Topological aspects of partial product spaces: a survey

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Topological methods in toric geometry, symplectic geometry and combinatorics:
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Western

Summary

- What is a generalized moment-angle complex/partial product space?
- Some history
- Calculating with the polyhedral product functor
- Stable splitting
- The (rational) homotopy Lie algebra; the (integral) Pontryagin algebra
- Some open problems

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Some references

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Definition (Partial product spaces)

- Let K be a simplicial complex on m vertices.
- For $1 \le i \le m$, let $* \hookrightarrow A_i \hookrightarrow X_i$ be based CW-complexes.
- For $\sigma \in K$, let
$$(\underline{X},\underline{A})^\sigma = \prod_{i=1}^{m} \begin{cases} X_i & \text{if } i \in \sigma; \\ A_i & \text{otherwise.} \end{cases}$$
- Let
$$\mathcal{Z}_K(\underline{X},\underline{A}) = \bigcup_{\sigma\in K} (\underline{X},\underline{A})^\sigma .$$
- Special case: if $X_i = X$ and $A_i = A$ for $1 \le i \le m$, write $\mathcal{Z}_K(X,A) := \mathcal{Z}_K(\underline{X},\underline{A})$.
- Another special case: if $A_i = *$ for all i, write $\underline{X}^K := \mathcal{Z}_K(\underline{X},\underline{A})$.

Generally: for each i, let $A_i \to X_i$ be a cofibration of cofibrant objects in a suitable model category. Let

$$\mathcal{Z}_K(\underline{X},\underline{A}) = \operatorname{colim}_{\sigma\in K} (\underline{X},\underline{A})^\sigma .$$

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first examples

Note

$$\prod_{i=1}^{m} A_i \subseteq \mathcal{Z}_K(\underline{X},\underline{A}) \subseteq \prod_{i=1}^{m} X_i.$$

Example

- Let $K = \Delta^{m-1}$, the full simplex. Then

$$\mathcal{Z}_K(\underline{X},\underline{A}) = X_1 \times X_2 \times \cdots \times X_m.$$

- Let $K = \overset{1}{\circ} \qquad \overset{2}{\circ}$. Then

$$\begin{aligned} \underline{X}^K &= (X_1 \times *) \cup (* \times X_2) \\ &= X_1 \vee X_2. \end{aligned}$$

- For $K = \partial\Delta^{m-1}$, $\underline{X}^K$ is the fat wedge of $X_1, \ldots, X_m$.

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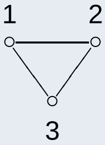
Example

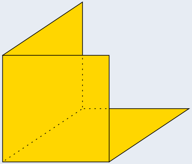
Let

$$K = \partial\Delta^2 \quad =$$



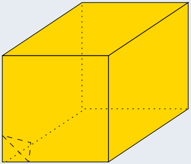
Then ($D^1 = [0, 1]$):

$$\mathcal{Z}_K(D^1, *) =$$



and

$$\mathcal{Z}_K(D^1, S^0) =$$



Example (Subspace arrangements)

For $k = \mathbb{C}$ and $k = \mathbb{R}$, $\mathcal{Z}_K(k, k^*)$ is a coordinate subspace arrangement:

$$\mathcal{Z}_K(k, k^*) = k^n - \bigcup_{\substack{S=\{x_{i_1}=\cdots x_{i_k}=0\}: \\ \{i_1,\ldots,i_k\notin K\}}} S.$$

[de Longueville, Schultz], [Goresky, MacPherson]

Example (Torus complexes)

Let $\Gamma = K^{(1)}$, and present the right-angled Artin group

$$G_\Gamma = \langle x_1, \ldots, x_m \mid x_i x_j = x_j x_i \text{ for } \{i, j\} \text{ an edge of } \Gamma \rangle .$$

Then

- $G_\Gamma = \pi_1((S^1)^K)$ [Kim, Roush] and
- $(S^1)^K$ is aspherical iff K is a flag complex. [Charney, Davis]

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Moment-angle complexes

- The partial product spaces $\mathcal{Z}_K(D^2, S^1)$ are the “classical” moment-angle complexes.
- If K is a simplicial d-sphere, $\mathcal{Z}_K(D^2, S^1)$ is a $m + d + 1$-dimensional manifold.
- The dual of a simple polytope P is a simplicial polytope. Write $\mathcal{Z}_P(D^2, S^1)$ for the corresponding moment-angle manifold.
- If $m + d + 1$ is even: [López de Medrano, Verjovsky; Bosio, Meersseman]
- Label faces of P with lattice vectors, a basis for $\mathbb{Z}^{d+1}$ around each vertex. Free T^{m-d+1} action on $\mathcal{Z}_P(D^2, S^1)$. The orbit space is a (quasi)toric manifold. [Davis, Januszkiewicz]

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Identification spaces

Definition

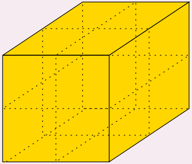
For a space X and simplicial complex K with m vertices, let

$$I_K(X) = X^{\times m} \times \mathcal{Z}_K(D^1, 0)/\sim,$$

where $\sim$ is: for $p \in \mathcal{Z}_K(D^1, 0)$, let $\sigma(p) = \{i \in [n] : p_i = 1\}$. Set $(x, p) \sim (x', p)$ if and only if $x_i = x_i'$ for all $i \notin \sigma(p)$.

Example

If K is a 3-cycle, then $I_K(\mathbb{Z}/2)$ consists of eight copies of $\mathcal{Z}_K(D^1, 0)$, identified:



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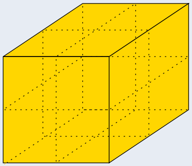

Proposition

For any X and simplicial complex K,

$$I_K(X) \cong \mathcal{Z}_K(CX, X),$$

where CX is the cone on X.

- Suppose K is a pure $d-1$-complex. If $\rho\colon G^{\times m} \to G^{\times d}$ is a group homomorphism with the property that $\rho|_{G^\sigma}$ is an isomorphism for each maximal $\sigma \in K$, form quotient $M_{K,G} = I_K(G)/\ker\rho$.
- $G = S^1$: quasitoric manifolds; $G = \mathbb{Z}/2$: small covers. [Davis, Januszkiewicz]

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Naturality: the polyhedral product functor

- If $f\colon (\underline{X},\underline{A}) \to (\underline{Y},\underline{B})$ is a map of pairs, then the induced map

$$\mathcal{Z}_f\colon \mathcal{Z}_K(\underline{X},\underline{A}) \to \mathcal{Z}_K(\underline{Y},\underline{B})$$

  preserves $\simeq$.

- For example:

$$\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*) \simeq \mathcal{Z}_K(D^2,S^1) \simeq \mathcal{Z}_K(ES^1,S^1);$$

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Lemma

Given fibrations

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ F' & \longrightarrow & E' & \longrightarrow & B' \end{array}$$

then $\mathcal{Z}_K(F,F') \to \mathcal{Z}_K(E,E') \to \mathcal{Z}_K(B,B')$ *is also a fibration, provided either* $B = B'$ *or* $F = F'$.

Example

For any topological group G, there are fibrations (in fact, $G^{\times m}$-bundles)

$$G^{\times m} \longrightarrow \mathcal{Z}_K(EG,G) \longrightarrow (BG)^K,$$

$$\mathcal{Z}_K(EG,G) \longrightarrow (BG)^K \longrightarrow BG^{\times m}.$$

So, e.g., $DJ(K) := (\mathbb{CP}^\infty)^K$ is the homotopy orbit space for the diagonal action of $T^m = (S^1)^{\times m}$ on $\mathcal{Z}_K(D^2,S^1)$. [Davis, Januszkiewicz]

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Cohomology

Lemma

1. *The inclusion* $\underline{X}^K \hookrightarrow \prod_{i=1}^m X_i$ *induces a surjection in cohomology:*

$$H^\cdot(\prod_{i=1}^m X_i, \mathbf{k}) \twoheadrightarrow H^\cdot(\underline{X}^K, \mathbf{k}).$$

2. $H^\cdot(\underline{X}^K, \mathbf{k}) \cong \varprojlim_{\sigma \in K} H^\cdot(\underline{X}^\sigma, \mathbf{k})$.

Useful cases:

- For $X = S^1$,

$$E := H^\cdot(T^m) = \bigwedge[x_1, \ldots, x_m] \twoheadrightarrow H^\cdot((S^1)^K) =: E/J_K,$$

- and for $X = BS^1 = \mathbb{CP}^\infty$,

$$S := H^\cdot(BT^m) = \mathbf{k}[x_1, \ldots, x_m] \twoheadrightarrow H^\cdot((BS^1)^K) =: S/I_K,$$

where I_K, J_K are generated by $x_{i_1}x_{i_2}\cdots x_{i_k}$, for $\{i_1, \ldots, i_k\} \notin K$.

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where I_K, J_K are generated by $x_{i_1} x_{i_2} \cdots x_{i_k}$, for $\{i_1, \dots, i_k\} \notin K$.

Cohomology

Lemma

1. *The inclusion* $\underline{X}^K \hookrightarrow \prod_{i=1}^m X_i$ *induces a surjection in cohomology:*

$$H^{\cdot}(\prod_{i=1}^m X_i, \mathbf{k}) \twoheadrightarrow H^{\cdot}(\underline{X}^K, \mathbf{k}).$$

2. $H^{\cdot}(\underline{X}^K, \mathbf{k}) \cong \varprojlim_{\sigma\in K} H^{\cdot}(\underline{X}^\sigma, \mathbf{k})$.

Useful cases:

- For $X = S^1$,

$$E := H^{\cdot}(T^m) = \bigwedge[x_1, \dots, x_m] \twoheadrightarrow H^{\cdot}((S^1)^K) =: E/J_K,$$

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- CDGA version: [Félix-Tanré]. Gives CDGA for $\mathcal{Z}_K(X, A)$ from X, A. So, e.g., $\operatorname{cat}(X^K) = \operatorname{cat}(X)(1 + \dim K)$ for simply-connected X.

Theorem (Buchstaber, Panov)

For any coefficients $\mathbf{k}$*, there is a bigraded algebra isomorphism*

$$H^{\cdot}(\mathcal{Z}_K(D^2, S^1), \mathbf{k}) \cong \operatorname{Tor}^S(S/I_K, \mathbf{k}).$$

- For $I \subseteq [m]$, let K_I denote induced subcomplex on vertices I. Additively,

$$\operatorname{Tor}^S(S/I, \mathbf{k}) \cong \bigoplus_{I \subseteq [m]} \widetilde{H}^{\cdot}(K_I). \quad \text{[Hochster]}$$

- Bigrading on left is from MHS on cohomology of the subspace complement. [Deligne, Goresky, MacPherson]
- Right-hand side has history: free resolutions of monomial ideals. [Eagon, Reiner, Welker]
- Works the same for (D^{2n}, S^{2n-1}), with a shift.

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equivariant cohomology

Note $G^{\times m}$ acts on $\mathcal{Z}_K(EG, G)$ and

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So, e.g., for $G = S^1$,

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If K is a simplicial d-sphere, same for $H^{\cdot}_{T^{m-d+1}}(M_{K,S^1})$, equivariant cohomology of the quasitoric manifold.

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Stable splitting

The suspension of a partial product space (often) has a nice description. [Bahri, Bendersky, Cohen, Gitler][1]

Definition

- If $(\underline{X}, \underline{A})$ are pairs of based CW-complexes and K is a simplicial complex, let

$$(\underline{X}, \underline{A})^{\widehat{\sigma}} = \bigwedge_{i=1}^{m} \begin{cases} X_i & \text{if } i \in \sigma; \\ A_i & \text{otherwise,} \end{cases}$$

$$\text{and} \qquad \widehat{\mathcal{Z}}_K(\underline{X}, \underline{A}) = \operatorname{colim}_{\sigma \in K} (\underline{X}, \underline{A})^{\widehat{\sigma}}$$

- Equals the image of $\mathcal{Z}_K(\underline{X}, \underline{A})$ in $X_1 \wedge \cdots \wedge X_m$.
- Let $\widehat{\underline{X}}^K := \widehat{\mathcal{Z}}_K(\underline{X}, \underline{*})$.

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Theorem (Bahri, Bendersky, Cohen, Gitler)

If each $A_i \hookrightarrow X_i$ is connected, there is a natural homotopy equivalence

$$\Sigma \mathcal{Z}_K(\underline{X}, \underline{A}) \xrightarrow{\simeq} \Sigma\Big(\bigvee_{I \subseteq [m]} \widehat{\mathcal{Z}}_K(\underline{X_I}, \underline{A_I})\Big).$$

- Let $\widehat{A}^I = \bigwedge_{i \in I} A_i$.

Theorem (Bahri, Bendersky, Cohen, Gitler)

If each $X_i \simeq *$, there is a homotopy equivalence

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Corollary (Moment-angle complexes)

We have

$$\Sigma \mathcal{Z}_K(D^2, S^1) \xrightarrow{\simeq} \Sigma \Big(\bigvee_{I \notin K} \Sigma^{2+|I|} |K_I| \Big).$$

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$$\Sigma(\underline{X}^K) \xrightarrow{\simeq} \Sigma\Big(\bigvee_{I \in K} \widehat{X}^I\Big)$$

- Order k-simplices by

$$(i_1 < i_2 < \cdots i_k) \prec (j_1 < j_2 < \cdots < j_k) \Leftrightarrow i_r < j_r \ \forall r.$$

- K is a shifted complex if simplices are an initial sequence in $\prec$.
- In this case, the previous result holds without suspension:

Theorem (Grbić, Theriault)

If K is shifted, $\mathcal{Z}_K(D^{n+1}, S^n)$ is homotopy equivalent to a wedge of spheres.

Remark

Not all (generalized) moment-angle complexes are homotopy equivalent to wedges of spheres.

Formality

- A (commutative) differential graded algebra (A, d) is formal if $(A, d) \simeq_{\text{q.i.}} (H^{\cdot}(A), 0)$.
- A space X is formal if the DGA $C^{\cdot}_{\text{sing}}(X, \mathbb{Z})$ is formal.
- A space X is $\mathbb{Q}$-formal if the CDGA $A^{\cdot}_{PL}(X)$ is formal.
- $DJ(K)$ is formal and $\mathbb{Q}$-formal. [Franz; Notbohm, Ray]
- Moment-angle complexes:

$$C^{\cdot}_{\text{sing}}(\mathcal{Z}_K(D^2, S^1), \mathbf{k}) \simeq_{\text{q.i.}} S/I_K \otimes_{\mathbf{k}} E,$$

  the Koszul complex of the Stanley-Reisner ideal. [Baskakov, Buchstaber, Panov]
- However, $\mathcal{Z}_K(D^2, S^1)$ need not be formal. [Baskakov]
- For $K \cong S^2$, $\mathcal{Z}_K(D^2, S^1)$ is “almost never” formal. [D., Suciu]

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The rational homotopy Lie algebra

- For simply-connected X, let $\mathfrak{g}(X) = \pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Note $\mathfrak{g}(DJ(K)) \cong \mathfrak{g}(\mathcal{Z}_K(D^2, S^1)) \times \mathbb{Q}^m$.
- Since $DJ(K)$ is $\mathbb{Q}$-formal,

$$U(\mathfrak{g}) = H_*(\Omega DJ(K), \mathbb{Q}) \cong \mathrm{Ext}_{S/I_K}(\mathbb{Q}, \mathbb{Q}).$$

- Right-hand side: Poincaré series?
- S/I_K is a Koszul algebra iff K is a flag complex.
- A semi-explicit presentation: [Berglund]

$$\mathfrak{g}(DJ(K)) \cong FH^*\mathcal{L}_\infty(K)$$

- [Dobrinskaya]: an operadic description of $H_*(\Omega X^K, \mathbf{k})$ in terms of $H_*(\Omega X)$ and a configuration space of points in $\mathbb{R}$ with partial collisions.
- Diagonal subspace arrangements?

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A 16-point triangulation K of S^6 yields a moment-angle complex $\mathcal{Z}_K(D^2, S^1)$ in $\mathbb{C}^{16}$ with an indecomposable Massey product, giving

$$\mathrm{Prim}_{10}(\mathrm{Ext}_{H^{\cdot}\mathcal{Z}_K(D^1,S^1)}(\mathbb{Q},\mathbb{Q})) \ncong \pi_{10}(\mathcal{Z}_K(D^2,S^1)) \otimes \mathbb{Q}.$$

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total: 1 26 148 404 645 645 404 148 26 1
    0: 1  .   .   .   .   .   .   .  . .
    1: .  .   .   .   .   .   .   .  . .
    2: .  .   .   .   .   .   .   .  . .
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    6: .  .  98   .   .   .   .   .  . .
    7: .  4   . 126   .   .   .   .  . .
    8: .  .  38   .  53   .   .   .  . .
    9: .  .   . 223   .  10   .   .  . .
   10: .  .   8   . 368   .   1   .  . .
   11: .  .   .  54   . 214   .   .  . .
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