Integrability and Laplacian growth: another view on the Schwarz potential

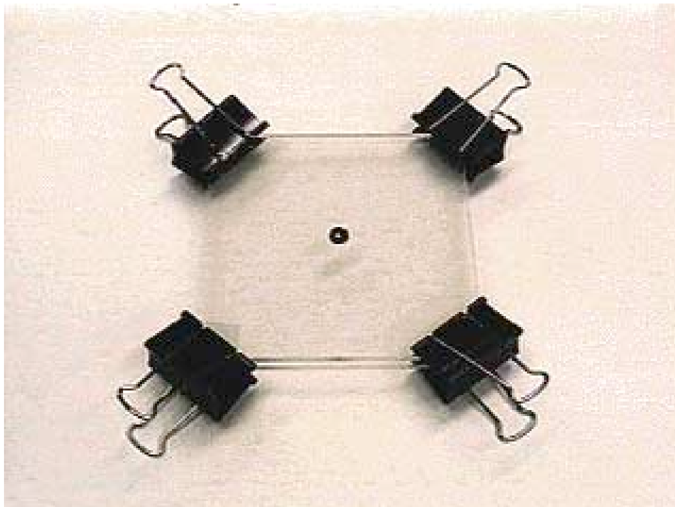
Razvan Teodorescu

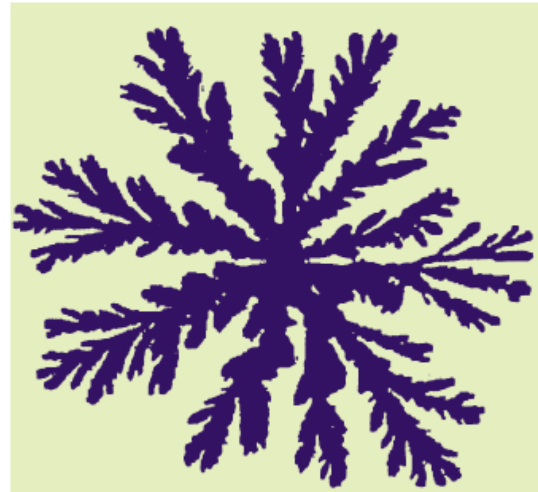
The Laplacian Growth problem

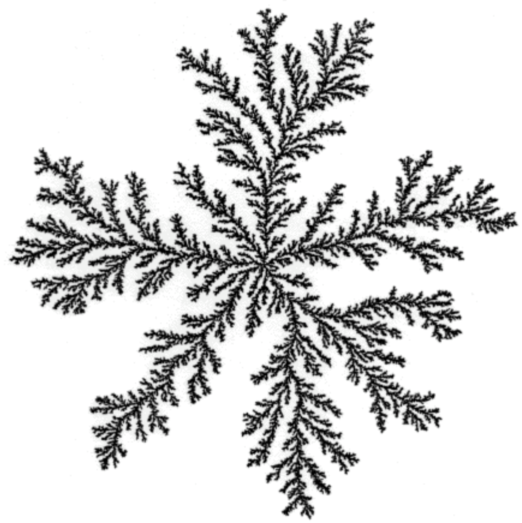
Let $D_+(t)$ be a simply-connected, bounded domain in $\mathbb{C}$, $\partial D_+(t)$ a real algebraic curve and $D_- := \mathbb{C} \setminus D_+(t)$:

$$(\mathbf{LG})\ \text{Laplacian Growth:} \begin{cases} \Delta p = 0 & \text{on } D_-(t) \setminus \{\infty\}, \\ p = 0 & \text{on } D_+(t) \\ V_n = -\partial_n p & \text{on } \partial D_-(t), \\ p \to -\log|z| & z \to \infty \end{cases}$$

Question: Is it possible to find a monotonic chain $\{D_+(t)\}$ such that $D(s) \subset D(t), (\forall) 0 < s < t \in [0, T] \subset \mathbb{R}$, satisfying **(LG)**?







Solutions from conformal mapping

Theorem [Polubarinova-Kochina, Galin, Kufarev cca. 1945] *Let* $z(w,t)$ *be the conformal map* $\mathbb{C}\setminus\mathbb{D} \xrightarrow{z(w,t)} D_-(t)$*, such that* $z'(\infty,t) = r(t) \in \mathbb{R}$*,* $z(\infty,t) = \infty$ *and denote* $w(z,t)$ *its inverse:*

$$z(w,t) = r(t)w + \sum_{k\geq 1} u_k(t)w^{-k}, \quad |w| \geq 1.$$

LG solution: $\boxed{p(z,t) = -\log|w(z,t)|, \quad V_n = |w'(z,t)|.}$

Consequence: Solutions exist as long as $|w'(z_*,t)| \to \infty$ only at points $z_* \in D_+(t)$.

Real fluid dynamics

Navier-Stokes:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v},$$

Small gap limit $b \to 0 \Rightarrow Re = \rho V b/\mu \to 0$, just Stokes:

$$\mu \nabla^2 \vec{v} = \vec{\nabla} p.$$

Poisseuille profile, averaging over the vertical direction:

$$\vec{v} = -\frac{b^2}{12\mu} \vec{\nabla} p = -K \vec{\nabla} p.$$

Richardson's theorem

Theorem [Richardson, 1972] *Harmonic moments of $D_-(t)$ do not change in time.*

$$\text{Moments}: t_k(t) = -\frac{1}{k\pi}\int_{D_-(t)} z^{-k}\mathrm{d}A(z), \quad t_0 = t = \frac{1}{\pi}\int_{D_+(t)} \mathrm{d}A(z).$$

$$\frac{\mathrm{d}t_k}{\mathrm{d}t} = \oint_{\partial D(t)} \frac{V_n}{z^k} d\ell = \oint_{\partial D(t)} (p\partial_n z^{-k} - z^{-k}\partial_n p) d\ell = -\int_{D_-(t)} z^{-k}\Delta p \,\mathrm{d}A(z).$$

Solutions revisited: $\boxed{z(w, t_0, \{t_k\}) = r(t_0, \{t_k\})w + \sum_{k\geq 1} u_k(t_0, \{t_k\})w^{-k}.}$

Note: Interior Richardson theorem by inversion: $\int_{D_+} z^k \mathrm{d}A(z)$ preserved.

Conformal map – harmonic moments relationships: an inverse moment problem

Area formula: $\boxed{t_0 = r^2 - \sum_{k\geq 1} k|u_k|^2}$

Example: the Joukowski map $z(w) = rw + u_0 + \frac{u}{w-a}$

Correspondence:

$$\begin{cases} t_0 &= r^2 - \frac{|u|^2}{(1-|a|^2)^2}, \\ \bar{\alpha} &= t_0 - r^2 + \frac{ur}{a^2}, \\ \beta &= \frac{r}{\bar{a}} + u_0 + \frac{u\bar{a}}{1-|a|^2} \\ \gamma &= \frac{\bar{u}}{\bar{a}} - \bar{u}_0, \end{cases}$$

$$V(z) := \sum_{k\geq 1} t_k z^k = \gamma z + \alpha \log\left(1 - \frac{z}{\beta}\right).$$

Existence of infinite-time solutions

Question: For which sets of values $\{t_k\}_{k=1}^{\infty}$ is it possible to find a solution valid for arbitrary $t \to \infty$?

Example: $t_3 \neq 0$, all others vanish:

$$z(w) = rw + 3t_3 r^2 w^{-2}, \quad t_0 = r^2 - 18|t_3|^2 r^4, \quad t_0 \leq t_c = \frac{1}{2}.$$

$$\frac{\mathrm{d}t_0}{\mathrm{d}r} = 0, \quad \text{at } t_0 = t_c, \quad \frac{\mathrm{d}z}{\mathrm{d}w} = 0, \quad \text{at } w = 1.$$

Known cases: circle, ellipse.

Schwarz function

- Schwarz function $S(z) = \bar{z}$ on boundary $\Gamma = \partial D$, with Laurent expansion around Γ:

$$S(z) = \sum_{k>0} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{p>0} \frac{v_p}{z^p}$$

$$\partial_{t_0} S(z, t_0) = -\partial_z p(z, t_0)$$

- meromorphic - *quadrature domains: Sakai, Gustafsson, Putinar*

$$\int_{D_+} f(z) \mathrm{d}A(z) = \sum_{k=1}^{n} \sum_{p=1}^{n_k} a_{kp} f^{(p)}(z_k), \quad (\forall) f \in L^1(D_+), \text{analytic.}$$

Inverse moment problem as determination of equilibrium measure (the Maxwell problem)

Find the support D of distribution $\rho(z)$ solving $\int_D \rho(z)\mathrm{d}A(z) = t_0$, and

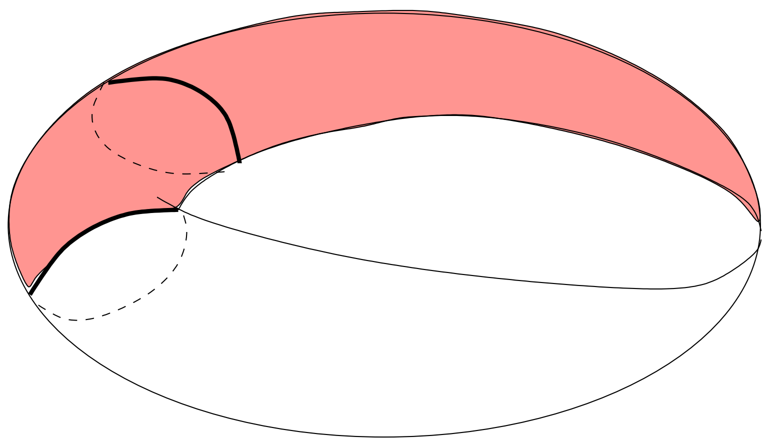
$$\frac{\delta}{\delta\rho(z)} \int_D \rho(z) \left[-|z|^2 + V(z) + \overline{V(z)} + \int_D \rho(\zeta) \log|z-\zeta|^2 \mathrm{d}A(\zeta) \right] \mathrm{d}A(z) = 0$$

- Smooth solution: characteristic function of D, $\boxed{\rho(z) = \chi_D(z)}$
- Equivalent exterior potential created by distribution of singularities of the Schwarz function (poles, cuts) $\rho_s(z)$

$$\int f(z)\rho(z)\mathrm{d}^2 z = \int f(z)\rho_s(z)\mathrm{d}^2 z, \quad f(z)\ L_1 - \text{integrable}$$

Schottky doubles: Laplacian Growth on Riemann surfaces

- Riemann surface: $F\left(\left(\frac{z+S(z)}{2}\right), \left(\frac{z-S(z)}{2i}\right)\right) = 0$, $\Gamma = \{F(x,y) = 0\}$
- Boundary Γ : $S(z) = \bar{z}$
- Singularities: branch points $S'(z) \to \infty$, double points $S_1(z) = S_2(z)$



Examples

<u>Ellipse</u> $z(w) = rw + \bar{t}_2 r w^{-1}$

$$zS - \frac{2(t_2 z^2 + \bar{t}_2 S^2)}{1+4|t_2|^2} - t_0 \frac{1-4|t_2|^2}{1+4|t_2|^2} = 0.$$

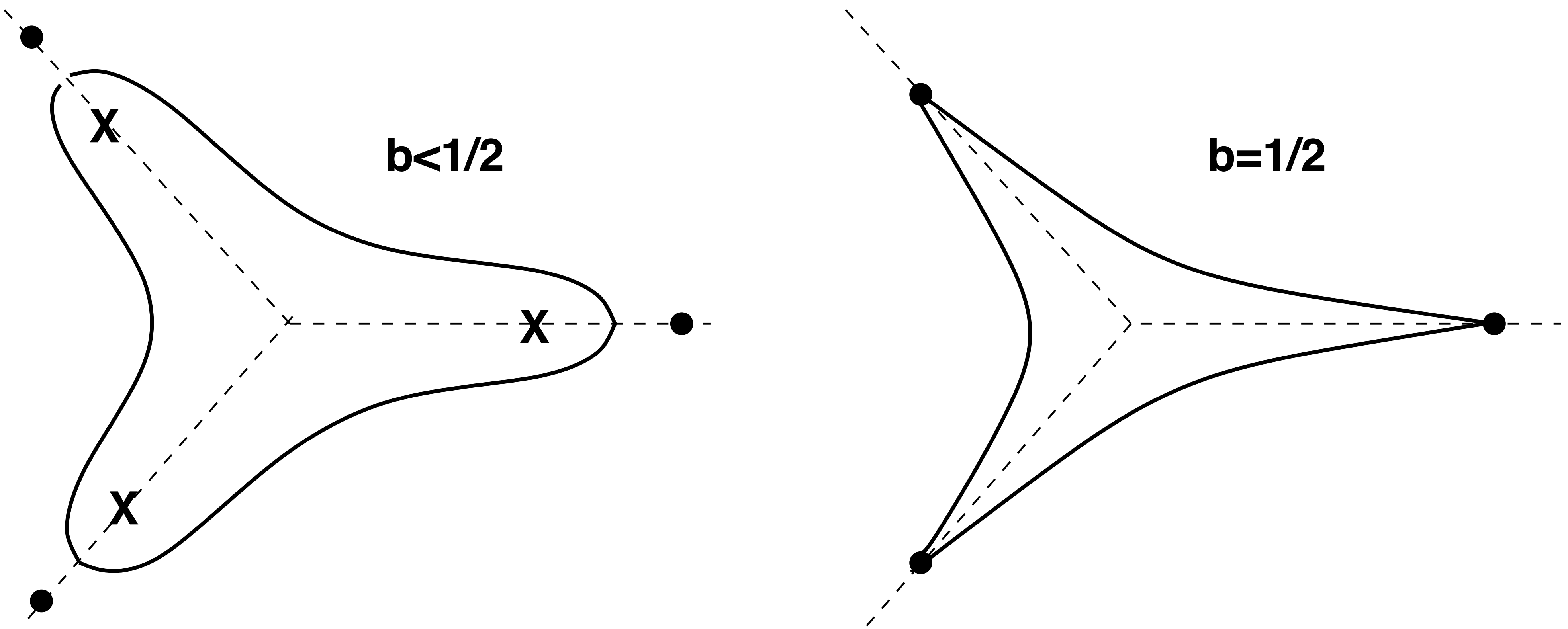
<u>Hypocycloid</u> $z(w) = rw + 3\bar{t}_3 r^2 w^{-2}$

$$(zS)^2 - \frac{S^3}{3t_3} - \frac{z^3}{3\bar{t}_3} + \frac{(1-9|t_3|^2 r^2)(1+18|t_3|^2 r^2)}{9|t_3|^2} zS - \frac{r^2(1-9|t_3|^2 r^2)^3}{9|t_3|^2} = 0.$$

<u>Joukowski</u> $z(w) = rw + u_0 + \frac{u}{w-a}$

$$z^2 S^2 - z^2 S\bar{\beta} - zS^2\beta + \left(|\bar{\beta}|^2 + \alpha + \bar{\alpha} - t_0\right) zS + z\bar{\beta}(t_0 - \alpha) + S\beta(t_0 - \bar{\alpha}) + h = 0.$$

Cusps as higher critical points



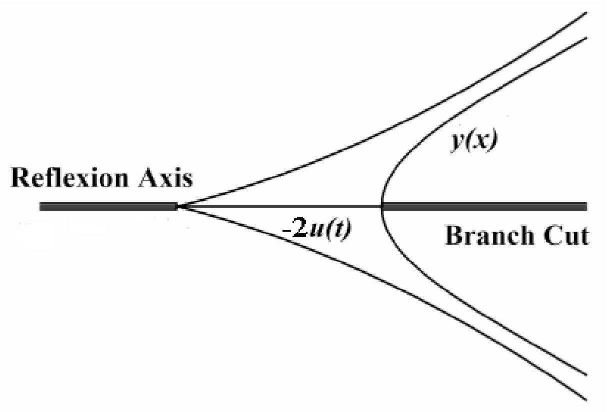
Coalescence of $2k+1$ branch points: $x^{2k+1} \sim y^2$ – cusp

The generic cusp singularity

Generic boundary singularity: branch point (inside) meets double point (outside). Consider the "reduced" Riemann surface.

- Local boundary: elliptic curve

$$y^2 = -4(\zeta - u)^2(\zeta + 2u), \quad u \to 0$$



Poisson structure of Laplacian growth

Poisson brackets:

$$\{f, g\}_{(t_0, \log w)} = w \left(\frac{\partial f}{\partial w} \frac{\partial g}{\partial t_0} - \frac{\partial g}{\partial w} \frac{\partial f}{\partial t_0} \right).$$

Hamiltonian:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{\log w, f\}$$

Polubarinova-Kochina's theorem:

$$\boxed{\{z(w,t), z^{\sharp}(w,t)\} = 1}$$

where $z^{\sharp}(w,t) = \bar{z}(w^{-1}, t)$.

Integrability from the Schwarz potential (Khavinson-Shapiro/Krichever-Mineev-Wiegmann-Zabrodin)

$$\mathrm{d}W = S\mathrm{d}z + p\mathrm{d}t_0, \quad \mathrm{d}^2W = 0, \quad \{t_k\} \text{ fixed.}$$

$$W(z, t_0, \{t_k\}) = t_0 \log |z|^2 + \sum_{k\geq 1} t_k z^k(w)_+ + \ldots,$$

$$z^k(w)_+ = \boxed{\text{regular part of Laurent expansion in } w}$$

$$\frac{\partial z}{\partial t_k} = \{z_+^k, z\}, \quad \frac{\partial z_+^p}{\partial t_k} - \frac{\partial z_+^k}{\partial t_p} = \{z^k(w)_+, z^p(w)_+\}$$

Theorem *As generating function for deformations in $\{t_k\}$, the total differential of the Schwarz potential is the Hirota 1-form of the K-P hierarchy, with the canonical Poisson bracket of Laplacian growth.*

Review of integrable hierarchies

Let $\mathcal{A}$ be the algebra of differential polynomials of the type $P = \partial^n + u_{n-2}\partial^{n-2} + \ldots + u_1\partial + u_0$, where $\partial =$ differential symbol, and work in the ring of pseudo-differential operators:

$$L = \sum_{-\infty}^{n} c_k \partial^k, \quad \partial^{-1}(\,.\,) \equiv \int (\,.\,) dx,$$

$$L_+ \equiv \sum_{0}^{n} c_k \partial^k, \quad L = L_+ + L_-.$$

Then the Kadomtsev-Petriashvilii hierarchy is given by

$$\mathcal{L} = \partial + u_0\partial^{-1} + u_1\partial^{-2} + \ldots$$

Review of integrable hierarchies

$$\frac{\partial \mathcal{L}}{\partial t_k} = [\mathcal{L}_+^k, \mathcal{L}], \quad k = 1, 2, \ldots$$

$$\mathrm{Zakharov - Shabat:} \quad [\partial_{t_k} - \mathcal{L}_+^k, \partial_{t_p} - \mathcal{L}_+^p] = 0, \quad (\forall)\, t_k, t_p.$$

Reductions: assume

$$(\mathcal{L}^2)_- = 0$$

Then

$$\mathcal{L} = L^{1/2}, \quad L = \partial^2 + 2u_0.$$

Korteweg-de Vries equation:

$$(\mathcal{L})_+^3 \equiv P = \partial^3 + \frac{3}{2}\left[u_0\partial + \partial u_0\right], \quad \frac{\partial L}{\partial t_3} = [P, L] \Rightarrow u_{t_3} = 6uu_x + u_{xxx}.$$

Boundary asymptotics from the Baker-Akhiezer function (Krichever/Novikov)

$$\mathcal{L}\psi = z\psi, \quad \frac{\partial \psi}{\partial t_k} = \mathcal{L}_+^k \psi, \quad \forall k \geq 1.$$

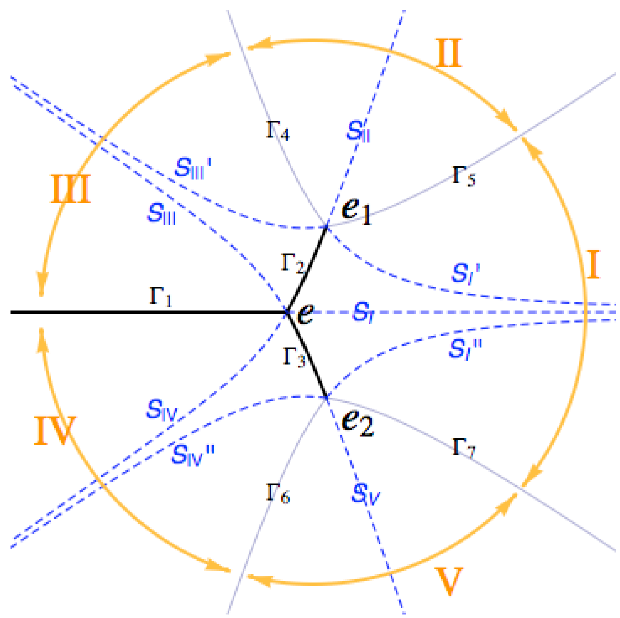
$$g(z, t_1, \dots) = \exp\left[\sum_1^\infty t_k z^k\right]$$

$$\psi(z, t_1, \dots) = g \cdot \frac{\tau(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, \dots)}{\tau(t_1, t_2, \dots)} = g \cdot \frac{\exp\left[\sum_1^\infty -\frac{1}{kz^k}\frac{\partial}{\partial t_k}\right]\tau(t_1, t_2, \dots)}{\tau(t_1, t_2, \dots)}$$

Krichever's solutions for the K-P hierarchy (fixed finite-genus torus):

$$\psi = g \cdot \frac{\theta(A(z) + \sum_q t_q \pi_q) - A(D) - K)\theta(A(\infty) - A(D) - K)}{\theta(A(z) - A(D) - K)\theta(A(\infty) + \sum_q t_q \pi_q - A(D) - K)}$$

Dimensional reduction of integrable hierarchies: cusp asymptotics



$$\psi^{LGWKB}(\lambda, t) = \frac{\sigma(\tau+\lambda)}{\sigma(\lambda)\sigma(\tau)} e^{-\zeta(\tau)\lambda} \cdot \frac{e^{\sum_{q=1}^{q=5} t_q \omega_q}}{\sqrt{\wp'(\lambda)}}$$

The Airy-Stokes-Liouville-Green-Wentzell-Kramers-Brillouin method

Hamilton-Jacobi equation for a particle in 1D potential:

$$\frac{\partial S}{\partial t} + V(q) + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 = 0,$$

Schrödinger equation

$$\left[\frac{(-i\hbar)^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right]\Psi_\hbar(q,t) = i\hbar\frac{\partial}{\partial t}\Psi_\hbar(q,t),$$

for the *semiclassical wave-function*

$$\Psi_\hbar(q,t) = e^{\frac{i}{\hbar}S(q,t)},$$

which is missing only the terms of order $\hbar$ when compared to the Hamilton-Jacobi equation,

$$\left[\frac{(-i\hbar)^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - i\hbar\frac{\partial}{\partial t}\right]\Psi_\hbar(q,t) =$$

$$= \left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) + \frac{\partial S}{\partial t}\right]\Psi_\hbar(q,t) -$$

$$-\frac{i\hbar}{2m}\frac{\partial^2 S}{\partial q^2}\Psi_\hbar(q,t).$$

Therefore, as long as $\frac{\hbar}{2m}\frac{\partial^2 S}{\partial q^2} \ll 1$, we can safely ignore that term and use the classical Hamilton-Jacobi equation to solve mechanics problems. However, if $\frac{\hbar}{2m}\frac{\partial^2 S}{\partial q^2} = O(1)$, the classical approximation breaks down and the Schrödinger equation must be used: <u>precisely what happens at a cusp</u>.

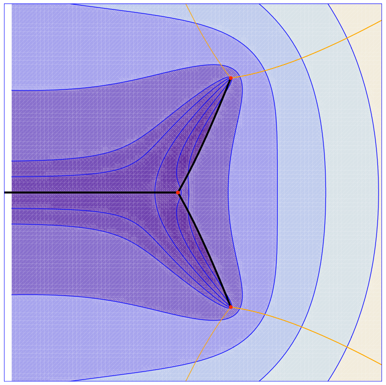
Hydrodynamics of the (2, 3)-cusp from wave-function

$$Y^2 = -4(X-u)^2(X+2u), \quad u \to 0$$

Parametrize: $X(k) = \wp(k|g_{2,3})$, $Y(k) = \wp'(k|g_{2,3})$, $g_2(t) = u^2$

For physical pressure:

$$\log \Psi = \int (pdt + Sdz) \Rightarrow p(k,t) = i\zeta(k) - i\frac{3g_3}{2t}k.$$



Elliptic case: details

Consider the linear problem

$$\left[\frac{\partial^2}{\partial t^2} - u\right]\psi(t,\lambda) = X(\lambda)\psi(t,\lambda),$$

$$\left[\partial_t^3 - \frac{3}{4}\{\partial_t, u\}, \partial_t^2 - u\right] = \epsilon.$$

Perturbatively in ϵ, the solution is

$$u(t) = 2\wp(t|g_{2,3}(t)),$$

where

$$X(\lambda|g_{2,3}) = \wp(\lambda|\omega), \quad Y(\lambda) = \partial_\lambda X(\lambda),$$

or equivalently represented in “Mumford-style”

$$\dot{L} = \begin{pmatrix} \frac{\ddot{u}}{6} & -\frac{\dot{u}}{3} \\ \frac{\dddot{u}}{6} + \frac{2\zeta\dot{u} - 4u\dot{u}}{9} & -\frac{\ddot{u}}{6} \end{pmatrix}, \; A' = \begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 0 \end{pmatrix},$$

and

$$[A,\ L] = \begin{pmatrix} \frac{\ddot{u}}{6} & -\frac{\dot{u}}{3} \\ \frac{2(\zeta+u)\dot{u}}{9} & -\frac{\ddot{u}}{6} \end{pmatrix}.$$

Thus,

$$0 = \dot{L} - A' - [A,\ L] = \begin{pmatrix} 0 & 0 \\ \frac{\dddot{u}}{6} - \frac{6u\dot{u}}{9} - \frac{2}{3} & 0 \end{pmatrix}.$$

The only non-trivial element of the matrix gives Painlevé I equation.

$$\dddot{u} - 4u\dot{u} - 4 = 0,$$

Regularized solution: modulated elliptic-functions solution of Painlevé I.

Hyperelliptic case: Mumford's recipe

Take a pair of 2×2 operators L, A, solving the linear problem

$$L\Psi = \mu\Psi,\ \partial_t\Psi = A\Psi,\ \partial_t L = [A,\, L]$$

Assume

$$L(\lambda) = \left[\begin{array}{cc} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda) \end{array} \right] \Rightarrow \mu^2 + \det L = 0,\ \mu(\lambda) = \pm i\sqrt{\det L(\lambda)}.$$

Isomonodromic deformation:

$$L\Psi = \mu\Psi + \epsilon\partial_\lambda\Psi, \quad [\epsilon\partial_\lambda - L,\, \partial_t - A] = 0, \quad 0 \le \epsilon \ll 1.$$

$$[\epsilon\partial_\lambda - L,\, \partial_t - A] = 0 \Rightarrow \epsilon(\partial_\nu L - \partial_\lambda A) + \partial_\tau L - [A, L] = 0,$$

where $\partial_t \longrightarrow \partial_\tau + \epsilon\partial_\nu$ (fast/slow variables). New solution: $u_\epsilon(\tau, k_i(\nu))$.

Hyperelliptic case: Abel-Jacobi inverse problems

$$\mu_i = \lambda^{g-i}\frac{d\lambda}{y}, \;\; M_{ij} = \int_{\alpha_j} \mu_i, \;\; \boldsymbol{\omega} = M^{-1}\boldsymbol{\mu}.$$

Period matrix $B_{ij} = \int_{\beta_j} \omega_i$ is symmetric and has positively defined imaginary part. The Riemann θ function:

$$\theta(\boldsymbol{z}|B) = \sum_{\boldsymbol{n}\in\boldsymbol{z}^g} e^{2\pi i(\boldsymbol{n}^t\boldsymbol{z}+\frac{1}{2}\boldsymbol{n}^t B\boldsymbol{n})}.$$

The g vectors $\boldsymbol{B}_k$ and vectors $\boldsymbol{e}_k$ define a lattice in $\mathbb{C}^g$. The $Jacobian$ variety of the curve Γ, is the quotient $J(\Gamma) = \mathbb{C}^g/(\mathbb{Z}^g + B\mathbb{Z}^g)$.

The Abel-Jacobi map associates to any point P on Γ, a point ($g-$ dimensional complex vector) on the Jacobian variety, through $\boldsymbol{A}(P) = \int_\infty^P \boldsymbol{\omega}$.