$\mathcal{H}_\infty$ Control for Descriptor Systems
A Structured Matrix Pencils Approach

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Outline

Introduction

We consider the system

$$
\begin{aligned}
E\dot{x} &= Ax + B_1 w + B_2 u, \qquad x(t_0) = x^0,\\
z &= C_1 x + D_{11} w + D_{12} u,\\
y &= C_2 x + D_{21} w + D_{22} u,
\end{aligned}
$$

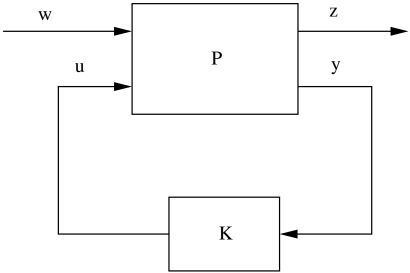
$E, A \in \mathbb{R}^{n,n}$, $B_i \in \mathbb{R}^{n,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$, $i, j = 1, 2$.

- E may be singular, $\operatorname{rank}(E) = r$
- $\lambda E - A$ regular, i.e. $\det(\lambda E - A)$ does not vanish identically
- x descriptor variable, w disturbance, u input, z controlled output, y measured output

The optimal $\mathcal{H}_\infty$ control problem

Determine a dynamic controller

$$\begin{aligned} \hat{E}\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}y, \\ u &= \hat{C}\hat{x} + \hat{D}y, \end{aligned}$$



with $\hat{E}, \hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$ such that the closed-loop system, formed by the given system combined with the controller, is internally stable and the closed-loop transfer function $T_{zw}(s)$ from w to z is minimized in the $\mathcal{H}_\infty$ norm.

Previous Work

The $\mathcal{H}_\infty$ control problem for descriptor systems has been studied using

- linear matrix inequalities [Rehm/Allgöwer]
- generalized Riccati equations [Takaba/Morihira/Katayama]

Since

- LMIs are non practical for large scale systems
- GREs are facing severe numerical difficulties

we are proposing a matrix pencil approach wich relies on the structure preserving computation of deflating subspaces of even matrix pencils, generalizing the results from [Benner/Byers/Mehrmann/Xu '04].

Additionally we would like to use only original system data as long as possible to prevent numerical errors.

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Two Subproblems

The modified optimal $\mathcal{H}_\infty$ control problem

For the descriptor system let Γ be the set of positive real numbers γ for which there exists an internally stabilizing dynamic controller such that the transfer function $T_{zw}(s)$ of the closed loop system satisfies $\|T_{zw}\|_\infty < \gamma$.
In the modified optimal $\mathcal{H}_\infty$ control problem we want to determine $\gamma_{mo} = \inf \Gamma$

The suboptimal $\mathcal{H}_\infty$ control problem

For a descriptor system and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$ determine an internally stabilizing dynamic controller such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$.

Modified optimal $\mathcal{H}_\infty$ Control

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Preliminary Assumptions

A1. The triple (E, A, B_2) is strongly stabilizable and the triple (E, A, C_2) is strongly detectable.

(E, A, B_2) is called strongly stabilizable, if it is both *finite dynamics stabilizable* i.e. $\text{rank}[\lambda E - A,\ B_2] = n$ and *impulse controllable* i.e. $\text{rank}[E,\ AS_\infty,\ B_2] = n$.

(E, A, C_2) is called strongly detectable, if it is both *finite dynamics detectable* i.e. $\text{rank}[\lambda E^T - A^T,\ C_2^T] = n$ and *impulse observable* i.e. $\text{rank}[E^T,\ A^T T_\infty,\ C_2^T] = n$.

Preliminary Assumptions

A1. The triple (E, A, B_2) is strongly stabilizable and the triple (E, A, C_2) is strongly detectable.

A2. $\operatorname{rank}\begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\omega \in \mathbb{R}$.

A3. $\operatorname{rank}\begin{bmatrix} A - i\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\omega \in \mathbb{R}$.

A4. For matrices T_∞, S_∞ with $\mathrm{Im}\, S_\infty = \ker E$ and $\mathrm{Im}\, T_\infty = \ker E^T$ the rank conditions

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} &= n + m_2 - \operatorname{rank} E, \\ \operatorname{rank}\begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} &= n + p_1 - \operatorname{rank} E \end{aligned}$$

holds.

Matrix Pencils

Matrix pencils we will use:

$$\lambda N_H + M_H(\gamma) =$$
$$\lambda \left[\begin{array}{cc|ccc} 0 & -E^T & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & -A^T & 0 & 0 & -C_1^T \\ -A & 0 & -B_1 & -B_2 & 0 \\ \hline 0 & -B_1^T & -\gamma^2 I & 0 & -D_{11}^T \\ 0 & -B_2^T & 0 & 0 & -D_{12}^T \\ -C_1 & 0 & -D_{11} & -D_{12} & -I \end{array}\right]$$

and

$$\lambda N_J + M_J(\gamma) =$$
$$\lambda \left[\begin{array}{cc|ccc} 0 & -E & 0 & 0 & 0 \\ E^T & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & -A & 0 & 0 & -B_1 \\ -A^T & 0 & -C_1^T & -C_2^T & 0 \\ \hline 0 & -C_1 & -\gamma^2 I & 0 & -D_{11} \\ 0 & -C_2 & 0 & 0 & -D_{21} \\ -B_1^T & 0 & -D_{11}^T & -D_{21}^T & -I \end{array}\right].$$

only contain data from the original system.
Even Pencils: $P(-\lambda)^T = P(\lambda)$.

Deflating Subspaces

Let

$$X_H(\gamma) = \begin{matrix} n \\ n \\ m_1 \\ m_2 \\ p_1 \end{matrix} \overset{r}{\begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \\ X_{H,3}(\gamma) \\ X_{H,4}(\gamma) \\ X_{H,5}(\gamma) \end{bmatrix}}, \quad X_J(\gamma) = \begin{matrix} n \\ n \\ p_1 \\ p_2 \\ m_1 \end{matrix} \overset{r}{\begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \\ X_{J,3}(\gamma) \\ X_{J,4}(\gamma) \\ X_{J,5}(\gamma) \end{bmatrix}}$$

Deflating Subspaces

Let $X \in \mathbb{R}^{n,k}$ with full column rank, then $\operatorname{Im} X$ is called *deflating subspace* for the pencil $\lambda E - A$ if there exists matrices $Y \in \mathbb{R}^{n,k}$, $R, U \in \mathbb{R}^{k,k}$ such that

$$(\lambda E - A)X = Y(\lambda R - U).$$

A deflating subspace is called *stable (semi-stable)* if all finite eigenvalues of $\lambda R - U$ are in the open (closed) left half plane.

Deflating Subspaces

Lagrangian Subspaces

Let $\mathcal{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

- A subspace $\mathcal{L}$ is called isotropic if $x^T \mathcal{J} y = 0$ for all $x, y \in \mathcal{L}$.
- An isotropic subspace with $\dim \mathcal{L} = n$ is called Lagrangian.

Main Result

Theorem

Consider a regular descriptor system of arbitrary index and the even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. Suppose that assumptions **A1–A4** hold.
Then there exists an internally stabilizing controller such that the transfer function from w to z satisfies $\|T_{zw}\|_\infty < \gamma$ if and only if γ is such that the conditions **C1–C4** hold.

Conditions for the General Case

C1. The index of both pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ is at most one.

C2. There exists a matrix $X_H(\gamma)$ such that

- **C2.a)** $\mathrm{im}\, X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H$;
- **C2.b)** $\mathrm{im} \begin{bmatrix} EX_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$ is a r-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
- **C2.c)** $\mathrm{rank}(EX_{H,1}(\gamma)) = r$.

C3. There exists a matrix $X_J(\gamma)$ such that

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C4. The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} \gamma X_{H,2}^T(\gamma) E X_{H,1}(\gamma) & X_{H,2}^T(\gamma) E X_{J,2}(\gamma) \\ X_{J,2}^T(\gamma) E^T X_{H,2}(\gamma) & \gamma X_{J,2}^T(\gamma) E^T X_{J,1}(\gamma) \end{bmatrix}.$$

is positive semidefinite and satisfies $\mathrm{rank}\mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$.

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Sketch of proof

The proof is mainly based on

- Existence of a preliminary index reducing feedback [Bunse-Gerstner/Byers/Mehrmann/Nichols '99]
- Weierstraß canonical form [Gantmacher '59]
- Pencil based approach for standard systems [Benner/Byers/Mehrmann/Xu '04]

Neither the computation of the index reducing feedback nor of the Weierstraß canonical form is necessary.

Computation

Procedure 1: (Classification of γ)

Input: Data of system, value $\gamma \geq 0$.
Output: Decision whether $\gamma < \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$.

1. Form the pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$.
2. Compute the deflating subspace matrices X_H and X_J associated with the eigenvalues in the closed left half plane.
3. IF the dimension of one/both of these subspaces is less than r, then $\gamma < \gamma_{mo}$,
   ELSE

   IF the rank of $EX_{H,1}$ and/or $E^T X_{J,1}$ is less than r, then $\gamma < \gamma_{mo}$,
   ELSE

   Form the matrix $\hat{\mathcal{Y}}$.
   IF $\hat{\mathcal{Y}}$ is not symmetric positive semi-definite and/or $\operatorname{rank} \hat{\mathcal{Y}} < \hat{k}_H + \hat{k}_J$, then $\gamma < \gamma_{mo}$,
   ELSE $\gamma \geq \gamma_{mo}$.

Computation

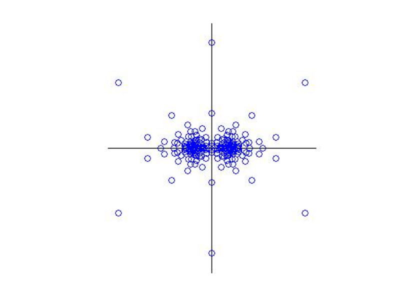
- The main part of the algorithm is the computation of the deflating subspaces
- These subspaces could be computed with the QZ-Algorithm, that however does not take advantage of the special structure of the matrix pencils or its eigensymmetry.
- Therefore we recommend a structure preserving algorithm to compute the eigenvalues and deflating subspaces of the even matrix pencils as has been introduced by [Benner/Byers/Mehrmann/Xu '99]

Spectral Properties

Hamiltonian eigensymmetry

Even pencils exhibit the Hamiltonian eigensymmetry:
if λ is a finite eigenvalue of $\mathcal{H} - \lambda\mathcal{S}$, then $\bar{\lambda}, -\lambda, -\bar{\lambda}$ are eigenvalues of $\mathcal{H} - \lambda\mathcal{S}$, too.

Typical Hamiltonian spectrum:



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sH/H Schur Form

Structured real skew-Hamiltonian/Hamiltonian Schur Form [Mehl '99]

Let $\mathcal{H} - \lambda\mathcal{S}$ be a regular real skew-Hamiltonian/Hamiltonian pencil. Under certain conditions on the purely imaginary and infinite eigenvalues there exists an (orthogonal) $\mathcal{J}$-congruence

$$\mathcal{J}\mathcal{Y}^T\mathcal{J}^T(\mathcal{H} - \lambda\mathcal{S})\mathcal{Y} = \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix} - \lambda \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix},$$

where H_{11} is quasi-upper triangular, S_{11} is upper triangular, H_{12} is symmetric, and S_{12} is skew-symmetric.

- Not every skew-Hamiltonian/Hamiltonian pencil has such a structured Schur form.
- Embedding in an extended pencil of double size resolves existence problem. [Benner/Byers/Mehrmann/Xu '99]

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Generalized Symplectic URV-Decomposition

Theorem

Let $\mathcal{H} - \lambda\mathcal{S}$ be a real regular skew-Hamiltonian/Hamiltonian pencil, then there exist orthogonal matrices $\mathcal{Q}_1, \mathcal{Q}_2$ such that

$$\begin{aligned} \mathcal{Q}_1^T \mathcal{H} \mathcal{Q}_2 &= \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}, \\ \mathcal{Q}_1^T \mathcal{S} \mathcal{J} \mathcal{Q}_1 \mathcal{J}^T &= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix} \in \mathbb{SH}_{2n}, \\ \mathcal{J} \mathcal{Q}_2^T \mathcal{J}^T \mathcal{S} \mathcal{Q}_2 &= \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^T \end{bmatrix} \in \mathbb{SH}_{2n}, \end{aligned}$$

where H_{11}, S_{11}, T_{11} are upper triangular and H_{22}^T is quasi-upper triangular.

The eigenvalues of $\mathcal{H} - \lambda\mathcal{S}$ are given by $\pm\Lambda(S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^T)^{\frac{1}{2}}$.

Embedding in Extended sH/H-Pencil (I)

Consider a skew-Hamiltonian/Hamiltonian pencil of the form

$$\mathcal{H} - \lambda \mathcal{S} = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} - \lambda \begin{bmatrix} A & B \\ C & A^T \end{bmatrix}$$

where B and C are skew-symmetric and G and H are symmetric.

Now let

$$\mathcal{B}_\mathcal{H} = \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{bmatrix}, \quad \mathcal{B}_\mathcal{S} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{bmatrix}, \tag{1}$$

and

$$\mathcal{Y}_r = \frac{\sqrt{2}}{2} \begin{bmatrix} I_{2n} & I_{2n} \\ -I_{2n} & I_{2n} \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}.$$

Then

$$\mathcal{Y}_r^T \mathcal{B}_\mathcal{H} \mathcal{Y}_r = \begin{bmatrix} 0 & \mathcal{H} \\ \mathcal{H} & 0 \end{bmatrix}, \quad \mathcal{Y}_r^T \mathcal{B}_\mathcal{S} \mathcal{Y}_r = \mathcal{B}_\mathcal{S}.$$

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Embedding in Extended sH/H-Pencil (II)

Set

$$\begin{aligned} \mathcal{B}_{\mathcal{H}}^r - \lambda \mathcal{B}_{\mathcal{S}}^r &:= \mathcal{P}^T \mathcal{Y}_r^T \left(\mathcal{B}_{\mathcal{H}} - \lambda \mathcal{B}_{\mathcal{S}} \right) \mathcal{Y}_r \mathcal{P} \\ &= \left[\begin{array}{cc|cc} 0 & F & 0 & G \\ F & 0 & G & 0 \\ \hline 0 & H & 0 & -F^T \\ H & 0 & -F^T & 0 \end{array} \right] - \lambda \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A & 0 & B \\ \hline C & 0 & A^T & 0 \\ 0 & C & 0 & A^T \end{array} \right] \end{aligned}$$

Computation of the Structured Schur Form

With $\tilde{\mathcal{Q}} = \mathcal{P}^T \operatorname{diag}(\mathcal{J}\mathcal{Q}_1\mathcal{J}^T, \mathcal{Q}_2)\mathcal{P}$, where $\mathcal{Q}_1, \mathcal{Q}_2$ are as in generalized SURV, we obtain

$$\mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}^r_{\mathcal{H}}\tilde{\mathcal{Q}} = \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^T & 0 & H_{12}^T & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^T & 0 \end{array}\right] =: \begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} \\ 0 & -\tilde{\mathcal{H}}_{11}^T \end{bmatrix},$$

$$\mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}^r_{\mathcal{S}}\tilde{\mathcal{Q}} = \left[\begin{array}{cc|cc} S_{11} & 0 & S_{12} & 0 \\ 0 & T_{11} & 0 & T_{12} \\ \hline 0 & 0 & S_{11}^T & 0 \\ 0 & 0 & 0 & T_{11}^T \end{array}\right] =: \begin{bmatrix} \tilde{\mathcal{S}}_{11} & \tilde{\mathcal{S}}_{12} \\ 0 & \tilde{\mathcal{S}}_{11}^T \end{bmatrix}.$$

Re-ordering the structured Schur decomposition $\Longrightarrow$

$$\begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^T \end{bmatrix} - \lambda \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{11}^T \end{bmatrix},$$

where $\Lambda(\mathcal{H}, \mathcal{S}) \cap \mathbb{C}^- \subset \Lambda(\mathcal{H}_{11}, \mathcal{S}_{11})$.

Structured Schur Form of Embedded sH/H-pencil

Theorem

Let $\mathcal{H} - \lambda\mathcal{S}$ be a skew-Hamiltonian/Hamiltonian pencil and consider the extended matrices $\mathcal{B}_\mathcal{H} = \text{diag}(\mathcal{H}, -\mathcal{H})$, $\mathcal{B}_\mathcal{S} = \text{diag}(\mathcal{S}, \mathcal{S})$.

a) There exist unitary $\mathcal{W}, \mathcal{V}$ such that

$$\begin{aligned} \mathcal{W}^T \mathcal{B}_\mathcal{H} \mathcal{V} &= \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & \mathcal{H}_{22} \end{bmatrix}, \\ \mathcal{W}^T \mathcal{B}_\mathcal{S} \mathcal{V} &= \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{22} \end{bmatrix}, \end{aligned}$$

where $\mathcal{H}_{11}, \mathcal{S}_{11} \in \mathbb{R}^{2n,2n}$ and

$$\begin{aligned} \Lambda(\mathcal{B}_\mathcal{S}, \mathcal{B}_\mathcal{H}) \cap \mathbb{C}^- &\subset \Lambda(\mathcal{S}_{11}, \mathcal{H}_{11}), \\ \Lambda(\mathcal{S}_{11}, \mathcal{H}_{11}) \cap \Lambda(\mathcal{B}_\mathcal{S}, \mathcal{B}_\mathcal{H}) \cap \mathbb{C}^+ &= \emptyset. \end{aligned}$$

Structured Schur Form of Embedded sH/H-pencil

Theorem

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b) Let $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{R}^{4n,2n} = \mathcal{V}(:, 1:2n)$, then

$$\mathrm{Def}_-(\mathcal{H}, \mathcal{S}) \subset \mathrm{range}\, V_1, \quad \mathrm{Def}_+(\mathcal{H}, \mathcal{S}) \subset \mathrm{range}\, V_2.$$

Equality holds if $\nexists$ eigenvalues $0, \infty$.

Computation

Computation of deflating subspaces

- Compute generalized symplectic URV of original pencils
- Embed pencils
- Compute structured Schur forms
- Reorder the eigenvalues
- Extract deflating subspaces from transformation matrices

Our experimental code for a γ-Iteration relying on this algorithm shows promising results.

Example

We consider the following example [Takaba/Morihira/Katayama, 94], [Rehm/Allgöwer, 98].

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ D_{21} = 1$$

- (E, A) is of index 2.
- goal: find the minimum value γ that satisfies the conditions **C1** – **C4**.
- γ_{opt} is calculated as $\gamma^\rho = 0.7678$ which is smaller than the calculated values using the LMI approach or the Riccati approach.

Suboptimal $\mathcal{H}_\infty$ Control

The modified optimal $\mathcal{H}_\infty$ control problem

For the descriptor system let Γ be the set of positive real numbers γ for which there exists an internally stabilizing dynamic controller such that the transfer function $T_{zw}(s)$ of the closed loop system satisfies $\|T_{zw}\|_\infty < \gamma$.
In the modified optimal $\mathcal{H}_\infty$ control problem we want to determine $\gamma_{mo} = \inf \Gamma$

The suboptimal $\mathcal{H}_\infty$ control problem

For a descriptor system and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$ determine an internally stabilizing dynamic controller such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$.

Suboptimal $\mathcal{H}_\infty$ Control

Theorem

Consider a regular descriptor system of arbitrary index. Suppose that assumptions **A1–A4** hold, $\gamma > \gamma_{mo}$ and $\bar{\sigma}(D_{11}) < \gamma$. Then the sub-optimal $\mathcal{H}_\infty$ control problem has an internally stabilizing controller such that the $\mathcal{H}_\infty$ norm of the closed loop is less than γ given by:

$$
\begin{aligned}
(-\lambda\hat{E} + \hat{A}) &= X_J^T \bar{\Pi}(\lambda) X_H \\
\hat{B} &= X_J^T \bar{B}_\Pi \\
\hat{C} &= \bar{C}_\Pi X_H \\
\hat{D} &= \bar{D}_\Pi
\end{aligned}
$$

Suboptimal $\mathcal{H}_\infty$ Control

$\bar{\Pi}(\lambda)$, $\bar{\Pi}_B$, $\bar{\Pi}_C$, $\bar{\Pi}_D$ are matrices containing original system data and a $m_2 \times p_2$ feedback matrix F such that $(E, A + B_2FC_2)$ is of index one.

- Computation of index reducing feedback necessary
- We also have formulas for the parametrized controller
- Then computation of kernel and cokernel of E is also necessary

Conclusions

Conclusions

- Existence conditions for $\mathcal{H}_\infty$ controllers in terms of the original system data
- Structure preserving Algorithm for the computation of the deflating subspaces
- Controller formulas in terms of the original system (plus Index reducing Feedback)