Singular Optimal Control, Lur'e Equations and Even Matrix Pencils

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Overview

Outline

Linear-quadratic optimal control problem

Minimize
$$\mathcal{J}(u,x_0) = \frac{1}{2}\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

subject to
$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

with (A,B) stabilizable, i.e. $\operatorname{rank}[\,A - sI\,,\ B\,] = n \ \ \forall s \in \overline{\mathbb{C}^+}$.

Definition

$\hat{u} : \mathbb{C}^+ \to \mathbb{C}^m$ is called *minimizer* if

$$\mathcal{J}(\hat{u},x_0) = \inf\{\mathcal{J}(u,x_0) : \ u \in L_2^{loc}(\mathbb{R}^+)\}.$$

The *optimal value* is

$$\mathcal{J}(x_0) = \inf\{\mathcal{J}(u,x_0) : \ u \in L_2^{loc}(\mathbb{R}^+)\}.$$

Usual assumption: $R \in \mathbb{C}^{m,m}$ is invertible.

Equivalent criteria for the existence of a minimizer $\hat{u} \in L_2^{loc}(\mathbb{R}^+)$:

- The algebraic Riccati equation (ARE)

$$A^* X + XA + Q - (S + XB)R^{-1}(S + XB)^* = 0$$

  has at least one solution $X = X^* \in \mathbb{C}^{n,n}$.

- The ARE has a *maximal solution* $X_+ = X_+^* \in \mathbb{C}^{n,n}$, i.e., for all other solutions X holds $X_+ \geq X$. In this case holds

$$\mathcal{J}(x_0) = x_0^* X_+ x_0, \quad \text{and} \quad \hat{u}(t) = -R^{-1}(S + X_+ B)^* e^{A - BR^{-1}(S+X_+B)^* t} x_0.$$

Further equivalent criteria for the existence of a minimizer $\hat{u} \in L_2^{loc}(\mathbb{R}^+)$:

- For all $u \in L_2^{loc}(\mathbb{R}^+)$ holds $\mathcal{J}(u,0) \geq 0$.
- The *Popov function*

$$\mathcal{P}(i\omega) = \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}$$

  is positive semidefinite ($\mathcal{P}(i\omega) \geq 0$) for all $\omega \in \mathbb{R}$.
- The Jordan form of the *Hamiltonian matrix*

$$H = \begin{bmatrix} A & 0 \\ -Q & -A^* \end{bmatrix} - \begin{bmatrix} B \\ S \end{bmatrix} R^{-1} \begin{bmatrix} S^* & -B^* \end{bmatrix}$$

  has the form

$$HT = T \begin{bmatrix} J_- & 0 & 0 \\ 0 & -J_- & 0 \\ 0 & 0 & J_i \end{bmatrix},$$

  where

  - $\sigma(J_-) \subset \mathbb{C}^-$,
  - $\sigma(J_i) \subset i\mathbb{R}$ and all Jordan blocks of J_i have even size.

Solution of ARE can be obtained via invariant subspaces of the Hamiltonian matrix H.

X is a solution of the ARE if and only if $X = X_2 X_1^{-1}$, where

- $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \widetilde{A} = H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, (im$[\, X_1^T \,,\, X_2^T \,]^T$ is *H-invariant*),
- $X_1 \in \mathrm{Gl}_n(\mathbb{C})$, (im$[\, X_1^T \,,\, X_2^T \,]^T$ is *1-regular*),
- $X_2^* X_1 = X_1^* X_2$ (im$[\, X_1^T \,,\, X_2^T \,]^T$ is *Lagrangian*).

Characterization of H-invariant Lagrangian subspaces:

Let $\mathcal{X} = \operatorname{im}[\, X_1^T \,,\, X_2^T \,]^T \subset \mathbb{C}^{2n}$ be an invariant subspace of the Hamiltonian matrix H, i.e.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \widetilde{A} = H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Then $\mathcal{X}$ is Lagrangian if $\widetilde{A}$ has the following properties:

- For all $\lambda, \mu \in \sigma(\widetilde{A}) \backslash i\mathbb{R}$ holds $\lambda + \overline{\mu} \neq 0$, and
- The Jordan blocks of $\widetilde{A}$ corresponding to the eigenvalues on $i\mathbb{R}$ have half the size as the corresponding Jordan blocks of H.

If, additionally $\sigma(\widetilde{A}) \subset \overline{\mathbb{C}^+}$, then $X_+ = X_2 X_1^{-1}$

Outline

Example 1:

Minimize
$$\mathcal{J}(u, x_0) = \frac{1}{2}\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

subject to
$$\dot{x}(t) = -u(t), \quad x(0) = x_0.$$

Observations:

- If $x_0 \neq 0$, then for all $u : \mathbb{R}^+ \to \mathbb{C}$ holds $\mathcal{J}(u, x_0) > 0$.
- For $u_n = n \cdot \chi_{[0,n^{-1}]} x_0$ holds $\lim_{n\to\infty} \mathcal{J}(u_n, x_0) = 0$.

Conclusions:

- The optimal value is given by $\mathcal{J}(x_0) = 0$ for all $x_0 \in \mathbb{C}$.
- There exists no minimizer $\hat{u} : \mathbb{R}^+ \to \mathbb{C}$.

Example 2:

Minimize $$\mathcal{J}(u,x_0)=\frac{1}{2}\int_0^\infty \begin{bmatrix}x(t)\\u(t)\end{bmatrix}^* \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \begin{bmatrix}x(t)\\u(t)\end{bmatrix} dt$$

subject to $$\dot{x}(t)=Ax(t)+Bu(t),\quad x(0)=x_0.$$

Observations:

- For all $u:\mathbb{R}^+\to\mathbb{C}^m$ holds $\mathcal{J}(u,x_0)=0$.

Conclusions:

- The optimal value is given by $\mathcal{J}(x_0)=0$ for all $x_0\in\mathbb{C}^n$.
- Every control $\hat{u}:\mathbb{R}^+\to\mathbb{C}^m$ is a minimizer.

Linear-quadratic optimal control problem

Minimize $$\mathcal{J}(u, x_0) = \frac{1}{2}\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

subject to $$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

with (A, B) stabilizable.

Result: (Willems 1972)

The *optimal value* is given by $\mathcal{J}(x_0) = x_0^* X_+ x_0$, where X_+ is the *maximal solution* of the Lur'e equations

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + S &= K^*L, \\ R &= L^*L. \end{aligned}$$

i.e., X_+ is a solution and all other solutions X fulfill $X_+ \geq X$.

Lur'e equations

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + S &= K^*L, \\ R &= L^*L. \end{aligned}$$

Unknowns: $L \in \mathbb{C}^{m,m}$, $K \in \mathbb{C}^{m,n}$, $X \in \mathbb{C}^{n,n}$.

Observation:

If R is invertible, then K and L can be eliminated, such that

$$A^*X + XA + Q - (XB + S)R^{-1}(XB + S)^* = 0. \qquad \text{(ARE)}$$

Lur'e equations

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + S &= K^*L, \\ R &= L^*L. \end{aligned}$$

Typical approach for singular R (Regularization):

Perturbed Lur'e equations with $\varepsilon > 0$

$$\begin{aligned} A^*X_\varepsilon + X_\varepsilon A + Q &= K_\varepsilon^* K_\varepsilon, \\ X_\varepsilon B + S &= K_\varepsilon^* L_\varepsilon, \\ R + \varepsilon I &= L_\varepsilon^* L_\varepsilon. \end{aligned}$$

$\rightsquigarrow$ Reformulation as ARE is possible.

Result: (Trentelman, 1987)

The maximal solutions $(X_\varepsilon)_+$ of the perturbed Lur'e equations fulfill

$$\lim_{\varepsilon \to 0} (X_\varepsilon)_+ = X_+.$$

Outline

Lur'e equations

$$
\begin{aligned}
A^*X + XA + Q &= K^*K, \\
XB + S &= K^*L, \\
R &= L^*L.
\end{aligned}
$$

with (A, B) stabilizable.

Necessary characterizations for solvability (Willems, 1972):

- For all $u \in L_2^{loc}(\mathbb{R}^+)$ holds $\mathcal{J}(u, 0) \geq 0$.
- The Popov function

$$
\mathcal{P}(i\omega) = \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}
$$

is positive semidefinite ($\mathcal{P}(i\omega) \geq 0$) for all $\omega \in \mathbb{R}$.

Lur'e equations

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + S &= K^*L, \\ R &= L^*L. \end{aligned}$$

with (A, B) stabilizable.

Sufficient characterizations for solvability (Clements, Anderson, Laub, Matson, 1997):

- The Popov function fulfills $\mathcal{P}(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ and there exists a $\gamma \in \mathbb{R}$ such that $\mathcal{P}(i\gamma) > 0$.
- (A, B) is controllable and $\mathcal{P}(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

Lur'e equations

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + S &= K^*L, \\ R &= L^*L. \end{aligned}$$

with (A, B) stabilizable.

Aim: Generalization of the Hamiltonian eigenspace correspondence to Lur'e equations.

Consider the *matrix pencil*

$$\lambda\mathcal{E} - \mathcal{A} = \begin{bmatrix} Q & \lambda I + A^* & S \\ -\lambda I + A & 0 & B \\ S^* & B^* & R \end{bmatrix}, \qquad \mathcal{E}, \mathcal{A} \in \mathbb{C}^{2n+m,2n+m}.$$

The pencil $\lambda\mathcal{E} - \mathcal{A}$ is *even*, that is $\mathcal{E} = -\mathcal{E}^*$, $\mathcal{A} = \mathcal{A}^*$.

Basics of matrix pencils

$$\lambda\mathcal{E} - \mathcal{A}, \qquad \mathcal{E}, \mathcal{A} \in \mathbb{C}^{N,N}.$$

Definition

- $\lambda\mathcal{E} - \mathcal{A}$ is called *regular* if there exists some $s \in \mathbb{C}$ such that $\det(s\mathcal{E} - \mathcal{A}) \neq 0$.
- A subspace $\operatorname{im} X_{\mathcal{L}} \subset \mathbb{C}^N$ (with $X_{\mathcal{L}} \in \mathbb{C}^{N,k}$ full column rank) is called *deflating subspace* there exist matrices $Y_{\mathcal{L}} \in \mathbb{C}^{n,k}$, $E_{\mathcal{L}}, A_{\mathcal{L}} \in \mathbb{C}^{k,k}$ such that for all $s \in \mathbb{C}$ holds

$$Y_{\mathcal{L}}(sE_{\mathcal{L}} - A_{\mathcal{L}}) = (s\mathcal{E} - \mathcal{A})X_{\mathcal{L}}.$$

Even Kronecker Form (Thompson 1976)

For an even matrix pencil there exists a $W \in \mathrm{Gl}_N(\mathbb{C})$ such that

$$W^*(\lambda E - A)W = \mathrm{diag}(\mathcal{B}_1(\lambda), \ldots, \mathcal{B}_k(\lambda))$$

where the pencils $\mathcal{B}_j(\lambda)$ are one of the following type:

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where the pencils $\mathcal{B}_j(\lambda)$ are one of the following type:

Type 1: Non-imaginary eigenvalues:

$$\mathcal{B}_j(\lambda) = \left[\begin{array}{cccc|cccc} & & & & \lambda - a & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & \lambda - a \\ \hline -\lambda - \overline{a} & & & & & & & \\ 1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & -\lambda - \overline{a} & & & & \end{array}\right]$$

with $a \in \mathbb{C}^-$.

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Type 2: Imaginary eigenvalues:

$$\mathcal{B}_j(\lambda) = \varepsilon_j \begin{bmatrix} & & & i\lambda + a \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ i\lambda + a & 1 & & \end{bmatrix}$$

with $a \in \mathbb{R}$ and

- $\varepsilon_j = 1$ (positive signature), or
- $\varepsilon_j = -1$ (negative signature).

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For an even matrix pencil there exists a $W \in \mathrm{Gl}_N(\mathbb{C})$ such that

$$W^*(\lambda E - A)W = \mathrm{diag}(\mathcal{B}_1(\lambda), \ldots, \mathcal{B}_k(\lambda))$$

where the pencils $\mathcal{B}_j(\lambda)$ are one of the following type:

Type 3: Infinite eigenvalues:

$$\mathcal{B}_j(\lambda) = \varepsilon_j \begin{bmatrix} & & & 1 \\ & & \iddots & i\lambda \\ & \iddots & \iddots & \\ 1 & i\lambda & & \end{bmatrix}$$

with

- $\varepsilon_j = 1$ (positive signature), or
- $\varepsilon_j = -1$ (negative signature).

Even Kronecker Form (Thompson 1976)

For an even matrix pencil there exists a $W \in \mathrm{Gl}_N(\mathbb{C})$ such that

$$W^*(\lambda E - A)W = \mathrm{diag}(\mathcal{B}_1(\lambda), \ldots, \mathcal{B}_k(\lambda))$$

where the pencils $\mathcal{B}_j(\lambda)$ are one of the following type:

Type 4: Singular block:

$$\mathcal{B}_j(\lambda) = \left[\begin{array}{cccc|cccc} & & & & \lambda & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \lambda & 1 \\ \hline -\lambda & & & & & & & \\ 1 & \ddots & & & & & & \\ & \ddots & -\lambda & & & & & \\ & & 1 & & & & & \end{array}\right]$$

Theorem: (R., slight extension of results of Clements, Glover 89)

Let

(i) (A, B) controllable, or

(ii) (A, B) stabilizable and the associated even matrix pencil be regular.

Then a maximal solution X_+ exists if and only if the even Kronecker form of the associated even matrix pencil has the following properties:

- All blocks corresponding to the infinite eigenvalues have odd size and negative signature.
- All blocks corresponding to the imaginary eigenvalues have even size and positive signature.

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Let

(i) (A, B) controllable, or

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Then a maximal solution X_+ exists if and only if the even Kronecker form of the associated even matrix pencil has the following properties:

- All blocks corresponding to the infinite eigenvalues have odd size and negative signature.
- All blocks corresponding to the imaginary eigenvalues have even size.

Technique for the proof:

Comparison of inertia of the blocks $\mathcal{B}_i(i\omega)$ in the even Kronecker form of $i\omega\mathcal{E} - \mathcal{A}$ and the fact that there exists some matrix $K(i\omega) \in \mathrm{Gl}_{2n+m}(\mathbb{C})$ such that

$$K^*(i\omega)(i\omega\mathcal{E} - \mathcal{A})K(i\omega) = \begin{bmatrix} 0 & -i\omega I + A^* & 0 \\ -i\omega I + A & 0 & 0 \\ 0 & 0 & \mathcal{P}(i\omega) \end{bmatrix}$$

Theorem: (R.)

Let there exist a deflating subspace $\mathrm{im}[\,X_1^T\,,\,X_2^T\,,\,X_3^T\,]^T \subset \mathbb{C}^{2n+m}$ of the associated even matrix pencil with

$$\operatorname{rank}(X_1) = n, \qquad (\mathrm{im}[\,X_1^T\,,\,X_2^T\,]^T \text{ is } \textit{1-regular}),$$

$$X_2^* X_1 = X_1^* X_2 \qquad (\mathrm{im}[\,X_1^T\,,\,X_2^T\,]^T \text{ is } \textit{Lagrangian}).$$

Then a solution of the Lur'e equation is given by $X = X_2 X_1^-$.

Converse direction also holds due to

$$\begin{bmatrix} Q & \lambda I + A^* & S \\ -\lambda I + A & 0 & B \\ S^* & B^* & R \end{bmatrix} \begin{bmatrix} I & 0 \\ X & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} X & K^* \\ -I & 0 \\ 0 & L^* \end{bmatrix} \begin{bmatrix} \lambda I - A & -B \\ K & L \end{bmatrix}.$$

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$$\begin{aligned} \mathrm{rank}(X_1) &= n, && (\mathrm{im}[\,X_1^T\,,\,X_2^T\,]^T \text{ is } \textit{1-regular}), \\ X_2^* X_1 &= X_1^* X_2 && (\mathrm{im}[\,X_1^T\,,\,X_2^T\,]^T \text{ is } \textit{Lagrangian}). \end{aligned}$$

Then a solution of the Lur'e equation is given by $X = X_2 X_1^-$.

Converse direction also holds due to

$$\begin{bmatrix} Q & \lambda I + A^* & S \\ -\lambda I + A & 0 & B \\ S^* & B^* & R \end{bmatrix} \begin{bmatrix} I & 0 \\ X & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} X & K^* \\ -I & 0 \\ 0 & L^* \end{bmatrix} \begin{bmatrix} \lambda I - A & -B \\ K & L \end{bmatrix}.$$

Now: Characterization of desired deflating subspaces via even Kronecker form:

Theorem (R.)

Assume that an even Kronecker form of the associated even matrix pencil is given by

$$W^*(\lambda\mathcal{E} - \mathcal{A})W = \operatorname{diag}(\mathcal{B}_1(\lambda), \ldots, \mathcal{B}_k(\lambda)), \qquad W = [\, W_1 \,, \ldots, \, W_k \,]$$

Then a deflating subspace $\mathcal{X} = \operatorname{im}[\, X_1^T \,, \, X_2^T \,, \, X_3^T \,]^T \subset \mathbb{C}^{2n+m}$ has the property that $\operatorname{im}[\, X_1^T \,, \, X_2^T \,]^T$ is 1-regular and Lagrangian if and only if

$$\mathcal{X} = \operatorname{im} \widetilde{W}_1 \oplus \cdots \oplus \operatorname{im} \widetilde{W}_k,$$

where

-

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$$\mathcal{X} = \operatorname{im} \widetilde{W}_1 \oplus \cdots \oplus \operatorname{im} \widetilde{W}_k,$$

where

- $\widetilde{W}_j$ either contains the first or the second half of the columns of W_j if $\mathcal{B}_j(\lambda)$ is a block corresponding to non-imaginary eigenvalues.

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$$W^*(\lambda\mathcal{E} - \mathcal{A})W = \operatorname{diag}(\mathcal{B}_1(\lambda), \ldots, \mathcal{B}_k(\lambda)), \qquad W = [\, W_1 \,, \,\ldots\,, \, W_k \,]$$

Then a deflating subspace $\mathcal{X} = \operatorname{im}[\, X_1^T \,, \, X_2^T \,, \, X_3^T \,]^T \subset \mathbb{C}^{2n+m}$ has the property that $\operatorname{im}[\, X_1^T \,, \, X_2^T \,]^T$ is 1-regular and Lagrangian if and only if

$$\mathcal{X} = \operatorname{im} \widetilde{W}_1 \oplus \cdots \oplus \operatorname{im} \widetilde{W}_k,$$

where

- ...
- $\widetilde{W}_j$ contains the second half of the columns of W_j if $\mathcal{B}_j(\lambda)$ is a block corresponding to imaginary eigenvalues.

Theorem (R.)

Assume that an even Kronecker form of the associated even matrix pencil is given by

$$W^*(\lambda\mathcal{E} - \mathcal{A})W = \mathrm{diag}(\mathcal{B}_1(\lambda), \ldots, \mathcal{B}_k(\lambda)), \qquad W = [\, W_1\, , \ldots , \, W_k \,]$$

Then a deflating subspace $\mathcal{X} = \mathrm{im}[\, X_1^T\, , \, X_2^T\, , \, X_3^T\,]^T \subset \mathbb{C}^{2n+m}$ has the property that $\mathrm{im}[\, X_1^T\, , \, X_2^T\,]^T$ is 1-regular and Lagrangian if and only if

$$\mathcal{X} = \mathrm{im}\, \widetilde{W}_1 \oplus \cdots \oplus \mathrm{im}\, \widetilde{W}_k,$$

where

- ...
- ...
- $\widetilde{W}_j$ contains the columns $l, \ldots, 2l+1$ of W_j if $\mathcal{B}_j(\lambda)$ is a block of size $2l+1$ corresponding to infinite eigenvalues.

Theorem (R.)

Assume that an even Kronecker form of the associated even matrix pencil is given by

$$W^*(\lambda\mathcal{E} - \mathcal{A})W = \operatorname{diag}(\mathcal{B}_1(\lambda), \dots, \mathcal{B}_k(\lambda)), \qquad W = [\, W_1 \,, \, \dots, \, W_k \,]$$

Then a deflating subspace $\mathcal{X} = \operatorname{im}[\, X_1^T \,, \, X_2^T \,, \, X_3^T \,]^T \subset \mathbb{C}^{2n+m}$ has the property that $\operatorname{im}[\, X_1^T \,, \, X_2^T \,]^T$ is 1-regular and Lagrangian if and only if

$$\mathcal{X} = \operatorname{im} \widetilde{W}_1 \oplus \cdots \oplus \operatorname{im} \widetilde{W}_k,$$

where

- ...
- ...
- ...
- $\widetilde{W}_j$ contains the columns $l, \dots, 2l+1$ of W_j if $\mathcal{B}_j(\lambda)$ is a singular block of size $2l+1$.

Theorem (R.)

Assume that an even Kronecker form of the associated even matrix pencil is given by

$$W^*(\lambda\mathcal{E} - \mathcal{A})W = \operatorname{diag}(\mathcal{B}_1(\lambda), \ldots, \mathcal{B}_k(\lambda)), \qquad W = [\, W_1 \,, \, \ldots \,, \, W_k \,]$$

Then a deflating subspace $\mathcal{X} = \operatorname{im}[\, X_1^T \,, \, X_2^T \,, \, X_3^T \,]^T \subset \mathbb{C}^{2n+m}$ has the property that $\operatorname{im}[\, X_1^T \,, \, X_2^T \,]^T$ is 1-regular and Lagrangian if and only if

$$\mathcal{X} = \operatorname{im} \widetilde{W}_1 \oplus \cdots \oplus \operatorname{im} \widetilde{W}_k,$$

where

- ...
- ...
- ...
- ...

If, particularly the second half of the columns of W_j for the blocks $\mathcal{B}_j(\lambda)$ corresponding to non-imaginary eigenvalues is chosen, then $X_+ = X_1 X_2^-$.

Outline

Linear-quadratic optimal control problem

Minimize $\mathcal{J}(u, x_0) = \frac{1}{2}\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$

subject to $\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$

and

$$\begin{aligned} A^* X_+ + X_+ A + Q &= K_+^* K_+, \\ X_+ B + S &= K_+^* L_+, \\ R &= L_+^* L_+, \end{aligned}$$

where X_+ is the maximal solution.

Theorem (Willems, 1972)

The optimal control in the distributional sense is given by $\hat{u}$ satisfying

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\hat{u}(t) + \delta_0 x_0, \\ 0 &= K_+ x(t) + L_+ \hat{u}(t). \end{aligned}$$

Conclusions from deflating subspace construction of an optimal control:

Corollary (R.)

Let the even Kronecker form of the associated even pencil be given by

$$W^*(\lambda\mathcal{E} - \mathcal{A})W = \operatorname{diag}(\mathcal{B}_1(\lambda), \ldots, \mathcal{B}_k(\lambda)), \qquad W = [\, W_1 \,, \, \ldots \,, \, W_k \,]$$

and let $2l + 1$ be the size of the largest block $\mathcal{B}_j(\lambda)$ corresponding to the infinite eigenvalues. Then an optimal control satisfies

$$\hat{u} \in \operatorname{span}\{\delta_0, \ldots, \delta_0^{(l-1)}\} \oplus L_2^{loc}(\mathbb{R}^+).$$

Moreover,

- an optimal control is unique if and only if no singular block is contained,
- there exists an optimal control with $\hat{u} \in \operatorname{span}\{\delta_0, \ldots, \delta_0^{(l-1)}\} \oplus L_2(\mathbb{R}^+)$ if and only if no block corresponding to imaginary eigenvalues is contained.

Outline

Further results:

- Existence of *minimal solutions*, if (A, B) is *anti-stabilizable*, i.e.,

$$\operatorname{rank}[\,A + sI\,,\ B\,] = n \quad \forall s \in \overline{\mathbb{C}^+}.$$

- Existence of solutions, if (A, B) is *sign-controllable*, i.e.,

$$\max\{\operatorname{rank}[\,A - sI\,,\ B\,], \operatorname{rank}[\,A + \bar{s}I\,,\ B\,]\} = n \quad \forall s \in \overline{\mathbb{C}^+}.$$

Conclusion

- Regular LQ optimal control problems lead to Riccati equations
  - Solution via invariant subspaces of Hamiltonian matrices
- Singular LQ optimal control problems lead to Lur'e equations
  - Solution via deflating subspaces of even matrix pencils
  - Conclusions for the uniqueness and structure of the optimal control