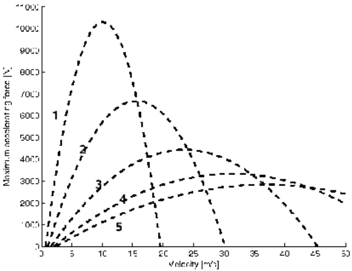
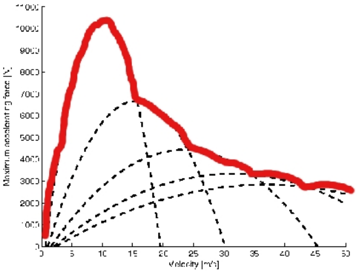


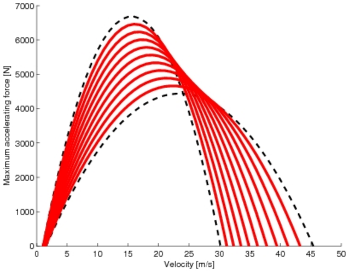


Compare maximum acceleration

Compare maximum acceleration for convex combinations, $a = 0.1 \ldots 0.9$

Inner convexification, use
$F_{\text{lr}} = F_{\text{lr}}(a \cdot i_{\text{g}}^2 + (1-a) \cdot i_{\text{g}}^3)$

where i_{g}^2 and i_{g}^3 are the transmission ratios of second resp. third gear



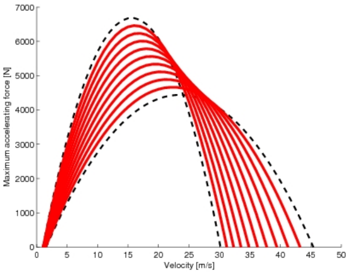
Compare maximum acceleration

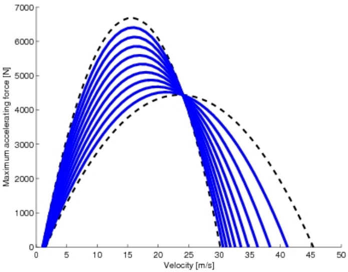
Compare maximum acceleration for convex combinations, $a = 0.1 \ldots 0.9$

Inner convexification, use
$F_{\mathrm{lr}} = F_{\mathrm{lr}}(a \cdot i_g^2 + (1-a) \cdot i_g^3)$

Outer convexification, use
$F_{\mathrm{lr}} = a \cdot F_{\mathrm{lr}}(i_g^2) + (1-a) \cdot F_{\mathrm{lr}}(i_g^3)$

where i_g^2 and i_g^3 are the transmission ratios of second resp. third gear





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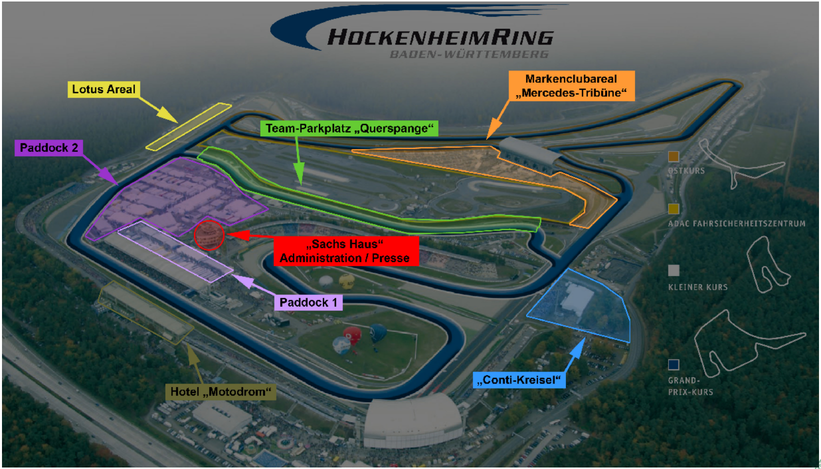
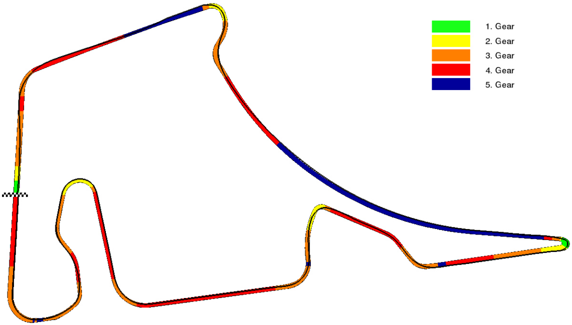


Problem characteristics

- Long optimization horizon, unstable $\Longrightarrow$ use Bock's direct multiple shooting method, 350 intervals
- $350 \cdot 5 =$ 1750 binary control variables
- $350 \cdot 3 =$ 1050 continuous control variables
- $\approx 7 \cdot 4300 \Rightarrow \approx$ 30.000 of discretized state variables
- Nonlinear dynamics, nasty engine speed + path constraints



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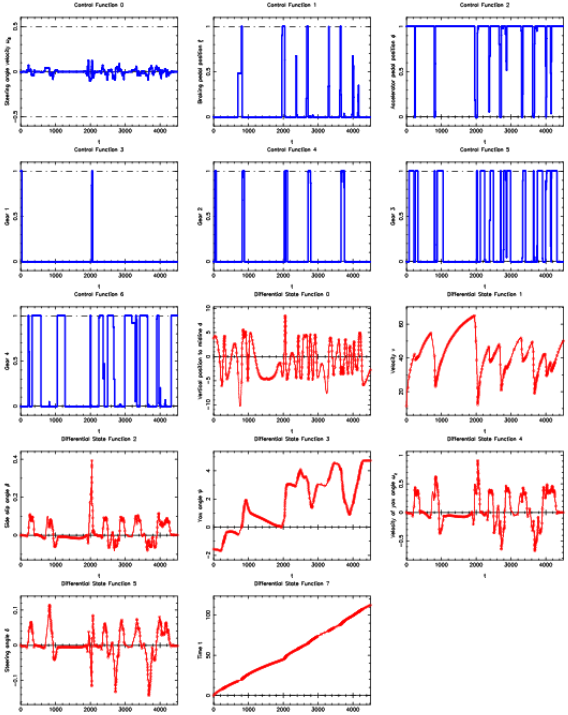
Control Function 0
Control Function 1
Control Function 2
Control Function 3
Control Function 4
Control Function 5
Control Function 6
Differential State Function 0
Differential State Function 1
Differential State Function 2
Differential State Function 3
Differential State Function 4
Differential State Function 5
Differential State Function 7

Outline

MIOC introduction

Methods for Mixed-Integer Optimal Control

...based on HJB / Dynamic Programming

...based on Direct Methods

MINLP

MS MINTOC

Switching Costs and Combinatorial Constraints

Switching Costs and Combinatorial Constraints

- What if constraints depend explicitly on integer variables?

$$0 \leq c(x(t), v(t), u(t))$$

- Example: switching restrictions / costs on fixed control grid
- Problem: Sum Up Rounding solution infeasible!

Switching Costs and Combinatorial Constraints

- What if constraints depend explicitly on integer variables?

$$0 \leq c(x(t), v(t), u(t))$$

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- Problem: Sum Up Rounding solution infeasible!

- One option: use MINLP solver
- Alternative: decompose problem into NLP and MILP, making use of approximation theorem!

Combinatorial Integral Approximation [S., Jung, Kirches, MMOR, to appear]

Idea: minimize $\max_t \left\| \int_0^t \alpha(\tau) - \omega(\tau) \, \mathrm{d}\tau \right\|_\infty$ with binary controls $\omega(\cdot) \approx p$ that fulfill combinatorial constraints.

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binary controls $\omega(\cdot) \approx p$ that fulfill combinatorial constraints.

$$\begin{aligned}
&\min_{p} \quad \max_{k=1..n_\omega} \max_{i=1..n_t} \left| \sum_{j=1}^{i} (q_{k,j} - p_{k,j}) \Delta t_j \right| \\
&\text{subject to} \\
&\quad \sigma_{k,\max} \geq \sum_{i=1}^{n_t - 1} |p_{k,i} - p_{k,i+1}|, \quad k = 1..n_\omega, \\
&\quad p_{k,i} \in \{0, 1\}, \quad k = 1..n_\omega, \; i = 1..n_t.
\end{aligned} \tag{3}$$

Given are: control discretization grid Δt_j,
maximum numbers of switches $\sigma_{k,\max}$, and relaxed solution $\alpha(\cdot) \approx q$

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- MINLP solution (Bonmin) $>>$ 1800 sec
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- MINLP solution (Bonmin) $>>$ 1800 sec
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- NLP + MILP solution (own code) $\approx$ 1.5 sec

How do we get this MILP solution so fast?

Cutting Planes?

- Number of facets of feasible polytope [PORTA – Christof, Löbel, Reinelt]

n_t	Control values $q_{1,j}$ fixed to: 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
4	16	18	23	33	29	33	23	18	16
5	21	31	87	189	54	189	87	31	21
6	30	60	745	612	248	612	745	60	30
7	47	150	4838	4840	922	4840	4838	150	47
8	83	899	37470	29884	4212	29884	37470	899	83

- depends heavily on relaxed solution q
- Hence cutting planes of limited use

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- depends heavily on relaxed solution q
- Hence cutting planes of limited use
- Instead use Branch&Bound without LP relaxations, making use of specific structure of inequalities

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Algorithm 1: Combinatorial Branch and Bound

```
Input  : Relaxed controls q, time grid {t_i}, i = 1..n_t, max. numbers of switches
         σ_{k,max}, k = 1..n_ω.
Result : Optimal solution (η*, s*, p*) of (3).
begin
    Create empty priority queue Q ordered by a.η (non-decreasing), if equal by a.d
    (non-increasing).
    Push an empty node (0, {}, {}, 0.0) into the queue.
    while Q is not empty do
        a = top node of Q and remove the node from Q.
        /* 1st solution found is optimal since best-first search
           is used                                                  */
        if a.d = n_t then
            Return optimal solution (a.η, a.s, a.p).
        /* Create child nodes, use strong branching.               */
        else
            forall possible permutations φ of {0,1}^{n_ω} do
                Create new node n with n.d = d + 1, n.p = a.p, n.s = a.s.
                Set n.p_{k,d+1} = φ_k, calculate n.s_{k,d+1}.
                if n.s fulfills switching constraint until time d + 1 then
                    n.η = max{ a.η, max_{k=1}^{n_ω} {± Σ_{j=1}^{d+1} (q_{k,j} − p_{k,j}) Δt_j} }
                    Push n into Q.
end
```

Where the listing uses plain-text math:

- Input: q, $\{t_i\}$, $i = 1..n_t$, $\sigma_{k,\max}$, $k = 1..n_\omega$
- Result: (η^*, s^*, p^*)
- $n.\eta = \max\left\{a.\eta, \max_{k=1}^{n_\omega}\{\pm \sum_{j=1}^{d+1}(q_{k,j} - p_{k,j})\Delta t_j\}\right\}$

Combinatorial Integral Approximation [S., Jung, Kirches, MMOR, to appear]

THEOREM. Assume p^{SUR} to be the solution obtained by Sum Up Rounding. The following claims hold for the optimal solution (η^*, s^*, p^*) of the MILP (3):

$$
\begin{aligned}
& (a)\ \eta^* < 0.5\ \delta t = 0.5 \min_{i=1..n_t} \Delta t_i \\
\Rightarrow\quad & (b)\ p^* = p^{\text{SUR}} \\
\Rightarrow\quad & (c)\ \sigma_{k,\max} \geq \sigma_k^{\text{SUR}} \quad \forall\ k = 1..n_\omega \\
\Rightarrow\quad & (d)\ \eta^* \leq 0.5\ \Delta t = 0.5 \max_{i=1..n_t} \Delta t_i
\end{aligned}
$$

where the solution $p^* = p^{\text{SUR}}$ in (b) is unique.

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where the solution $p^* = p^{\text{SUR}}$ in (b) is unique.

Hence: MILP solution approximates MINLP solution when $n_t \to \infty$ and if $\sigma_{k,\max}$ are large enough

CIA summary

- replace 1 MINLP by 1 NLP + 1 MILP
- generic approach (not only for switching constraint)
- for given grid Δt the solutions may be different,
  but asymptotic convergence as $\Delta t \to 0$
- MILP solution can be used as fast UB in MINLP
- MILP solution can be used to initialize Switching Time Optimization
- MILP structure (e.g., max switching) may allow for ultrafast strategies

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Conclusions

- Many applications (with optimization potential) are MIOCPs
- Open benchmark library: http://mintoc.de

- Many different approaches compete
- Speaker's personal favorite: MS MINTOC
  - exact bounds and error estimates
  - can be combined with state-of-the-art numerics
  - extensions towards NMPC, global, robust, multi-objective, DAE, PDE, combinatorial constraints, . . . are possible

- Survey article [S. Sager. Reformulations and algorithms for the optimization of switching decisions in nonlinear optimal control. Journal of Process Control, 2009.]
- Literature: http://mathopt.de

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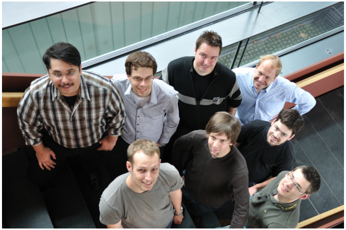

Thank you for your attention!

Questions?

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