Laplacian growth, elliptic growth, and singularities of the Schwarz potential

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The Schwarz function of a curve:

- Let $\Gamma:=\{q(x,y)=0\}$, q is real-analytic.
- The Schwarz function $S(z)$ is the unique function complex-analytic near Γ that coincides with $\bar{z}$ on Γ.
- If $q(x,y)$ is a polynomial, $S(z)$ can be obtained by changing variables:

  $z = x + iy$, $\bar{z} = x - iy$

  and solving for $\bar{z}$.

The Schwarz function of a curve: Example

C. Neumann's oval

$$\left(x^2+y^2\right)^2 = a^2\left(x^2+y^2\right)+4\mathrm{b}^2 x^2$$

$$\left(z\,\overline{z}\right)^2 = a^2\left(z\,\overline{z}\right)+\mathrm{b}^2\left(z+\overline{z}\right)$$

$$S(z)=\frac{z\left(a^2+2\mathrm{b}^2\right)+z\sqrt{4\mathrm{a}^4+4\mathrm{a}^2 b^2+4\mathrm{b}^2 z^2}}{2\left(z^2-b^2\right)}$$

∇

The Schwarz **Potential**: A generalization of the Schwarz **Function**

- Define the Schwarz potential of an analytic surface as the unique solution of the Cauchy problem (Khavinson-Shapiro):

$$\Delta W(\underline{x}) = 0 \quad (\text{near } \Gamma)$$

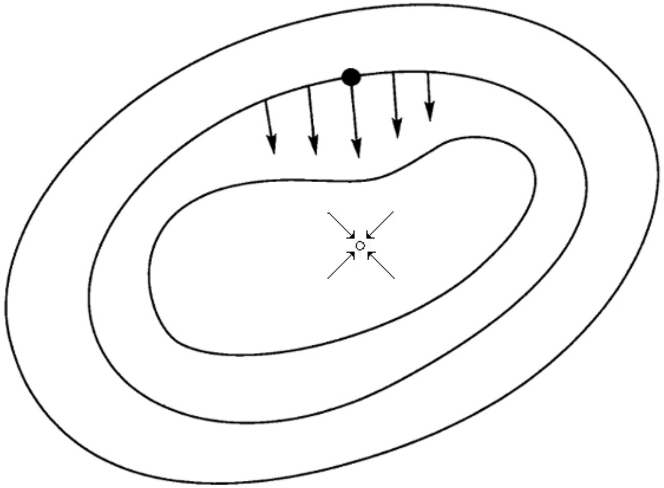
$$W(\underline{x}) = \|\underline{x}\|^2/2, \quad \nabla W(\underline{x}) = \underline{x} \quad (\text{on } \Gamma)$$

- In the plane we have

$$S(z) = W_x(x, y) - i\,W_y(x, y) = 2\,W_z(x, y).$$

Laplacian Growth with sink at $\underline{x}_0$

$$V = -\nabla p$$
$$\Delta p = 0$$
$$p\,|_{\partial\Omega} = 0$$
$$p(\underline{x} \to \underline{x}_0) \sim -Q\,K(\underline{x} - \underline{x}_0)$$



where V is the normal velocity of $\partial\Omega$, *and* K *is the fundamental solution of Laplace's equation.*

Often Ω is considered unbounded and the sink is placed at $\underline{x}_0$ = infinity, so that p solves an *exterior* Dirichlet problem.

Several sources and sinks $\underline{x}_i$

$$V = -\nabla p$$

$$\Delta p = 0 \, , \; p(\underline{x} \to \underline{x}_i) \sim -Q_i K(\underline{x} - \underline{x}_i)$$

$$p|_{\partial\Omega} = 0$$

Integrable structure: Given a list of total quantities to be injected or removed from each source and sink, the end result does not depend on the order of work of the sources and sinks.

2D case related to soliton theory by K-M-W-Z.

2-D Laplacian Growth and the Schwarz function: Dynamics of Singularities

Suppose *S(z,t)* is the Schwarz function of $\partial\Omega_t$, and *p(z,t)* is the "pressure". Then

$$S_t(z,t) = -4 \cdot p_z(z,t)$$

Thus "singularities of the Schwarz function do not depend on time except for simple poles stationed at the sources/sinks".

Laplacian Growth and the Schwarz potential:

Suppose *W(x,t)* is the Schwarz potential of $\partial\Omega_t$, and *p(x,t)* is the "pressure". Then we have (*n* = spatial dim):

$$W_t(x,t) = -n \cdot p(x,t)$$

- Relates a solution of a "mathematically-posed" Cauchy Problem to that of a "physically-posed" Dirichlet Problem.
- Related to Richardson's Theorem by way of theory of "quadrature domains".
- Proof uses vector calculus to show that both sides of the equation solve the same Cauchy problem.

Elliptic Growth and a generalized Schwarz potential:

"Elliptic Growth" allows for the porous medium to have a non-constant "filtration" coefficient λ and "porosity" ρ.

Let L := L u = div(λ ρ grad u).

The "pressure" satisfies L (P) = 0 , and the velocity of the boundary is - λ grad P

Suppose $q(x)$ solves L q = n ρ.
Then given a curve (or surface), define the Elliptic SP:

$$L\, u = 0 \quad (\, near\ \Gamma\,)$$

$$u = q\,, \quad \text{grad}\, u = \text{grad}\, q \quad (\, on\ \Gamma\,)$$

Elliptic Growth and a generalized Schwarz potential:

For the elliptic growth, we have the exact same equation:

$$u_t(x,t) = -n \cdot p(x,t)$$

Examples of 2D LG illustrated through singularities of Schwarz function:

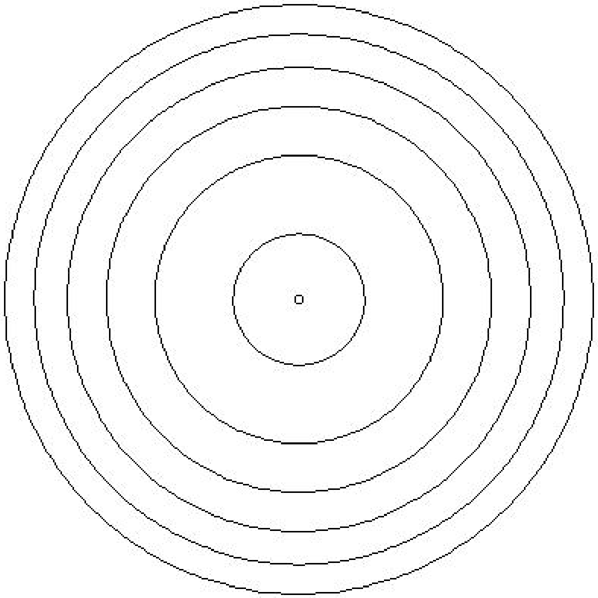
(1.) Concentric circles

$S(z,t) = t / z$, $S_t(z,t) = 1/z$

$$x^2 + y^2 = r^2$$

$$z \cdot \bar{z} = r^2$$

$$S(z) = r^2 / z$$



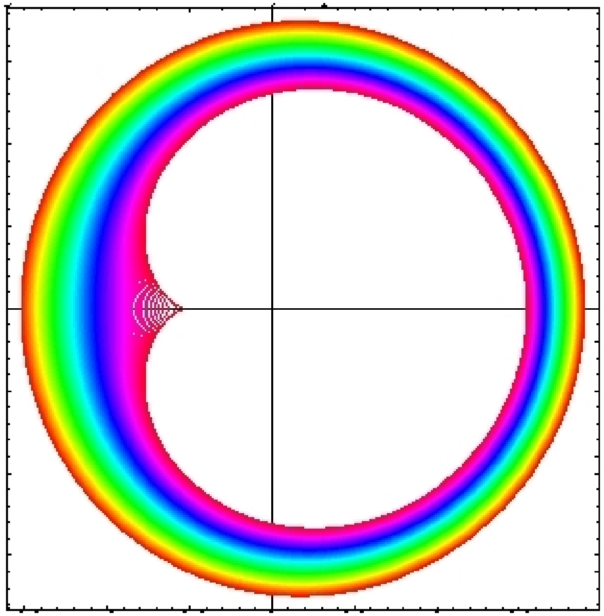
Examples of 2D LG illustrated through the dynamics of singularities:

(2.) Start with the two parameter family of Pascal's "Limacon", for which $S(z) = a/z + b/z^2 + H(z)$, where $H(z)$ is analytic inside the domain.

Choose b = const. and $a(t) = t$:

$S_t(z,t) = 1/z + H_t(z)$.

Physical breakdown: develops a cusp (limacon becomes a cardioid).



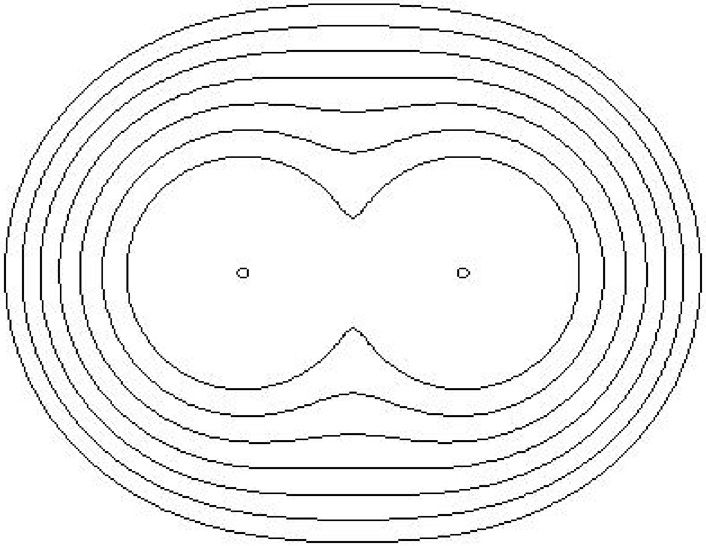
Examples of 2D LG illustrated through the dynamics-of-singularities:

3. “C. Neumann's Oval”

$$\left(x^2+y^2\right)^2=a^2\left(x^2+y^2\right)+4b^2x^2$$

$$\left(z\,\bar{z}\right)^2=a^2\left(z\,\bar{z}\right)+b^2\left(z+\bar{z}\right)$$

$$S\left(z\right)=\frac{z\left(a^2+2b^2\right)+z\sqrt{4a^4+4a^2b^2+4b^2z^2}}{2\left(z^2-b^2\right)}$$



$S(z,t) = t/(z-1) + t/(z+1) + H(z)$

“two sinks”: $S_t(z,t) = 1/(z-1) + 1/(z+1) + H_t(z)$

Axially-symmetric examples in R^4

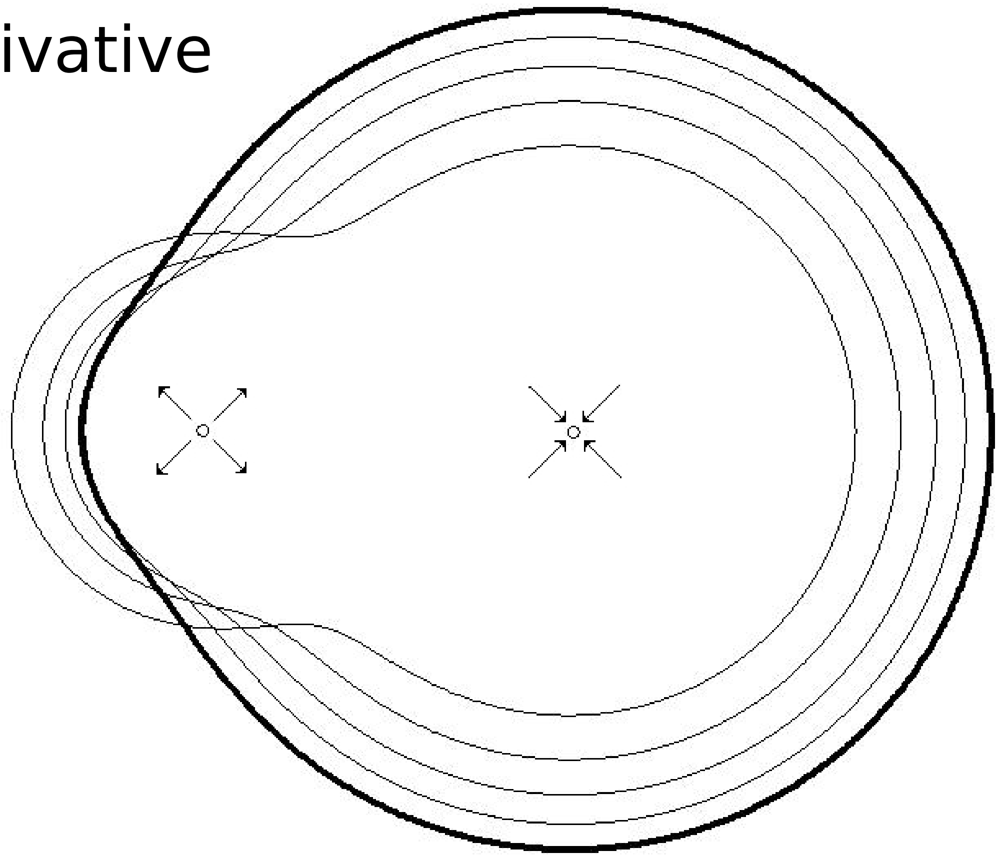
For axially-symmetric hypersurfaces in R^4 the Schwarz potential *W(x,y)* can be calculated exactly using the fact that *yW(x,y)* is a harmonic function in the plane (*x* = symmetry-axis coord. and *y* = dist. to symmetry-axis).

L. Karp outlined the details of the procedure and worked out explicit examples showing singularities of certain surfaces of revolution generated by quadrature domains.

The examples considered by L. Karp lead to solutions involving multi-poles.

Can we find examples with simple sources/sinks?

- Profile of an axially-symmetric solution in R^4, with one source and one sink:
- Schwarz potential has a singular segment joining the source and sink (which is killed by time-derivative).
- Pumping rates must be chosen carefully or the time-derivative will include the singular segment.
- But recall: the end result is independent of rates and only depends on total amounts.



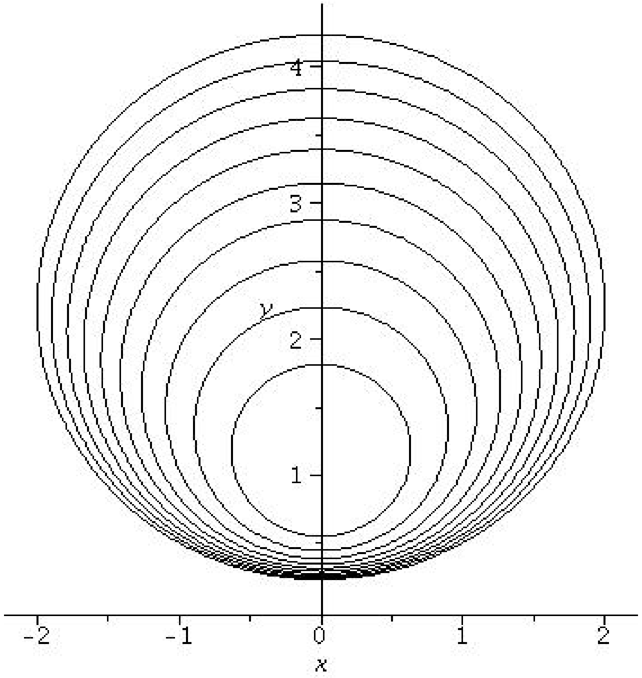
A conjecture regarding a comment of H. S. Shapiro:

Conjecture: In dimensions higher than two, there exist quadrature domains (in the classical, restricted sense) that are not algebraic.

Here, “QD” means that the SP has finitely many isolated singularities of finite order inside the domain, and “algebraic” means “zero set of a polynomial”. In two dimensions any QD is algebraic.

Some exact solutions to elliptic growth.
For $\lambda = y^m$ and $\rho = y^{2-m}$

Theorem: For m a positive integer, the exact solutions from LG with polynomial conformal map (Galin) can be generated as an elliptic growth (with the above filtration and porosity) driven by multi-poles of finite order.



- Resembles results of Loutsenko and Yermolayeva.
- For negative m, Theorem fails in an interesting way.

We can still obtain exact solutions with fixed multi-poles, but singularities of (harmonic) Schwarz function move.

Toward three-dimensional examples: A view from C^n

- The Schwarz potential solves a Cauchy problem. Since the data and initial surface are both algebraic, problem can be lifted to C^n .
- Leray's Principle describes propagation of singularities from characteristic points on the surface in C^n . But it is only locally rigorous.
- G. Johnsson made Leray's principle globally rigorous for quadratic surfaces.
- Johnsson's proof inverts a linear system of equations that becomes nonlinear for higher-degree surfaces.

R^3: Singularity set of SP for Neumann Ovaloid

$$\left(x^2+y^2+z^2\right)^2=\left(a^2x^2+4b^2\left(y^2+z^2\right)\right)$$

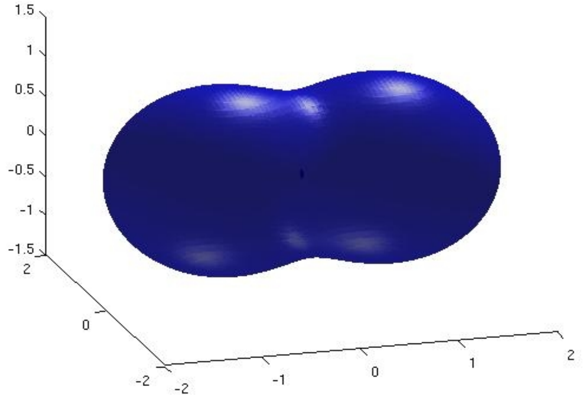
- Theorem: The Schwarz potential of the family of surfaces above is real-analytic throughout the interior domain except on the segment joining (-b,0,0) and (b,0,0).

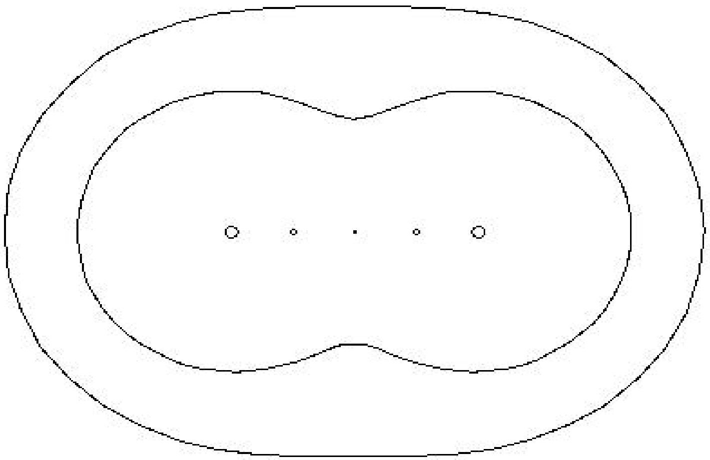
- In R^2 it is immediate.
- In R^4 it was an example done by L. Karp.
- For proof in R^n in general we used C^n techniques to establish the analytic extension without actually being able (so far) to calculate the Schwarz potential.

Ideas in proof using a c^2 argument:

1. Method of globalizing families from proof of the Bony-Shapira Theorem.
2. Zerner's local extension theorem.
3. Extension theorem of Ebenfelt, Khavinson, and Shapiro.

Three-dimensional Neumann Ovaloid: approximate description of sinks





Having located the singularity set for the SP, one can choose some points on the segment and interpolate a quadrature formula. This gives a good approximate description of the initial and final domains shown as a Laplacian growth with, say, 5 sinks.

i.e. we approximately solve the inverse problem of describing the driving mechanism required to generate an initial and final domain from the family of Neumann Ovaloids.

Schwarz Potential for axially-symmetric case: Integral formula (following Garabedian's book)

$$u(z,w)=\frac{1}{2}\left(A(S(w),w;z,w)S(w)w+A(z,S(z);z,w)S(z)z\right)$$

$$+\frac{1}{2}\int_{S(w)}^{w}\frac{n-2}{S(t)-t}A(S(t),t;z,w)S(t)t+A(S(t),t;z,w)S(t)-A_t(S(t),t;z,w)S(t)tdt$$

$$-\frac{1}{2}\int_{z}^{S(z)}\frac{n-2}{s-S(s)}A(s,S(s);z,w)S(S)s+A(s,S(s);z,w)S(s)-A_s(s,S(s);z,w)S(s)sds$$

where *S(z)* is the Schwarz function of the profile curve, and *A(s, t ; z, w)* is the Riemann function for the axially-symmetric Laplacian in complex characteristic coordinates: *A(s, t ; z, w)* involves a Gaussian Hypergeometric function.

Schwarz Potential Conjecture:

SP Conjecture (Khavinson, Shapiro): Suppose *u* solves the Cauchy problem for Laplace's equation posed on a non-singular analytic surface *Γ* with real-entire data. Then the singularity set of u is contained in the singularity set of the Schwarz potential *w*.

Consequence: Given a Laplacian growth, the same "movie" can be produced amid elliptic growth If $\alpha = \lambda\rho = 1$ and ρ is entire. The driving singularities are at the same positions as the singularities of the (harmonic) Schwarz function.

Elliptic SP Conjecture: Suppose α is entire and that *v* solves the Cauchy problem on a non-singular analytic surface for div(α grad *u*) = 0 with entire data. Then the singularity set of *u* is contained in the singularity set of *v*, where *v* solves the Cauchy problem with data *q* the solution of

$$\mathrm{div}(\alpha \text{ grad } q) = 1.$$

Future directions:

- Formulate general principles governing singularities of axially-symmetric SP in terms of SF of the profile curve.
- Study examples that are not axially-symmetric.
- Exact solutions in special geometry (cone, cylindrical channel, half-space)? Self-similar solutions?
- Applications to quasi-geostrophic vortex dynamics? A missing ingredient: stability analysis.
- Explore integrable structure of n-dim LG. Perhaps consider first the axially-symmetric case.
- Is there a connection to KP hierarchy?
- Scaling limit of cusps that can form:

(a) stable direction: bubble break-up.

(b) unstable direction: “catastrophic” cusps

References:

- J.-M. Bony, P. Shapira, *Existence et prolongement des solutions holomorphes des equations aux d eriv ees partielles*, 1972
- D. Crowdy, *Quadrature domains and fluid dynamics,* 2005
- P. Ebenfelt, D. Khavinson, H.S. Shapiro, *Extending solutions of holomorphic PDEs across real hypersurfaces,* 1998
- J. Escher, G. Simonett, *Classical solutions of multidimensional Hele-Shaw models,* 1997
- P. Garabedian, *Partial differential equations,* 1964
- B. Gustafsson, H. S. Shapiro, *What is a quadrature domain?,* 2004
- S. Howison, *Complex variable methods in Hele-Shaw problems,* 1992
- G. Johnsson, *The Cauchy problem in R^n for linear second order PDEs with data on a quadric surface,* 1994
- L. Karp, *Construction of quadrature domains in R^n from quadrature domains in R^2,* 1992
- I. Loutsenko, O. Yermolayeva, Non-Laplacian growth: Exact results, 2007
- *I. Krichever, M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin*, *Laplacian growth and Whitham equations of soliton theory,* 2003
- E. Lundberg, *Laplacian Growth, elliptic growth, and Singularities of the Schwarz Potential*, submitted.
- H. S. Shapiro, *The Schwarz function and its generalization to higher dimensions,* 1992
- A. N. Varchenko, P. I. Etingoff, *Why the boundary of a round drop becomes a curve of order four,* 1992
- M. Zerner, *Domaine d'holomorphie des fonctions vérifiant une équation aux dérivées partielles, 1971*