Cohomological rigidity problem, topological toric manifolds and face numbers of simplicial cell manifolds

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Content of this talk

§0. **Toric manifolds and fans**

§1. **Cohomological rigidity problem and related problems**

§2. **Topological toric manifolds**

§3. **Face numbers of simplicial cell manifolds**

§0. Toric manifolds and fans

Definition

A toric variety X^n of $\dim_{\mathbb{C}} = n$

$\Longleftrightarrow$

a normal algebraic variety of $\dim_{\mathbb{C}} = n$ with an effective action of $(\mathbb{C}^*)^n$ having an open dense orbit.

$(\mathbb{C}^*)^n =$ **open dense orbit** $\subset X^n \curvearrowleft (\mathbb{C}^*)^n$

Toric manifold $\overset{def}{=}$ **compact smooth toric variety**

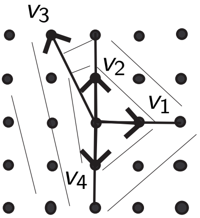
Fundamental theorem in toric geometry

$\{$ toric varieties $X^n \curvearrowleft (\mathbb{C}^*)^n\} \quad \Longleftrightarrow \quad \{$fans in $\mathbb{R}^n\}$

Hirzebruch surface

$F_a = P(\mathbb{C} \oplus \mathcal{O}(a)) \to \mathbb{C}P^1$

$(a = -2)$



$K = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\},$
$\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}\}$

A (simplicial) fan Δ_X may be viewed as a pair $(K, \{v_i\})$.

Correspondence $X \to \Delta_X$ when X is a toric manifold

Let $X_1, \dots, X_m$ be invariant divisors of a toric manifold X each is fixed under some $\mathbb{C}^*$-subgroup of $(\mathbb{C}^*)^n$.

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We deduce two data.

[1] $K := \{I \subset \{1, \dots, m\} \mid \bigcap_{i \in I} X_i \neq \emptyset\}$
abstract simplicial complex of dim $n-1$

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[2] $v_i \in \mathbb{Z}^n = \mathrm{Hom}_{alg}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ **is characterized by**

1. $v_i(\mathbb{C}^*)$ **fixes** X_i **pointwise,**
2. $v_i(g)_*(\xi) = g\xi$ **for** $\xi \in (\tau X|_{X_i})/\tau X_i$.

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The pair $(K, \{v_i\}_{i=1}^m)$ is essentially the fan Δ_X.

Interpretation via equivariant cohomology

Let X be a toric manifold of $\dim_{\mathbb{C}} = n$.

$$H^*_{(\mathbb{C}^*)^n}(X) := H^*(E(\mathbb{C}^*)^n \times_{(\mathbb{C}^*)^n} X)$$

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(1) $K := \{I \subset \{1, \ldots, m\} \mid \bigcap_{i \in I} X_i \neq \emptyset\}$

- $x_i :=$ **Poincaré dual of $X_i \in H^2_{(\mathbb{C}^*)^n}(X)$.**

Lemma

$H^*_{(\mathbb{C}^*)^n}(X) = \mathbb{Z}[x_1, \ldots, x_m]/(\prod_{i \in I} x_i \mid I \notin K)$ as rings

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$$H^*_{(\mathbb{C}^*)^n}(X) := H^*(E(\mathbb{C}^*)^n \times_{(\mathbb{C}^*)^n} X)$$ **algebra over** $H^*(B(\mathbb{C}^*)^n)$

through $\pi^*\colon H^*(B(\mathbb{C}^*)^n) \to H^*_{(\mathbb{C}^*)^n}(X)$.

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Lemma

There exists a unique $v_i \in H_2(B(\mathbb{C}^*)^n)$ for each i satisfying

$$\pi^*(u) = \sum_{i=1}^{m} \langle u, v_i \rangle x_i \in H^2_{(\mathbb{C}^*)^n}(X) \qquad \text{for } \forall u \in H^2(B(\mathbb{C}^*)^n)$$

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- $v_i \in H_2(B(\mathbb{C}^*)^n) = [B\mathbb{C}^*, B(\mathbb{C}^*)^n] = \mathrm{Hom}_{alg}(\mathbb{C}^*, (\mathbb{C}^*)^n) = \mathbb{Z}^n$

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$$H^*(B(\mathbb{C}^*)^n) \xrightarrow{\pi^*} H^*_{(\mathbb{C}^*)^n}(X) \twoheadrightarrow H^*(X)$$

Theorem (Danilov(1978 general case)-Jurkiewicz(projective case))

Let $\Delta_X = (K, \{v_i\}_{i=1}^m)$. Then $H^*(X) = \mathbb{Z}[x_1, \ldots, x_m]/\mathcal{I}$ where $\deg x_i = 2$ and the ideal $\mathcal{I}$ is generated by

1. $\prod_{i \in I} x_i$ for $I \notin K$
2. $\sum_{i=1}^m \langle u, v_i \rangle x_i$ for $u \in \mathbb{Z}^n = H^2(B(\mathbb{C}^*)^n)$

A simple observation

$$H^*(F_a) = \mathbb{Z}[x,y]/(x^2, y(y+ax)) \qquad (a \in \mathbb{Z})$$

One can easily check

$$H^*(F_a) \cong H^*(F_b) \iff a \equiv b\ (2) \iff F_a \cong F_b \text{ diffeo}$$

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These are toric versions of Poincaré conjecture.

No counterexamples are known and there are some partial affirmative solutions.

Bott manifolds

A Bott tower of height n is a sequence of $\mathbb{C}P^1$-bundles

$$M_n \xrightarrow{\mathbb{C}P^1} M_{n-1} \xrightarrow{\mathbb{C}P^1} \cdots \xrightarrow{\mathbb{C}P^1} M_2 \xrightarrow{\mathbb{C}P^1} M_1 \xrightarrow{\mathbb{C}P^1} M_0 = \{\textbf{a point}\}$$

where $M_i = P(\underline{\mathbb{C}} \oplus L_i) \to M_{i-1}$ and $L_i \to M_{i-1}$ is a $\mathbb{C}$-line b'dle.

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Theorem

Cohomological rigidity holds for Bott manifolds M_n's when $n = 3$ (Choi-M-Suh) and $n = 4$ (Choi).

Problem (Invariance of Pontryagin classes)

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- **Affirmative for Bott manifolds (Choi, 2010)**

Real toric manifolds

$X(\mathbb{R}) =$ **the set of real points in a toric manifold** X

Example

1. When $X = \mathbb{C}P^n$, $X(\mathbb{R}) = \mathbb{R}P^n$
2. When $X = F_a$ (Hirzebruch surface), $X(\mathbb{R})$ is a torus or Klein bottle.

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- $X(\mathbb{R})$ **is not simply conn. and often an aspherical manifold.**
- $H^*(X(\mathbb{R}); \mathbb{Z}_2) = H^{2*}(X; \mathbb{Z}) \otimes \mathbb{Z}_2$

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Not true in general but true in some cases.

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- **We call M_n a real Bott manifold.**
- $M_1 = \mathbb{R}P^1 = S^1$, M_2 **is a torus or Klein bottle.**
- M_n **admits a flat Riemannian metric.**

• $L_i \to M_{i-1}$ **is characterized by** $w_1(L_i) \in H^1(M_{i-1};\mathbb{Z}_2) \cong \mathbb{Z}_2^{i-1}$**.**
Putting $w_1(L_i)$ **in the** i**-th column, we obtain**

$$A = \begin{pmatrix} 0 & A_2^1 & A_3^1 & \dots & A_{n-1}^1 & A_n^1 \\ 0 & 0 & A_3^2 & \dots & A_{n-1}^2 & A_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & A_n^{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \qquad (A_j^i \in \mathbb{Z}_2 = \{0,1\})$$

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- $H^*(M(A);\mathbb{Z}_2) = \mathbb{Z}_2[x_1,\dots,x_n]/(x_j^2 + x_j\sum_{i=1}^{j-1} A_j^i x_i \mid j = 1,\dots,n)$

Theorem (Choi-M-Oum, 2010)

Let A and B be upper triangular binary matrices with zero diagonals. The following are equivalent.

1. $M(A) \cong M(B)$ diffeo,
2. $H^*(M(A);\mathbb{Z}_2) \cong H^*(M(B);\mathbb{Z}_2)$ as rings,
3. $A \leftrightsquigarrow B$ via three matrix operations (Op1), (Op2), (Op3)
4. $D_A \leftrightsquigarrow D_B$ via three graph operations (Op1)', (Op2)', (Op3)'

where D_A is the labeled acyclic digraph associated with A.

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where D_A is the labeled acyclic digraph associated with A.

(Op1) is conjugation by a permutation matrix
(Op2) is a variant of simultaneous column addition
(Op3) is a row addition under certain condition

Example (The case $n = 3$)

There are $2^3 = 8$ upper triangular binary matrices of size 3 with zero diagonal entries.

$1^\star$. The zero matrix of size 3. $M(A) = (S^1)^3$.

2.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$M(A) = S^1 \times (\text{Klein bottle})$.

$3^\star$.

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M(A) = P(\gamma \oplus \gamma) \xrightarrow{T^2} S^1.$$

4.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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$\{$**Upper triangular binary matirces of size** $n\}/(Op1),(Op2),(Op3)$

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The number of real Bott manifolds.

n	1	2	3	4	5	6	7	8	9	10
$Diff_n$	1	2	4	12	54	472	8,512	328,416	?	?
Ori_n	1	1	2	3	8	29	222	3,607	131,373	?
$Symp_n$		1		2		6		31		416

Theorem (Unique decomposition property)

The decomposition of real Bott manifolds into a product of indecomposable real Bott manifolds is unique up to permutation of factors.

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If $S^1 \times M \cong S^1 \times M'$ for real Bott manifolds M, M', then $M \cong M'$.

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The cancellation property does not hold for compact flat Riemannian manifolds in general (Charlap 1965).

Theorem (H. Ishida 2010)

The following are equivalent for real Bott manifolds M.

1. M admits a Kähler structure.
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Conclusion.
(Real) Bott manifolds have several “rigidity” properties. So (real) toric manifolds probably have some rigidity property.

§2. Topological toric manifolds
(by Ishida-Fukukawa-M, 2010)

Local charts of a toric manifold

A toric manifold $X^n \curvearrowleft (\mathbb{C}^*)^n$ has invariant local charts $\{(U_\sigma, \varphi_\sigma)\}$ such that

$$\varphi_\sigma\colon U_\sigma \xrightarrow{\approx} \mathbb{C}^n \curvearrowleft (\mathbb{C}^*)^n \quad \text{sum of 1-dim algebraic rep's.}$$

Transition functions are Laurent monomials

$$(w_1, \dots, w_n) \to \Big(\prod_{j=1}^n w_j^{a_{1j}}, \ \dots, \ \prod_{j=1}^n w_j^{a_{nj}}\Big) \quad (a_{ij} \in \mathbb{Z})$$

Definition (Topological toric manifold)

A closed smooth manifold $M^{2n} \curvearrowleft (\mathbb{C}^*)^n$ is topological toric if

1. the action has an open dense orbit,
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$$\{\textbf{toric manifolds}\} \subsetneq \{\textbf{topological toric manifolds}\}$$

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It can be obtained by gluing four $\mathbb{C}^2$ as follows.

$$
\begin{array}{ccccc}
\mathbb{C}^2 & (w_1^{-1}, w_1^{-1}\bar{w}_1 w_2) & \longleftarrow & (w_1, w_2) & \mathbb{C}^2 \\
 & \downarrow & & \downarrow & \\
\mathbb{C}^2 & (\bar{w}_1^{-1}\bar{w}_2, \bar{w}_1 w_1^{-1}\bar{w}_2^{-1}) & \longleftarrow & (w_1 w_2^{-1}, w_2^{-1}) & \mathbb{C}^2
\end{array}
$$

Transition funcitons are Laurent monomials in w_1, w_2 and $\bar{w}_1, \bar{w}_2$.

A (simplicial) fan was a pair $(K, \{v_i\})$ where

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A topological fan is an ordinary fan when $b_i = c_i$.

Quotient construction of toric manifolds work in our setting. To $\Delta = (K, \{\beta_i\}_{i=1}^m)$ topological fan, we have

1. $U(K) := \mathbb{C}^m \backslash Z \curvearrowleft (\mathbb{C}^*)^m$
2. $\beta_i \in \mathrm{Hom}_{smooth}(\mathbb{C}^*, (\mathbb{C}^*)^n) = \mathbb{C}^n \times \mathbb{Z}^n$ **define**

$$\lambda := \prod_{i=1}^m \beta_i : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^n.$$

Then $X(\Delta) := U(K)/\ker\lambda \curvearrowleft (\mathbb{C}^*)^m/\ker\lambda = (\mathbb{C}^*)^n$

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Theorem (Ishida-Fukukawa-M, 2010)

The correspondence $\Delta \to X(\Delta)$ gives a bijection:

$$\{\text{complete non-singular topological fans}\}$$
$$\to \{\text{Omnioriented topological toric manifolds}\}$$

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For instance, $M/(S^1)^n$ is a manifold with corners s.t.

1. **every face (even $M/(S^1)^n$ itself) is contractible,**
2. **any intersection of faces is connected unless empty.**

So, $M/(S^1)^n$ looks like a simple polytope and hence $H^*(M)$ is generated by $H^2(M)$ as a ring.

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But $M/(S^1)^n$ is not necessarily a simple polytope.
$\exists$ a topological toric manifold M^8 s.t. $\partial(M/(S^1)^4)$ is dual to the Barnette sphere (a non-polytopal simplicial 3-sphere).

Quasitoric manifolds (by Davis-Januszkiewicz, 1991)

Definition

A quasitoric manifold is a closed smooth manifold M^{2n} with smooth action of $(S^1)^n$ s.t.

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Theorem (Ishida-Fukukawa-M, 2010)

$$\{\text{Quasitoric manifolds}\}$$
$$\subsetneqq \{\text{top. toric manifolds with restricted compact torus actions}\}$$

up to equivariant homeomorphism.

§3. Face numbers of simplicial cell manifolds

P: an n-polytope.

$f_i = f_i(P) = \#$**of i-dim faces of P,** $(f_0, f_1, \dots, f_{n-1})$ f**-vector**

The h-vector $(h_0, h_1, \dots, h_n)$ of P is defined by

$$\sum_{i=0}^{n} h_i t^{n-i} = \sum_{j=0}^{n} f_{j-1}(t-1)^{n-j} \quad (f_{-1} = 1)$$

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g-theorem (Billera-Lee, Stanley, 1980)

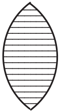
An integer vector $(h_0, h_1, \dots, h_n)$ with $h_0 = 1$ is the h-vector of a simplicial n-polytope iff the following hold.

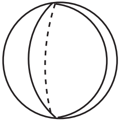
1. $h_i = h_{n-i}$ for $\forall i$ (Dehn-Sommerville eq's)
2. $1 = h_0 \leq h_1 \leq \cdots \leq h_{[n/2]}$
3. $h_{i+1} - h_i \leq (h_i - h_{i-1})^{\langle i \rangle}$ for $1 \leq i \leq [n/2] - 1$

Torus manifolds and simplicial cell spheres

S^{2n} $(n \geq 2)$ cannot be a topological toric manifold because $H^*(S^{2n})$ is not generated by $H^2(S^{2n})$. However, S^{2n} admits a smooth action of $(S^1)^n$ and $S^{2n}/(S^1)^n$ is a manifold with corners s.t.

1. **every face is contractible, but**
2. **intersections of faces can be disconnected.**





S^4/T^2 S^6/T^3

More generally,

Theorem (Panov-M, 2006)

If a (torus) manifold $M^{2n} \curvearrowleft (S^1)^n$ satisfies $H^{odd}(M) = 0$, then $M/(S^1)^n$ is a manifold with corners and

1. every face is acyclic, and
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1. every face is acyclic, and
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The dual of $\partial(M/(S^1)^n)$ is (often) a simplicial cell $(n-1)$-sphere.

Theorem (Stanley 1991, Masuda 2005)

An integer vector $(h_0, h_1, \dots, h_n)$ with $h_0 = 1$ is the h-vector of a simplicial cell $(n-1)$-sphere $\mathcal{P}$ iff the following hold.

1. $h_i = h_{n-i}$ for $\forall i$ (Dehn-Sommerville eq's)
2. $h_i \geq 0$ for $1 \leq i \leq n-1$
3. If $h_i = 0$ for some i, then $\sum_{i=0}^{n} h_i$ is even.

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Example

If $\mathcal{P}$ is obtained by gluing two 2-simplices along their boundary $(= \partial(S^6/(S^1)^3)^*)$, then

$$(h_0, h_1, h_2, h_3) = (1, 0, 0, 1).$$

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Idea of proof of necessity. Suppose $\mathcal{P} = \partial(M^{2n}/(S^1)^n)^*$ **for some** $M^{2n} \curvearrowleft (S^1)^n$ **with** $H^{odd}(M) = 0$**. Then** $h_i = \operatorname{rank} H^{2i}(M)$**. (1) and (2) follow from this. Moreover**

$$w(M) = \prod(1 + x_i) \pmod 2 \quad \textbf{for } x_i \in H^2(M).$$

If $h_i = 0$ **for some** $1 \leq i \leq n-1$**, then** $w_{2n}(M) = 0$ **and hence**

$$0 = w_{2n}(M)[M] \equiv \chi(M) = \sum h_i.$$

Face numbers of simplicial cell manifolds

Problem

Fix a manifold N and characterize h-vectors of all simplicial cell complexes homeomorphic to N.

This is solved when N is

- S^{n-1} **(Stanley, M)**
- $\mathbb{R}P^{n-1}$ **and** $S^p \times S^q$ **(Murai 2010)**

and studied when N is

- D^n **(Kolins 2010)**

Those results suggest

A naive conjecture

For a closed manifold N of dim $n-1$, $\exists r_i(N) \in \mathbb{Z}$ s.t. an integer vector $(h_0, h_1, \ldots, h_n)$ with $h_0 = 1$ is the h-vector of a simplicial cell complexes homeomorphic to N iff the following hold.

1. $h_{n-i} - h_i = (-1)^i \binom{n}{i}(\chi(N) - \chi(S^{n-1}))$ for $\forall i$ (DS eq's),
2. $h_i \geq r_i(N)$ for $1 \leq i \leq n-1$,
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Novik-Swartz show that each h_i has a lower bound. It is best possible in some cases but not in general.