Two problems in Hele-Shaw free boundary flows

Robb McDonald
Department of Mathematics
University College London

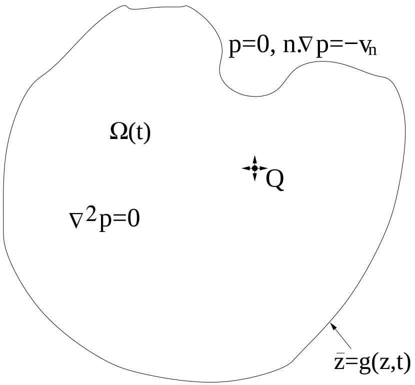
I: Computation of Hele-Shaw flows near obstacles

II: Generalized Hele-Shaw flows: a Schwarz function approach

Part I: Computation of Hele-Shaw flows near obstacles

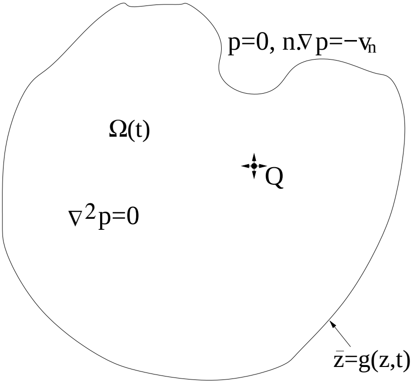
(see McDonald, *Theor. Comp. Fluid Dyn.* 2010)

Hele-Shaw free boundary problem



Part I: Computation of Hele-Shaw flows near obstacles
(see McDonald, *Theor. Comp. Fluid Dyn.* 2010)

Hele-Shaw free boundary problem

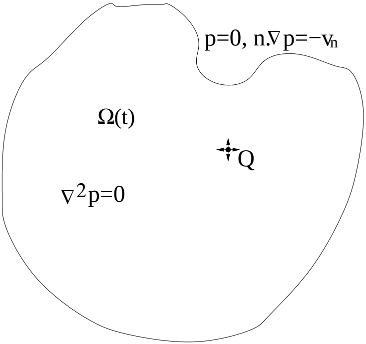


The Baiocchi transform.

Define the real-valued function on Ω (e.g. Cummings *et al.* 1999):

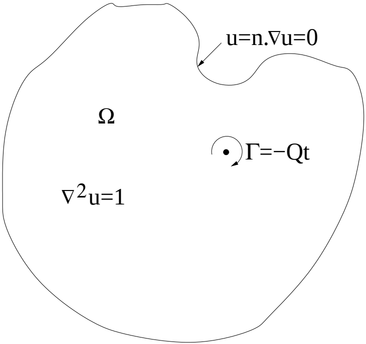
$$u(z,\bar{z},t) = \frac{1}{4}\left(z\bar{z} - h(z,t) - \overline{h(z,t)}\right)$$

where $g(z,t) = h'(z,t)$.
$\implies \nabla^2 u = 1$ in Ω and $u_z = u_{\bar{z}} = 0$ on $\partial\Omega$.
Also, $\partial u/\partial t = p$.



Hele-Shaw problem

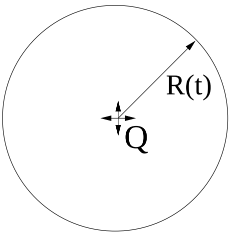
$\longrightarrow$



Baiocchi ("*vortical*") problem

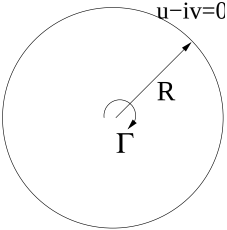
Example: circular blob with source at the origin

Hele-Shaw problem:



$$p = -\frac{Q}{2\pi}\log|z| \implies$$
$$R(t) = \sqrt{Qt/\pi}$$

Vortical problem:



(Note: zero net circulation)

$$U - iV = \begin{cases} -\frac{i}{2}(\bar{z} - R^2/z) & \text{if} \quad z \in \Omega \\ 0 & \text{if} \quad z \notin \Omega, \end{cases}$$

$$\implies \Gamma = -\pi R^2 = -Qt.$$

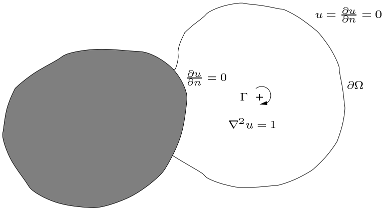
Hele-Shaw free boundary flows near obstacles

Previous work:

(i) Exact solutions (infinite walls, wedges, corners, etc.): e.g. Richardson (*JFM* 1981, *EJAM* 2001), Cummings (*EJAM* 1999), Gustafsson & Vasil'ev (2006).

(ii) Numerical solutions e.g. Bogoyavlenskiy & Cotts (*Phys. Rev. E* 2004)–random walk method.

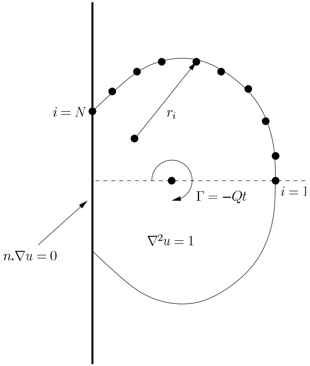
Hele-Shaw flows near obstacles: Baiocchi formulation



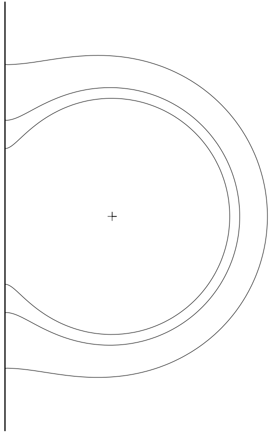
Task: seek steady vortex patch enclosing a point vortex (with zero net circulation) such that *tangential* 'velocity' vanishes on the obstacle boundary $\Longrightarrow$ contour dynamics.

- Contour dynamics with boundaries e.g. Johnson & McDonald *Proc. Roy. Soc.* (2004), *JFM* (2005); Crowdy & Surana *JFM* (2007).
- Contour dynamics: computation of steady solutions e.g. Deem & Zabusky *PRL* (1978), McDonald *Phys. Fluids* 2005.

Infinite straight wall



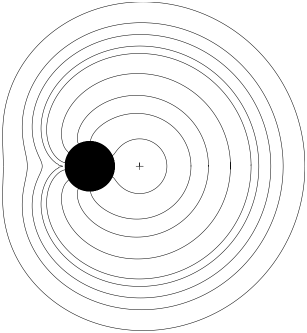
unknowns: r_i, $i = 1, N$
equations $|u_i - iv_i| = 0$, $i = 1, N$
Solve by Newton's method.



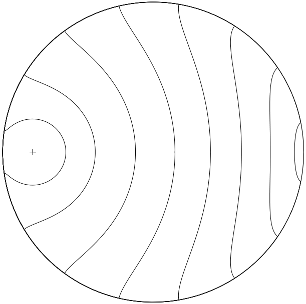
cf. Richardson's explicit solution, *JFM* 1981.

Circular boundary

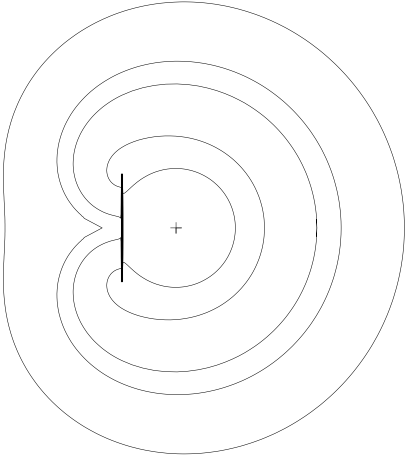
exterior source



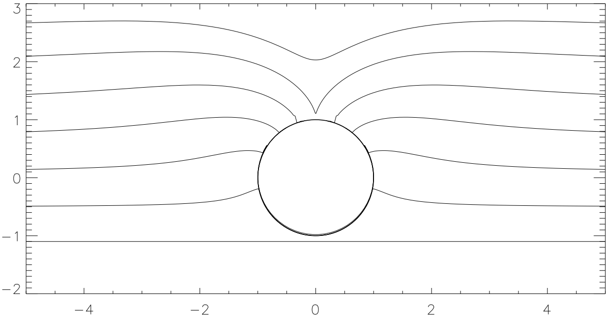
interior source



Finite plate



Circular disk encountering an infinite free boundary



II: Generalized Hele-Shaw flows: a Schwarz function approach

(see McDonald, submitted to *Eur. J. Appl. Math.*)

Hele-Shaw flows subject to an external potential $\Psi(x, y)$ (generalized Hele-Shaw flows) satisfy the free boundary problem (see Entov & Etingof, *Eur. J. Appl. Math.* 2007)

$$\nabla^2\phi = \sum_{j=1}^{N} Q_j \delta(x - x_j, y - y_j), \quad (x_j, y_j) \in \Omega,$$

$$\phi = \Psi(x, y), \quad (x, y) \in \partial\Omega,$$

$$v_n = \frac{\partial \phi}{\partial n}, \quad (x, y) \in \partial\Omega,$$

Q_j are the hydrodynamic source strengths and v_n is the normal velocity of the boundary.

Previous studies

(i) centrifugal potential ($\Psi = \omega r^2/2$)

- Entov, Etingof & Kleinbock 1995 (moment based method)
- Magdaleno, Rocco & Casademunt 2000 (modified Polubarinova-Galin eq.)
- Crowdy 2002 (Cauchy transform)

(ii) centrifugal potential, uniform gravity, ‘point charges’, etc.

- Entov & Etingof 2007 (moments, conformal mapping; steady solutions)

Derivation of the Schwarz function equation

Let $\bar{z} = g(z,t)$ on $\partial\Omega$

$$\bar{v} = v\frac{\partial g}{\partial z} + \frac{\partial g}{\partial t} \quad \text{on} \quad \partial\Omega. \quad (v = U + iV)$$

Since $\phi = \Psi(z,\bar{z}) = \Psi(z, g(z,t))$ on $\partial\Omega$, then tangent to $\partial\Omega$

$$\Re\left[\bar{v}\frac{\partial z}{\partial s}\right] = \frac{1}{2}\left[\bar{v}\frac{\partial z}{\partial s} + v\frac{\partial \bar{z}}{\partial s}\right] = \frac{\partial \Psi}{\partial s}. \qquad (1)$$

Using (Davis 1974)

$$\frac{\partial z}{\partial s} = \left(\sqrt{\frac{\partial g}{\partial z}}\right)^{-1}, \quad \frac{\partial \bar{z}}{\partial s} = \sqrt{\frac{\partial g}{\partial z}},$$

(1) becomes

$$\bar{v} + v\frac{\partial g}{\partial z} = 2\sqrt{\frac{\partial g}{\partial z}}\frac{\partial \Psi}{\partial s}. \qquad (2)$$

Derivation of the Schwarz function equation (cont.)

On $\partial\Omega$ $\bar{z} = g(z,t)$ and hence (2) gives

$$\begin{aligned}
\bar{v} + v\frac{\partial g}{\partial z} =& 2\sqrt{\frac{\partial g}{\partial z}}\left[\frac{\partial \Psi}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial \Psi}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial s}\right], \\
=& 2\frac{\partial \Psi}{\partial z} + 2\frac{\partial g}{\partial z}\frac{\partial \Psi}{\partial \bar{z}} \\
=& 2\frac{\partial}{\partial z}\Psi(z, g(z,t)). \qquad (3)
\end{aligned}$$

Adding (1) and (3)

$$2\bar{v} = \frac{\partial g}{\partial t} + 2\frac{\partial \Psi}{\partial z}. \qquad (4)$$

Finally, using $\bar{v} = \partial w/\partial z$ ($w \equiv$ complex potential) in (4) gives on $\partial\Omega$

$$\boxed{\frac{\partial w}{\partial z} = \frac{1}{2}\frac{\partial g}{\partial t} + \frac{\partial \Psi}{\partial z}} \qquad (5)$$

Example 1: Evolution of a blob in a centrifugal potential

(see Crowdy, *SIAM J. Appl. Math.* 2002)

Here $\Psi = \omega|z|^2/2 = \omega z g/2$ ($\omega \equiv const.$) and the governing equation becomes

$$2\frac{\partial w}{\partial z} = \frac{\partial g}{\partial t} + \omega\frac{\partial}{\partial z}(zg). \tag{6}$$

Consider the conformal map from the unit ζ-disk to $\Omega(t)$

$$z = \frac{R\zeta}{\zeta^2 - a^2}, \tag{7}$$

where $R(t)$ and $a(t)$, $|a(t)| > 1$, are real functions to be found. Note

$$\begin{aligned} g(z,t) &= -\frac{R\zeta/a^2}{\zeta^2 - a^{-2}} \\ &= -\frac{R}{2a^2}\left(\frac{1}{\zeta - a^{-1}} + \frac{1}{\zeta + a^{-1}}\right), \end{aligned} \tag{8}$$

has simple poles at $\zeta = \pm a^{-1}$. Let $z(a^{-1}) = z_0(t) = Ra/(1 - a^4)$.

Example 1: Blob in a centrifugal potential (cont.)

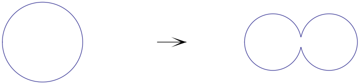
$$\text{As} \quad \zeta \to a^{-1}: \quad \frac{1}{\zeta - a^{-1}} = \frac{z_\zeta(a^{-1})}{z - z_0} + \frac{z_{\zeta\zeta}(a^{-1})}{2z_\zeta(a^{-1})} + \mathrm{O}(z - z_0),$$

and finding the Laurent expansion of (6) about $z = z_0$ ($\partial_z w$ is regular since there are no hydrodynamic singularities) gives

$$\frac{R^2(1+a^4)}{(1-a^4)^2} = const,$$

$$\dot{z}_0(t) = \omega z_0(t), \tag{9}$$

(c.f. Crowdy 2002).



(generalisation: $z = R\zeta/(\zeta^N - a^N)$ etc.)

Example 2: Taylor-Saffman bubble

What is the shape and speed of an air bubble in an infinite Hele-Shaw cell?

- steady bubble with speed U in positive $\Re z$ direction
- fluid speed at infinity is unity
- in bubble frame: $w \to (1-U)z$ as $z \to \infty$
- $\Psi = -Ux = -U(z+g)/2$ (c.f. uniform 'gravitational' field)

Schwarz function equation becomes:

$$2(1-U) = -U\left(1 + \frac{\partial g}{\partial z}\right). \tag{10}$$

Note $g(z) \to (U-2)z/U$ as $z \to \infty \implies \partial\Omega$ is an ellipse (Millar 1990).

Example 2: Taylor-Saffman bubble (cont.)

Map from the unit ζ-disk to outside of elliptical bubble

$$z = \frac{a}{\zeta} + b\zeta, \tag{11}$$

where $a > b > 0$ are real constants; ellipse aspect ratio is $(a - b)/(a + b)$. The Schwarz function of the ellipse is

$$g(z,t) = \frac{b}{a}z + \frac{a^2 - b^2}{a}\zeta. \tag{12}$$

Equating (12) with the behaviour $g(z) \to (U - 2)z/U$ as $z \to \infty$ gives $b/a = (U - 2)/U$ c.f. Taylor & Saffman (1959).

Example 3: Hydrodynamic dipole with two electric 'charges'

- steady flow
- hydrodynamic dipole of strength μ at $z = 0$
- point charges with strengths E at $z = a$ ($a \in \Re$, $a > 0$) and $-E$ at $z = -a$

$$\Psi = \frac{E}{4\pi} \log \frac{(z-a)(g-a)}{(z+a)(g+a)}. \tag{13}$$

As $z \to 0$, $2\partial_z w = \dot{g} + 2\partial_z \Psi$ becomes

$$\frac{E}{4\pi} \log \left[\frac{g-a}{g+a} \right] = \frac{\mu}{2\pi z} + const. \tag{14}$$

Let $z(\zeta)$ be the map from the unit ζ-disk to Ω s.t. as $z \to 0$, $z = z_\zeta(0)\zeta + \mathrm{O}(\zeta^2)$

Example 3: Dipole with two electric 'charges' (cont.)

Hence

$$\frac{g-a}{g+a} = k \exp\left(\frac{2\mu}{E z_\zeta(0)\zeta}\right), \tag{15}$$

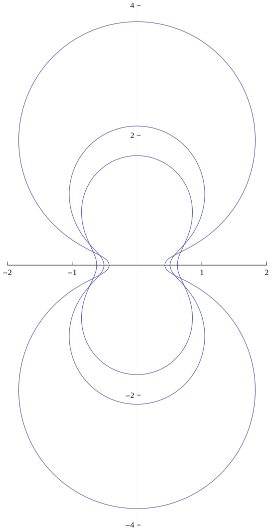
where k is a constant. Taking the complex conjugate of (15) and using $\bar{z} = g$ and $\bar{\zeta} = \zeta^{-1}$ on $\partial\Omega$ gives

$$z = -a \tanh\left(\sqrt{\frac{-\mu}{aE}}\zeta\right). \tag{16}$$

Note $\operatorname{sgn}(\mu E) < 0$.

Example 3: Dipole with two electric 'charges' (cont.)

Free boundary shapes given by $z = -a \tanh\left(\sqrt{-\mu/aE}\zeta\right)$ for a hydrodynamic dipole of strength μ at $z = 0$ and electric point sources of strength $\pm E$ at $z = a$, for $\mu/E = -1$ and $a = 0.48$ (largest), 0.59 and 0.76 (smallest).



Example 4: Elliptical bubble in strain and centrifugal potential

- $w \to -Mz^2/2\pi$ as $z \to \infty$
- $\Psi = \omega z g/2$
- Recall $z = a/\zeta + b\zeta \implies g = bz/a + (a^2 - b^2)\zeta/a$, where $a(t) > b(t)$.

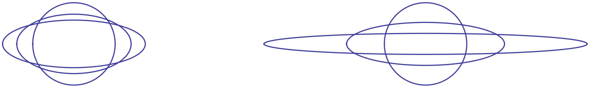
Thus as $z \to \infty$, $\partial_z w = \dot{g}/2 + \partial_z \Psi$ becomes

$$\frac{d}{dt}\left(\frac{b}{a}\right) + 2\omega \frac{b}{a} = -\frac{2M}{\pi}, \tag{17}$$

which has solution

$$\frac{b}{a} = -\frac{M}{\pi\omega} + \left(\frac{M}{\pi\omega} + \frac{b(0)}{a(0)}\right)\exp(-2\omega t). \tag{18}$$

Example 4: Elliptical bubble in strain and centrifugal potential (cont.)



Collapse of an elliptical bubble in a centrifugal potential field with $\omega = 1$ and strain field of strength M. The bubble is initially circular with unit radius. On the left $M = -\pi/2$ and the times shown are $t = 0$, 0.5 and ∞. On the right $M = -3\pi/2$ and the bubble is shown for times $t = 0$, 0.24 and 0.44. In this case the bubble collapses in finite time $t \approx 0.55$.

Moments and the Schwarz function equation

For finite Ω the moments M_k, $k = 0, 1, 2, \cdots$, are

$$M_k = \iint_{\Omega} z^k dA = \frac{1}{2i} \oint_{\partial\Omega} z^k g(z,t) dz. \tag{19}$$

Differentiating (19) w.r.t. time and using $\partial_z w = \dot{g}/2 + \partial_z \Psi$

$$\begin{aligned} \frac{dM_k}{dt} &= \frac{1}{2i} \oint_{\partial\Omega} z^k \frac{\partial g}{\partial t} dz, \\ &= \frac{1}{i} \oint_{\partial\Omega} z^k \left(\frac{\partial w}{\partial z} - \frac{\partial \Psi}{\partial z} \right) dz, \\ &= \sum_{j=1}^{N} Q_j z_j^k + \frac{k}{i} \oint_{\partial\Omega} z^{k-1} \Psi dz. \end{aligned} \tag{20}$$

Moments and the Schwarz function equation (cont.)

For the centrifugal potential, $\Psi = \omega z g/2$ on $\partial\Omega$ and (20) becomes

$$\frac{dM_k}{dt} = \sum_{j=1}^{N} Q_j z_j^k + \frac{\omega k}{2i} \oint_{\partial\Omega} z^k g dz,$$
$$= \sum_{j=1}^{N} Q_j z_j^k + \omega k M_k. \qquad (21)$$

For a uniform gravitational field, $\Psi = U(z+g)/2$ on $\partial\Omega$, and (20) becomes

$$\frac{dM_k}{dt} = \sum_{j=1}^{N} Q_j z_j^k + UkM_{k-1}, \qquad (22)$$

(see Entov, Etingof & Kleinbock, *EJAM*, 1995).

Remarks

Part I: Computation of Hele-Shaw flows near obstacles

- Compare Baiocchi ("*vortical*") formulation with direct boundary integral method.
- Two fluids?
- Non-Laplacian growth near obstacles e.g. ocean flows $\Longrightarrow$ Helmholtz equation.

Part II: Generalized Hele-Shaw flows

- Further exact solutions and other background fields e.g. inverse square law.
- Applications e.g. tumour growth, nano/micro fluidics.
- Two fluids?