BIRS Workshop
Evolutionary Games

Equilibrium transitions in stochastic models of finite populations

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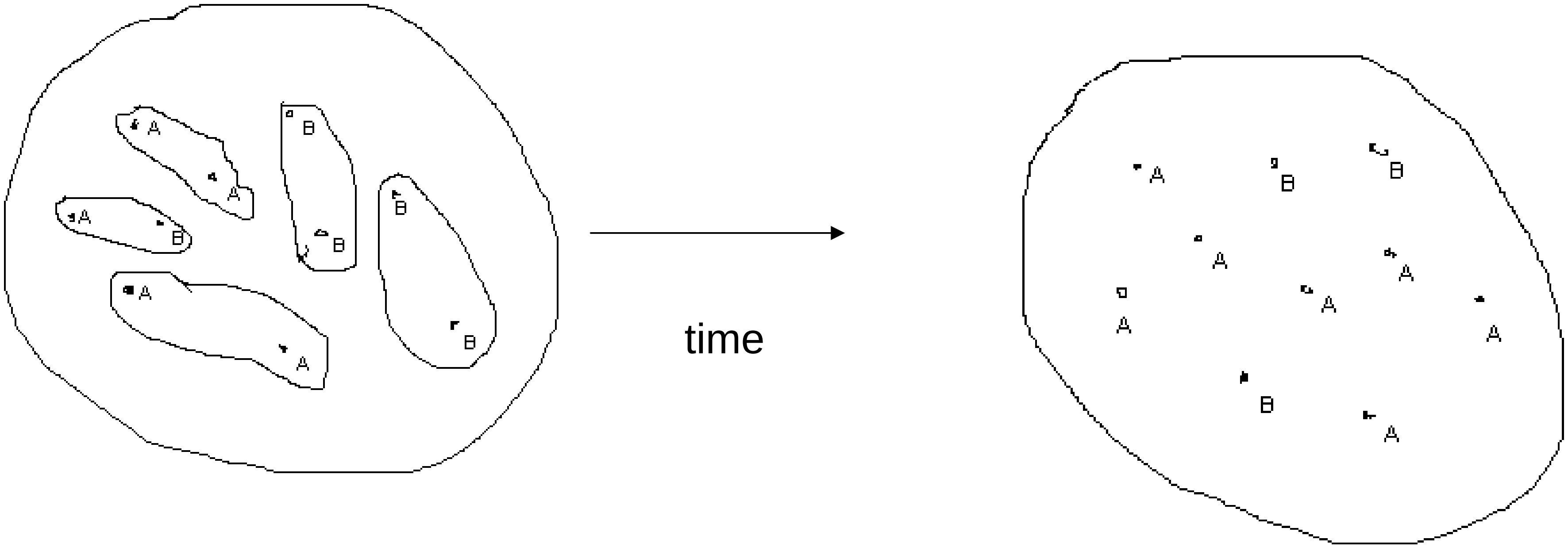
Population dynamics



A and B are two possible behaviors, fenotypes or strategies of each individual

Matching of individuals

everybody interacts with everybody

random pairing of individuals

space – structured populations

Main Goals

Equilibrium selection in case of multiple Nash equilibria

Evolutionary stability of cooperation

Dependence of the long-run behavior of population on

--- its size

--- mutation level

--- topology of interactions

Equilibrium transitions

Stochastic dynamics of finite unstructured populations

n - # of individuals

z_t - # of individuals playing A at time t

$\Omega = \{0,\ldots,n\}$ - state space

selection

$z_{t+1} > z_t$ if „average payoff" of A > „average payoff" of B

mutation

each individual may mutate and switch to the other strategy with a probability ε

Markov chain with n+1 states and a unique stationary state μ^{ε}_{n}

Definition

$Z \in \Omega$ is **stochastically stable** if $\lim_{\epsilon \to 0} \mu_n^{\epsilon}(Z) > 0$

extinctions

fixations

another approach: fixation probabilities
in systems with absorbing states

Previous results

Playing against the field, Kandori-Mailath-Rob 1993

	A	B
A	a	b
B	c	d

a>c, d>b, a>d,
a+b<c+d

$$\pi_A(z_t) = \frac{a(z_t - 1) + b(n - z_t)}{n - 1}$$

$$\pi_B(z_t) = \frac{cz_t + d(n - z_t - 1)}{n - 1}$$

(A,A) and (B,B) are Nash equilibria

A is an efficient strategy
B is a risk-dominant strategy

$$z_{t+1} > z_t \;\; if \;\; \pi_A(z_t) > \pi_B(z_t)$$

$$z_{t+1} < z_t \;\; if \;\; \pi_A(z_t) < \pi_B(z_t)$$

$$z_{t+1} = z_t \;\; if \;\; \pi_A(z_t) = \pi_B(z_t)$$

$$z_{t+1} = z_t \;\; if \;\; z_t = 0 \;\; or \;\; z_t = n$$

Theorem

For sufficiently large n, strategy B is stochastically stable, that is

$$\lim_{\epsilon \to 0} \mu_n^{\epsilon}(0) = 1$$

Random matching of players, Robson - Vega Redondo, 1996

p_t # of crosspairings

$$\tilde{\pi}_A(z_t, p_t) = \frac{a(z_t - p_t) + bp_t}{z_t}$$

$$\tilde{\pi}_B(z_t, p_t) = \frac{cp_t + d(n - z_t - p_t)}{n - z_t}$$

Theorem

For sufficiently large n, strategy A is stochastically stable, that is

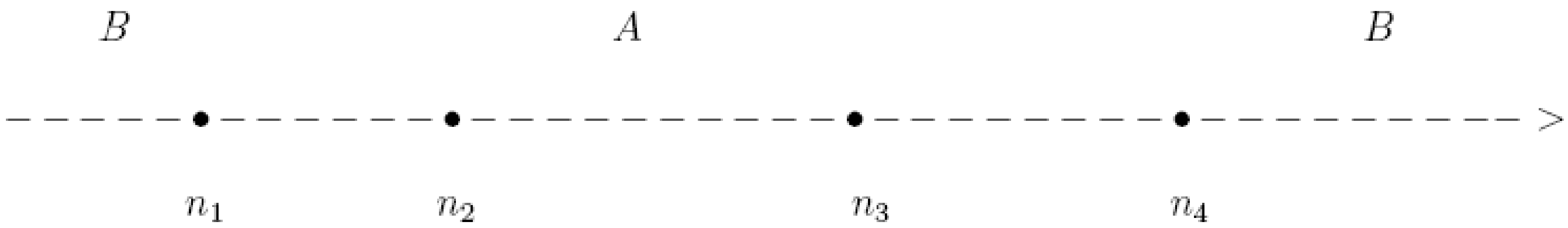
$$\lim_{\epsilon \to 0} \mu_n^{\epsilon}(n) = 1$$

Our results, JM J. Theor. Biol, 2005

so far n was fixed and $\epsilon \to 0$

now ϵ is fixed and $n \to \infty$

Theorem (random matching model)



Spatial games with local interactions

$\Lambda \subset \mathbf{Z}^d$

$S = \{1, ..., k\}$ – *set of strategies*

$\Omega_\Lambda = S^\Lambda$ – *set of population states*

N_i – *neighbourhood of the i – th player*

$U : S \times S \to \mathbf{R}$ – *payoff matrix*

Let $X \in \Omega_\Lambda$, *then* $\nu_i(X) = \sum_{j \in N_i} U(X_i, X_j)$ – *payoff of the i – th player*

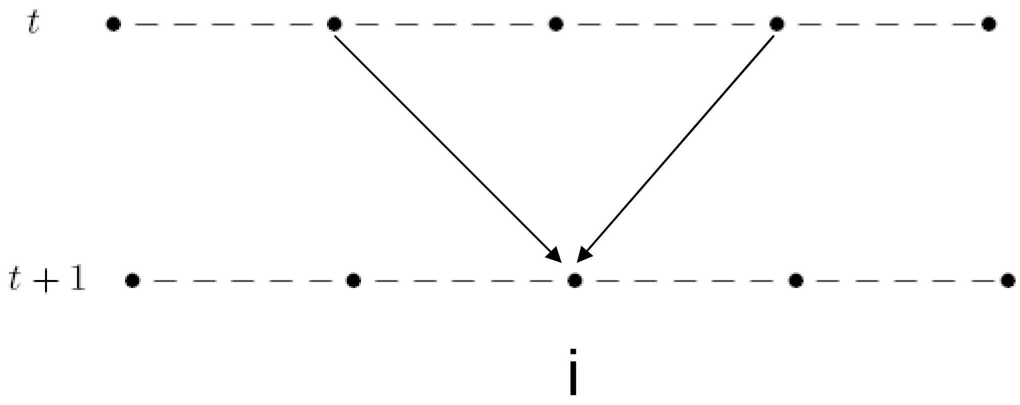
Definition

$X \in \Omega_\Lambda$ is a **Nash configuration** if for every $i \in \Lambda$ and $Y_i \in S$,

$$\nu_i(X_i, X_{-i}) \geq \nu_i(Y_i, X_{-i})$$

Deterministic dynamics

the best-response rule



imitation

Stochastic dynamics

a) perturbed best response

with the probability 1-ε, a player chooses the best response
with the probability ε a player makes a mistake

b) log-linear rule or Boltzmann updating

$$p_i^\epsilon(X_i^{t+1}|X_{-i}^t) = \frac{e^{\frac{1}{\epsilon}\nu_i(X_i^{t+1},X_{-i}^t)}}{\sum_{Y_i \in S} e^{\frac{1}{\epsilon}\nu_i(Y_i,X_{-i}^t)}}$$

$$lim_{\epsilon \to 0} p_i^\epsilon = Br$$

Example 1   JM, J. Phys. A 2004

square lattice with nearest-neighbour interactions, log-linear rule

		A	B	C
	A	1.5	0	1
U =	B	0	2	1
	C	1	1	2

X^A , X^B , X^C Nash configurations

$$\mu_\Lambda^\epsilon(X) = \frac{e^{\frac{1}{\epsilon}\sum_{ij} U(X_i,X_j)}}{\sum_{Z\in\Omega_\Lambda} e^{\frac{1}{\epsilon}\sum ijU(Z_i,Z_j)}}$$

$$\lim_{\epsilon\to 0} \mu_\Lambda^\epsilon(X^k) = 1/2 \ \ , k = B, C$$

$$\lim_{\Lambda\to\mathbf{Z}^2} \mu_\Lambda^\epsilon(X) = 0 \ for \ every \ X \in \Omega = S^{\mathbf{Z}^2}$$

Gibbs states

$$\lim_{\Lambda \to Z^2} \mu_\Lambda^\epsilon = \mu^\epsilon$$

$$U = \begin{array}{c|ccc} & A & B & C \\ A & 1.5 & 0 & 1 \\ B & 0 & 2 & 1 \\ C & 1 & 1 & 2 \end{array}$$

Theorem

$$\mu^\epsilon(X_0 = C) = 1 - \delta(\epsilon)$$

$$\delta(\epsilon) \to 0 \;\; when \;\; \epsilon \to 0$$

Proof

counting lowest cost excitations

BBBBBBB
BBBCBBB
BBBBBBB

CCCCCCC
CCCACCC
CCCCCCC

CCCCCCC
CCCBCCC
CCCCCCC

Example 2

JM, J. Phys. A 2004

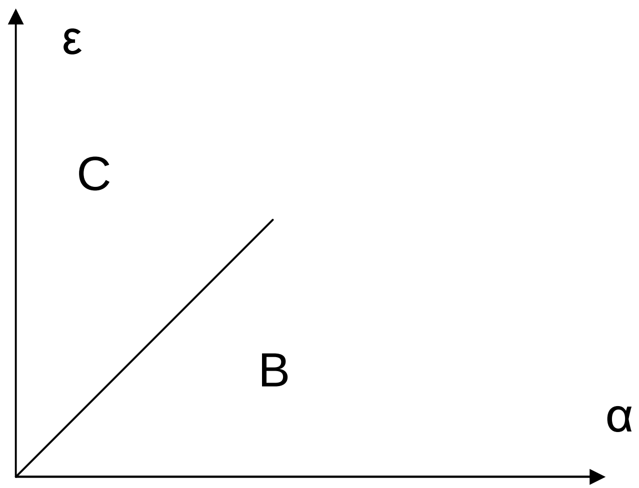
	A	B	C
A	0	0.1	1
B	0.1	2+α	1.1
C	1.1	1.1	2

without A B is stochastically stable

A is a dominated strategy

with A

where $\alpha > 0$



stochastic stability

number of players fixed, noise $\rightarrow$ 0

ensemble stability

noise fixed, number of players $\rightarrow$ ∞

Snow Drift

with Agata Powałka and Christoph Hauert

b - prize
c - cost

	C	D
C	b-c/2	b-c
D	b	0

r= c/(2b-c)

replicator dynamics

dx/dt = 2/(c-2b) x(1-x) (x – (1-r))

x = 1 - r is the mixed Nash equlibrium

letters to nature

NATURE | VOL 428 | 8 APRIL 2004 | www.nature.com/nature

Spatial structure often inhibits the evolution of cooperation in the snowdrift game

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pairwise comparisons

randomly chosen players imitate randomly chosen neighbors with probability proportional to the difference of payoffs

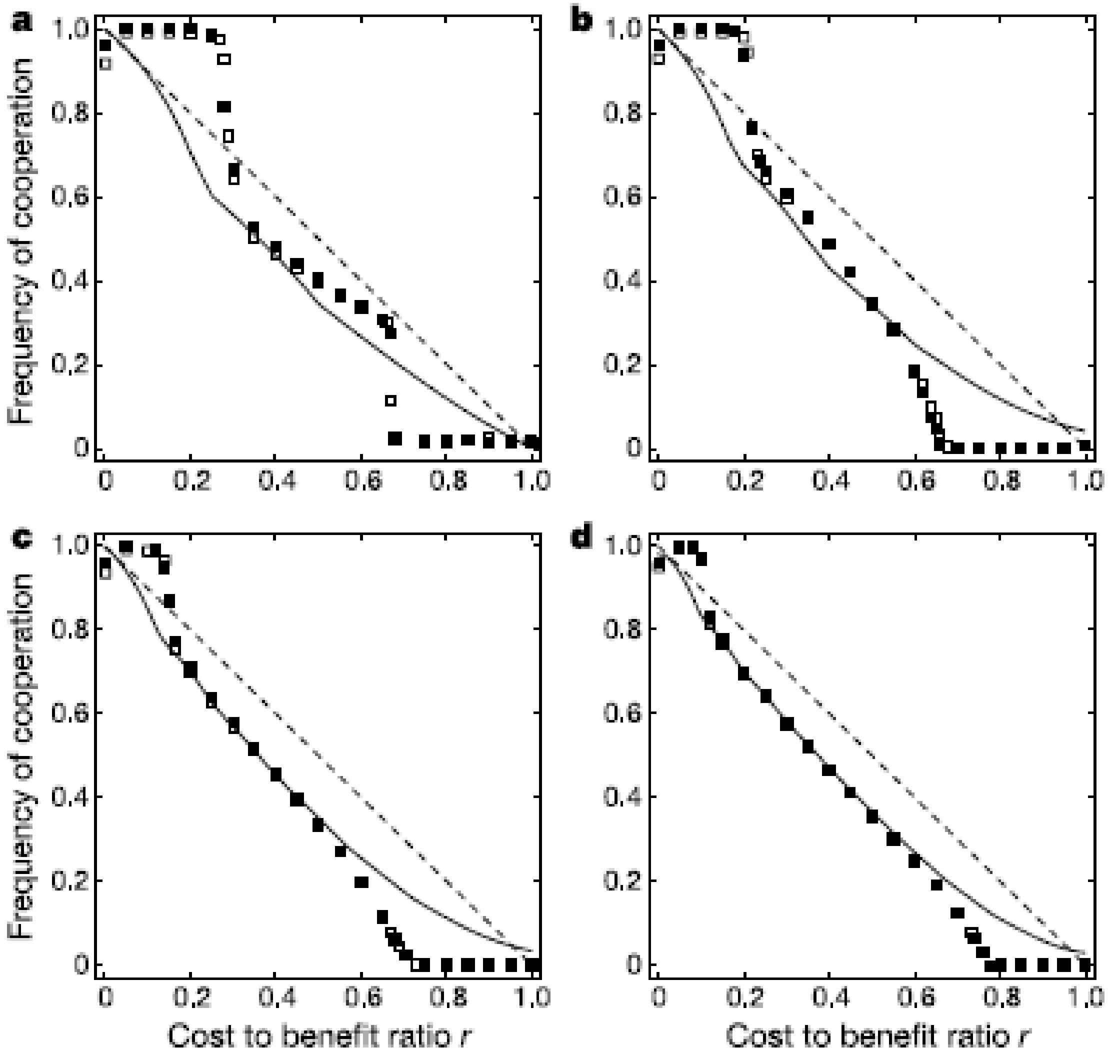
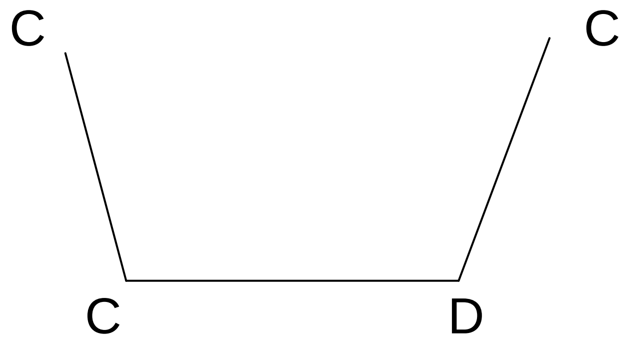


Figure 1 Frequency of cooperators as a function of the cost-to-benefit ratio $r = c/(2b - c)$ in the snowdrift game for different lattice geometries. **a**, Triangular lattice, neighbourhood size $N = 3$; **b**, square lattice, $N = 4$; **c**, hexagonal lattice, $N = 6$; **d**, square lattice, $N = 8$. For small r, spatial structure promotes cooperation; however, for



random matching model

	C	D
C	b-c/2	b-c
D	b	0

$$E(C) = b - c + (k-1)(x(b-c/2) + (1-x)(b-c)),$$

$$E(D) = b + (k-1)xb.$$

From $E(C) = E(D)$ it follows that

$$x = 1 - r\frac{k+1}{k-1}$$

extinction threshold for cooperation

$$r = \frac{k-1}{k+1}$$

If the neighborhood of imitation is idependent of the neighborhood of interaction, then

$$E(C) = k(x(b - c/2) + (1 - x)(b - c)),$$

$$E(D) = kxb.$$

and E(C) = E(D)

gives the replicator dynamic coexistence

$x = 1 - r$

Prisoner's Dilemma on random graphs

joint work with Bartosz Sułkowski
and Jakub Łącki

	C	D
C	3	0
D	5	1

(D,D) is the only Nash equilibrium

Erdos – Renyi random graphs

Each pair of vrtices is joint by an edge with probability ε

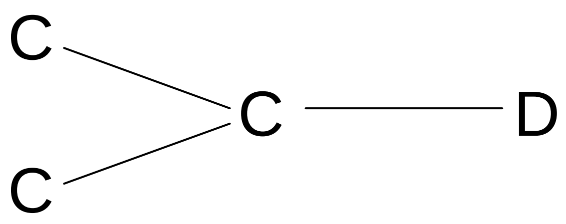
Distribution of vertex degrees is Poissonian

Scale-free graphs of Barabasi-Albert

Preferential   linking

Distribution of vertex degrees $\sim k^{-\lambda}$

imitation dynamic



	C	D
C	3	0
D	5	1

	C	D
C	2	-1
D	4	0

left players earn 3
middle player 6
right player 5

D changes into C

left players earn 2
middle player 3
right player 4

middle C changes into D

spatial game with linking cost   Matsuda 2007

	C	D
C	1	0
D	T	0

	C	D
C	$1-\gamma$	$-\gamma$
D	$T-\gamma$	$-\gamma$

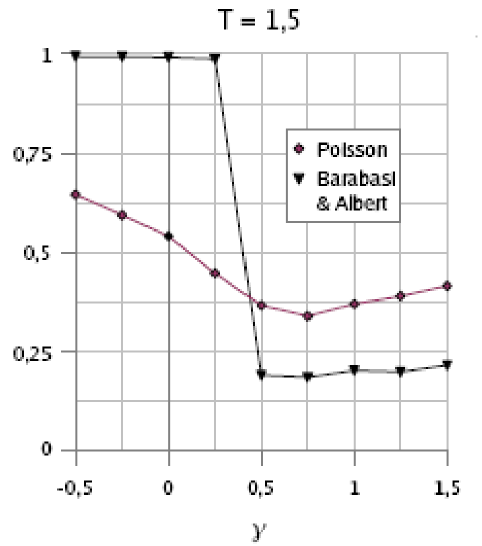
γ - linking cost

imitation dynamic

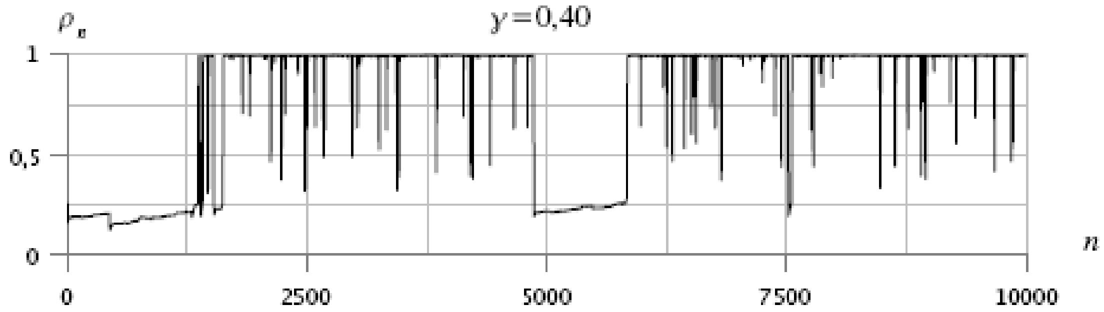
a random player imitates the best strategy in the neighborhood
with theprobability $1 - \varepsilon$

makes a mistake with the probability ε

level of cooperation



time series



phase transition ?

That’s it for today