

TWO-DIMENSIONAL SHAPES AND LEMNISCATES

P. EBENFELT, D. KHAVINSON, AND H. S. SHAPIRO

1. INTRODUCTION

The newly emerging field of vision and pattern recognition focuses on the study of 2-dimensional “shapes”, i.e., simple, smooth, closed curves in the plane. A common approach to describing shapes consists of defining “natural” distances between them, embedding the shapes into a metric space and then studying the mathematical structure of the latter. Of course, the resulting metric space must faithfully represent the continuous variability of shapes and reflect in their classification a similarity between them, i.e., not make a distinction between the shapes obtained from one another by scaling or translation (cf. [3] and many references therein). Also, one may consult [12] for extensions to analysis of “surfaces”, more specifically, surfaces of a human face.

Another idea which has apparently been pioneered by A. Kirillov [9, 10] and developed by Mumford and Sharon [11] and many others, consists of representing each shape by its “fingerprint”, an orientation-preserving diffeomorphism of the unit circle. In this context every shape defines a unique equivalence class of such diffeomorphisms (up to a right composition with a Möbius automorphism of the unit disk onto itself.) More precisely, let Γ be a smooth, simple closed curve (a Jordan curve) in \mathbb{C} , Ω_- the region enclosed by Γ (i.e. the bounded component of $\mathbb{C} \setminus \Gamma$), and $\Omega_+ := \hat{\mathbb{C}} \setminus \bar{\Omega}_-$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Let $\Phi_- : \mathbb{D} \rightarrow \Omega_-$, $\Phi_+ : \mathbb{D}_+ \rightarrow \Omega_+$, where $\mathbb{D} = \{|z| < 1\}$ is the unit disk, \mathbb{D}_+ its complement in $\hat{\mathbb{C}}$, and Φ_{\pm} are conformal maps (whose existence is guaranteed by the Riemann mapping theorem). We accept the normalization $\Phi_+(\infty) = \infty$ and $\Phi'_+(\infty) > 0$, where the latter means that Φ_+ has a Laurent series expansion in neighborhood of infinity,

$$\Phi_+(z) = az + \sum_{k=0}^{\infty} a_k z^{-k},$$

with $a > 0$. From now on we shall assume Γ to be a C^∞ curve, and hence Φ_{\pm} extend in a C^∞ fashion to the boundaries of their respective domains. Let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle and consider $k := \Phi_+^{-1} \circ \Phi_- : \mathbb{T} \rightarrow \mathbb{T}$, an orientation preserving C^∞ -diffeomorphism

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of the unit circle onto itself. We can think of the derivative k' as a 2π -periodic function on \mathbb{R} . We have then $k(x + 2\pi) = k(x) + 2\pi$ and $k' > 0$. Obviously, k is uniquely determined by Γ up to a Möbius automorphism of \mathbb{D} , i.e. up to right composition $k \circ \phi$ with

$$(1.1) \quad \phi(z) = \lambda \frac{z - a}{1 - \bar{a}z}, \quad |\lambda| = 1, \quad a \in \mathbb{D}.$$

The equivalence class of the diffeomorphism k under the action of the Möbius group of automorphisms (1.1) is called the *fingerprint* of Γ . Moreover, if $\tilde{\Gamma}$ denotes the curve $A(\Gamma)$, where A is the affine transformation

$$(1.2) \quad A(z) = az + b, \quad a > 0, \quad b \in \mathbb{C},$$

then the fingerprint of $\tilde{\Gamma}$ equals that of Γ , as is easily verified. Thus, we have a map \mathcal{F} from the set of all smooth Jordan curves Γ modulo scaling and translation as in (1.2), *shapes*, into the set of all orientation preserving diffeomorphisms k of the circle modulo Möbius automorphisms (1.1) of the unit disk, *fingerprints*. The following theorem was first explicitly stated in [9, 10], although as is noted in [9, 10, 11] it follows more or less directly from the results of Ahlfors and Bers [1] on solutions of Beltrami equation (cf. [14]).

Theorem 1.1. *The map \mathcal{F} is a bijection.*

Using this theorem and well-developed software packages, e.g., [6], Mumford and Sharon show how in principle one may recover (i.e., approximate) Γ from its fingerprint k and vice-versa, by approximating Γ by polygons and using the Schwarz–Christoffel formula. The experimental data presented in [11] looks stunningly convincing.

Note in passing that if we relax significantly the smoothness hypothesis on Γ , the map $\Gamma \rightarrow k$ from closed curves to circle *homeomorphisms* is neither onto nor one-to-one (cf. [5]).

In this paper, we present a somewhat “ideologically” different explanation of why Kirillov’s theorem is true, motivated by a well-known theorem of Hilbert (cf. [15, Ch. 4]) stating that any smooth curve can be approximated (with respect to the Hausdorff distance in the plane (2.6)) by polynomial lemniscates. A considerable advantage in this approach lies in the observation that the fingerprint of a polynomial lemniscate of degree n is particularly simple, an n^{th} root of a (finite) Blaschke product of degree n (Theorem 2.2). We then show that every smooth, orientation-preserving diffeomorphism of the circle can be approximated in the C^1 -norm by these former simple ones (Theorem 2.3). To complete the picture in this approach, we show that a diffeomorphism of the unit circle given by the n^{th} root of a Blaschke product of degree n is the fingerprint of a unique polynomial lemniscate of degree n (Theorem 3.1). Although we have not been able yet to reconstruct effectively a lemniscate from its fingerprint on the circle, we still hope that via associated finite Blaschke products lemniscates could serve as natural and convenient “coordinates” in the enormous space of smooth shapes.

In Section 2, we discuss fingerprints of polynomial lemniscates and show that the diffeomorphisms induced by roots of finite Blaschke products approximate all smooth diffeomorphisms of the circle. In Section 3, we prove that the diffeomorphisms induced by roots of finite Blaschke products are fingerprints of lemniscates. We end with some questions and remarks assembled in §4.

2. LEMNISCATES AND THEIR FINGERPRINTS

A (polynomial) lemniscate of degree n is a subset $\Gamma \subset \mathbb{C}$ of the form

$$\{z \in \mathbb{C} : |P(z)| = 1\},$$

where $P(z)$ is a polynomial in z of degree n . We let $\Omega_- := \{z \in \mathbb{C} : |P(z)| < 1\}$ and $\Omega_+ := \hat{\mathbb{C}} \setminus \overline{\Omega_-} = \{z \in \mathbb{C} : |P(z)| > 1\} \cup \{\infty\}$. An immediate consequence of the maximum modulus theorem is that Ω_+ can have no bounded components and, hence, is a connected open subset containing a neighborhood of ∞ in $\hat{\mathbb{C}}$. We shall say that Γ is a *proper* lemniscate of degree n if Γ is smooth ($P'(z) \neq 0$ on Γ) and Ω_- is connected. Note that the interior Ω_- of a proper lemniscate of degree n (or, for a general smooth lemniscate, each component of Ω_-) is also simply connected, since its complement is connected.

Recall that the critical points of a polynomial $P(z)$ of degree n are the zeros of its derivative $P'(z)$. Let z_1, \dots, z_{n-1} be the critical points (repeated according to their multiplicity) of $P(z)$. The values w_1, \dots, w_{n-1} , where $w_k = P(z_k)$ for $k = 1, \dots, n-1$, are called the critical values of $P(z)$. The multiplicity of a critical value w is the number of times it appears in the list w_1, \dots, w_{n-1} . For a smooth lemniscate Γ of degree n , the property of being proper can be characterized by the critical values of its defining polynomial.

Proposition 2.1. *Let $P(z)$ be a polynomial of degree n and assume that the lemniscate $\Gamma = \{z \in \mathbb{C} : |P(z)| = 1\}$ is smooth. The following are equivalent:*

- (i) *The open set $\Omega_- = \{z \in \mathbb{C} : |P(z)| < 1\}$ is connected (i.e. Γ is a proper lemniscate of degree n).*
- (ii) *All the critical values w_1, \dots, w_{n-1} of $P(z)$ satisfy $|w_k| < 1$.*

Proof. First note that no critical value can satisfy $|w_k| = 1$ since Γ is smooth. Let r denote the number of critical points, counted with multiplicities, in Ω_- . Statement (ii) above is then equivalent to $r = n - 1$. The equivalence of (i) and (ii) is a simple consequence of the classical Riemann-Hurwitz formula. Indeed, let $\Omega_1, \dots, \Omega_k$ denote the components of Ω_- , and d_j and r_j for $j = 1, \dots, k$ the number of zeros and critical points, respectively, of P in Ω_j . We then have $d_1 + \dots + d_k = n$ and $r_1 + \dots + r_k = r$. Now, let f_j denote the restriction of P to Ω_j . Each f_j is then a d_j -to-1 ramified covering (proper analytic map) $f_j: \Omega_j \rightarrow \mathbb{D}$ with total ramification number r_j . Since each Ω_j , as well as \mathbb{D} , is simply connected and hence has Euler characteristic 1, the Riemann-Hurwitz formula in this setting (see e.g. [13]) states that

$$-1 = -d_j + r_j, \quad j = 1, \dots, k.$$

By summing over j , we obtain $-k = -n + r$ or $r = n - k$. Thus, we have $r = n - 1$ if and only if the number of components of Ω_- is one. \square

Consider a proper lemniscate $\Gamma = \{z \in \mathbb{C} : |P(z)| = 1\}$ of degree n . Clearly, there is no loss of generality in assuming that the degree n coefficient of $P(z)$ is real and positive, i.e.

$$(2.1) \quad P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0, \quad a_n > 0, \quad a_k \in \mathbb{C} \text{ for } k = 0, \dots, n-1.$$

Let $\xi_1, \dots, \xi_n \in \Omega_-$ be the zeros of P (repeated with multiplicity), and $\Phi_- : \mathbb{D} \rightarrow \Omega_-$ a Riemann mapping. Then, $P \circ \Phi_-$ is an n -to-1 ramified covering $\mathbb{D} \rightarrow \mathbb{D}$ and hence must be a finite Blaschke product of degree n , i.e.,

$$(2.2) \quad B_1(z) := (P \circ \Phi_-)(z) = e^{i\theta} \prod_1^n \frac{z - a_j}{1 - \bar{a}_j z}, \quad a_j = \Phi_-^{-1}(\xi_j), \quad \theta \in \mathbb{R}.$$

Indeed, $(P \circ \Phi_-)/B_1$ is analytic in \mathbb{D} , does not vanish there, is continuous in the closed disk and has modulus one on $\mathbb{T} := \partial\mathbb{D}$, and thus is a unimodular constant.

Now let $\Phi_+^{-1} : \Omega_+ \rightarrow \mathbb{D}_+$ be the conformal mapping from the exterior $\Omega_+ \subset \hat{\mathbb{C}}$ of Γ onto the exterior of the unit disk $\mathbb{D}_+ \subset \hat{\mathbb{C}}$, normalized by $\Phi_+^{-1}(\infty) = \infty$, $(\Phi_+^{-1})'(\infty) > 0$. We claim that

$$(2.3) \quad \Phi_+^{-1}(w) = \sqrt[n]{P(w)},$$

where we choose a suitable branch of the n th root (i.e. $\sqrt[n]{1} = 1$) to comply with our normalization of $(\Phi_+^{-1})'(\infty) > 0$. To see this, we note that $P \circ \Phi_+$ has a pole of order n at ∞ , no other poles in \mathbb{D}_+ , and maps \mathbb{D}_+ to itself sending the boundary $\mathbb{T} = \partial\mathbb{D}_+$ to itself; i.e. $P \circ \Phi_+$ is a ramified n -to-1 covering $\mathbb{D}_+ \rightarrow \mathbb{D}_+$. It follows, as above, that $B_2 := P \circ \Phi_+$ is a Blaschke product of degree n and since B_2 has all its poles at ∞ we conclude that $B_2(z) = cz^n$, where c is a unimodular constant. Since $(\Phi_+^{-1})'(\infty) > 0$ and the highest degree coefficients a_n of P is positive, we deduce that in fact $c = 1$ and $B_2(z) = z^n$. The identity (2.3) follows readily. Since the fingerprint of Γ is given by $k = \Phi_-^{-1} \circ \Phi_+$, we obtain the following theorem.

Theorem 2.2. *Let P be a polynomial of degree n such that $\Gamma = \{z \in \mathbb{C} : |P(z)| = 1\}$ is a proper lemniscate of degree n . Let Ω_- be the interior of Γ and $\Phi_- : \mathbb{D} \rightarrow \Omega_-$ a Riemann mapping. Then, the fingerprint $k : \mathbb{T} \rightarrow \mathbb{T}$ of Γ is given by*

$$(2.4) \quad k(z) = \sqrt[n]{B(z)}$$

where B is the Blaschke product

$$(2.5) \quad B(z) = e^{i\theta} \prod_{k=1}^n \frac{z - a_k}{1 - \bar{a}_k z}, \quad a_k = \Phi_-^{-1}(\xi_k), \quad \theta \in \mathbb{R},$$

and ξ_1, \dots, ξ_n denote the zeros of P repeated according to multiplicity.

In view of Hilbert's theorem (cf. [15, Ch. 4]) that every smooth curve can be approximated by lemniscates in the Hausdorff metric, which measures the distance between two curves C_1, C_2 as (cf. [3])

$$(2.6) \quad \text{dist}(C_1, C_2) = \sup_{z \in C_2} \inf_{w \in C_1} |z - w| + \sup_{z \in C_1} \inf_{w \in C_2} |z - w|,$$

our next goal is to address the following two questions:

- (I) Do the diffeomorphisms k given by (2.4) approximate in some reasonable metric all orientation preserving diffeomorphisms of the unit circle?

If the answer to (I) is in the affirmative, then to complete our (alternative) approximate visualization of Theorem 1.1 we have to answer the following question

- (II) Does each diffeomorphism (2.4) represent the fingerprint of a lemniscate?

The following theorem answers (I).

Theorem 2.3. *The (algebraic) diffeomorphisms (2.4) approximate all orientation preserving diffeomorphisms Ψ of the circle \mathbb{T} in the C^1 -norm.*

Remark 2.4. As usual, the C^1 -norm on \mathbb{T} means

$$\|f\|_{C^1} = \sup_{\theta} (|f(e^{i\theta})| + |f'(e^{i\theta})|).$$

Proof. First, we note that it suffices to verify the theorem for real-analytic diffeomorphisms Ψ since the latter are dense in the C^1 -norm in the set of all diffeomorphisms of \mathbb{T} . Let $\Psi(\theta) = \exp(i\psi(\theta))$, where the real-valued function $\psi(\theta)$ is strictly monotone increasing and $\psi' > 0$ is 2π -periodic. Of course, $\psi(\theta + 2\pi) = \psi(\theta) + 2\pi$. To approximate $\psi(\theta)$ on $[0, 2\pi]$ in C^1 -norm by arguments of functions in (2.4) it suffices to approximate uniformly on \mathbb{T} a (positive) function ψ' such that $\int_0^{2\pi} \psi' d\theta = 2\pi$ by functions

$$\frac{1}{n} \frac{d}{d\theta} \arg B(e^{i\theta}),$$

where B is a Blaschke product of degree n . Note that a straightforward calculation yields

$$(2.7) \quad \frac{1}{2\pi} \frac{d}{d\theta} \left(\frac{1}{n} \arg B(e^{i\theta}) \right) = \frac{1}{2\pi} \left\{ \frac{1}{n} \sum_{j=1}^n P(e^{i\theta}, a_j) \right\},$$

where for

$$(2.8) \quad z = re^{i\phi}, \quad P(e^{i\theta}, z) := \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)}$$

denotes the Poisson kernel evaluated at z .

Following [8], we argue now as follows. Since ψ' is positive and real-analytic on \mathbb{T} , we can approximate it by a positive trigonometric polynomial $h(\theta) = \sum_{-N}^N a_k e^{ik\theta} > 0$ maintaining the normalization $\int_{\mathbb{T}} h d\theta = 2\pi$. Observe the following:

- (i) We can consider $h(\theta)$ to be boundary values of a bounded positive harmonic function $H(r, \theta) := \sum_{-N}^N a_k r^{-|k|} e^{ik\theta}$ in $\mathbb{C} \setminus \mathbb{D}$ (in other words, replacing z^k, \bar{z}^k by $\frac{1}{z^k}$ and $\frac{1}{\bar{z}^k}$ in the expansion of h in terms of z, \bar{z}).
- (ii) $H(r, \theta)$ extends as a positive harmonic function to a slightly larger domain $\mathbb{D}_-^R := \{|z| > R, R < 1\}$ — this is obvious in view of the continuity of H and the compactness of \mathbb{T} .

Representing $h(\theta) = H(e^{i\theta})|_{r=1}$ in \mathbb{D}_-^R via the Poisson integral of its boundary values on $\{|z| = R < 1\}$ and taking into account the change in orientation we easily obtain

$$(2.9) \quad \begin{aligned} h(e^{i\theta}) &= H(e^{i\theta})|_{r=1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - R^2}{1 + R^2 - 2R \cos(\theta - \varphi)} H(Re^{i\varphi}) d\varphi \\ &= \int_{|z|=R, R<1} P(e^{i\theta}, z) d\mu(z), \end{aligned}$$

where $\mu > 0$ is a probability measure supported on a compact subset of \mathbb{D} , i.e., on the circle $\{|z| = R < 1\}$. Every such measure μ is a weak* limit of discrete atomic probability measures with n atoms, $n \rightarrow \infty$, having equal charges $1/n$ at these atoms. This last observation together with (2.7) finishes the proof of the assertion that ψ' , with $\int_0^{2\pi} \psi' d\theta = 2\pi$, is uniformly approximable on \mathbb{T} by functions

$$\frac{1}{n} \frac{d}{d\theta} \arg B(e^{i\theta}),$$

where B is a Blaschke product of degree n . The remaining part of the theorem is easily derived from it, so we shall omit it. \square

3. ROOTS OF BLASCHKE PRODUCTS AS FINGERPRINTS

In this section, we shall prove a converse to Theorem 2.2, which answers question (II) above:

Theorem 3.1. *Let B be a Blaschke product of degree n ,*

$$(3.1) \quad B(z) = e^{i\theta} \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z}, \quad |a_j| < 1.$$

Then there is a proper lemniscate $\Gamma \subset \mathbb{C}$ of degree n such that its fingerprint $k: \mathbb{T} \rightarrow \mathbb{T}$ is given by

$$(3.2) \quad k(z) = \sqrt[n]{B(z)}.$$

If $\tilde{\Gamma} \subset \mathbb{C}$ is any other C^1 -smooth Jordan curve with the same fingerprint, then there is an affine linear transformation $T(z) := az + b$, with $a > 0$ and $b \in \mathbb{C}$, such that $\tilde{\Gamma} = T(\Gamma)$.

Proof. The uniqueness part of the theorem is of course a consequence of Theorem 1.1, but for the readers' convenience we shall reproduce the simple proof here. Suppose that two C^1 -smooth Jordan curves $\Gamma, \tilde{\Gamma} \subset \mathbb{C}$ have the same fingerprint, and let $\Phi_-, \tilde{\Phi}_-, \Phi_+, \tilde{\Phi}_+$ be the corresponding Riemann mappings $\mathbb{D}_- \rightarrow \Omega_-, \mathbb{D}_- \rightarrow \tilde{\Omega}_-, \mathbb{D}_+ \rightarrow \Omega_+, \mathbb{D}_+ \rightarrow \tilde{\Omega}_+$, respectively (following the notation introduced above in an obvious way). Since both Jordan curves are assumed C^1 -smooth, all Riemann mappings extend continuously and homeomorphically to the corresponding boundaries. (For this conclusion, weaker conditions than that of being C^1 suffice, but mere continuity does not; cf. [5].) The fact that the two fingerprints are equal means that $\Phi_+^{-1} \circ \Phi_- = \tilde{\Phi}_+^{-1} \circ \tilde{\Phi}_-$ on $\mathbb{T} = \partial\mathbb{D}_- = \partial\mathbb{D}_+$. This can be rewritten as $\tilde{\Phi}_+ \circ \Phi_+^{-1} = \tilde{\Phi}_- \circ \Phi_-^{-1}$ on $\Gamma = \partial\Omega_-$. We conclude that the conformal mapping $\tilde{\Phi}_+ \circ \Phi_+^{-1}: \Omega_+ \rightarrow \tilde{\Omega}_+$ can be extended as a conformal mapping $T: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ by defining it as $\tilde{\Phi}_- \circ \Phi_-^{-1}$ in $\bar{\Omega}_-$. Since $T(\infty) = \infty$ and $T'(\infty) > 0$, we conclude that $T(z) = az + b$ with $a > 0$ and $b \in \mathbb{C}$. This proves the uniqueness modulo affine linear transformations of the type described in the theorem.

To prove the existence part of the theorem, we shall consider a modification of the map \mathcal{F} defined in the introduction in the setting of lemniscates and Blaschke products. A proper lemniscate Γ of degree n is the set of points $z \in \mathbb{C}$ that satisfy $|P(z)| = 1$, where P is a polynomial in z of degree n whose highest order coefficient is positive (see (2.1)) and all of whose critical values belong to \mathbb{D} (see Proposition (2.1)). It follows immediately from (2.3) that the polynomial P is uniquely determined by Γ . We shall let \mathcal{L} denote the subset of $\mathbb{R} \times \mathbb{C} \times \dots \times \mathbb{C}$ (with n factors of \mathbb{C}) that, under the map

$$(3.3) \quad (a_n, a_{n-1}, \dots, a_0) \mapsto P(z) := a_n z^n + \dots + a_0,$$

yields polynomials whose lemniscates are proper of degree n . Clearly, \mathcal{L} is open. One can also easily prove that \mathcal{L} is connected by using Proposition 2.1 in the following way. Let P be a polynomial of degree n corresponding to a point in \mathcal{L} and denote by Γ the corresponding proper lemniscate of degree n . By Proposition 2.1, the critical values of P all have modulus less than one. Consider the lemniscates Γ_R defined by $|P(z)| = R$, or equivalently by $|P_R(z)| = 1$ where $P_R(z) := P(z)/R$, for $R \geq 1$. Clearly, the critical values of P_R all belong to the open disk of radius $1/R \leq 1$ and hence the lemniscates Γ_R are proper lemniscates of degree n . Now, pick $r > 0$ such that

$$(3.4) \quad na_n |z|^{n-1} - ((n-1)|a_{n-1}| |z|^{n-2} + (n-2)|a_{n-2}| |z|^{n-3} + \dots + |a_1|) > 0, \quad \forall |z| \geq r,$$

and then pick $R > 0$ such that

$$(3.5) \quad \frac{1}{R} (a_n |z|^n + |a_{n-1}| |z|^{n-1} + \dots + |a_0|) \leq \frac{1}{2}, \quad \forall |z| \leq r.$$

Finally, consider the lemniscates Γ_R^t defined by the polynomials

$$P_R^t(z) := \frac{1}{R} (a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)), \quad 0 \leq t \leq 1.$$

It follows immediately from (3.4) and (3.5) that the critical values of P_R^t , for $0 \leq t \leq 1$, have modulus less than 1 and, hence, the Γ_R^t are all proper lemniscates of degree n . Note that Γ_R^0 is a circle. Thus, we conclude that any proper lemniscate Γ of degree n can be deformed through proper lemniscates of degree n to a circle. It follows that the open subset \mathcal{L} is connected.

As explained in the introduction, the fingerprint of a shape remains unchanged under translations and scaling. In other words, given a proper lemniscate $\Gamma = \{z: |P(z)| = 1\}$ of degree n with fingerprint $k: \mathbb{T} \rightarrow \mathbb{T}$, let Γ_{ab} be the image of Γ under the inverse of the affine linear transformation $T_{ab}(z) := az + b$, with $a > 0$ and $b \in \mathbb{R}$, and let k_{ab} denote its fingerprint. Then, we have $k = k_{ab}$. We shall consider the space of equivalence classes $\{\Gamma_{ab}\}_{a>0, b \in \mathbb{C}}$ of proper lemniscates Γ of degree n under this action of the group $\mathcal{G} := \{T(z) = az + b: a > 0, b \in \mathbb{C}\}$. Note that if Γ is defined by $|P(z)| = 1$, then Γ_{ab} is defined by $|P_{ab}(z)| = 1$, where

$$(3.6) \quad \begin{aligned} P_{ab}(z) &:= P(az + b) = a_n(az + b)^n + \dots + a_0 \\ &= a^n a_n z^n + (na^n b + a_{n-1})z^{n-1} + \dots + P(b). \end{aligned}$$

Thus, in each equivalence class there is a unique polynomial of the form (2.1) with $a_n = 1/n$ and $a_{n-1} = 0$. In other words, we can parametrize the space of equivalence classes of proper lemniscates of degree n by a subset $\mathcal{EL} \subset \mathbb{C}^{n-1}$ under the identification

$$(3.7) \quad (a_0, \dots, a_{n-2}) \mapsto P(z) = \frac{1}{n}z^n + a_{n-2}z^{n-2} + \dots + a_0.$$

The subset $\mathcal{EL} \subset \mathbb{C}^{n-1}$ is clearly open and connected by the same arguments as above. We also note that there is a finite, n -to-1, holomorphic (polynomial) mapping $\Lambda: \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ induced by the action of \mathcal{G} and defined as follows. For $(\tilde{a}_1, \dots, \tilde{a}_{n-1}) \in \mathbb{C}^{n-1}$, consider the polynomial

$$(3.8) \quad \tilde{P}(z) := \frac{1}{n}z^n + \tilde{a}_{n-1}z^{n-1} + \dots + \tilde{a}_1z.$$

The image $\Lambda(\tilde{a}_1, \dots, \tilde{a}_{n-1})$ are the coefficients (a_0, \dots, a_{n-2}) of the unique polynomial of the form

$$(3.9) \quad P(z) = \frac{1}{n}z^n + a_{n-2}z^{n-2} + \dots + a_0$$

in the equivalence class of \tilde{P} under the action of \mathcal{G} described in (3.6). (The reader may want to write down this map explicitly.) Let $\tilde{\mathcal{EL}}$ denote the inverse image of \mathcal{EL} under this map. For future reference, we remind the reader of the following well-known property, which will be used below, of a finite holomorphic mapping (or a branched covering) H from one complex manifold X to another Y (endowed with some metrics). *The inverse images of H in X depend continuously on the values in Y* in the following sense: Let $w_0 \in Y$ and let $H^{-1}(w_0)$ denote the (finite) set of inverse images. For any $\epsilon > 0$, there exists $\delta > 0$ such that if w belongs to a δ -ball centered at w_0 , then the set of inverse

images $H^{-1}(w)$ belongs to the union of ϵ -balls centered at the points of $H^{-1}(w_0)$. We shall say that $H^{-1}(w)$ converges to $H^{-1}(w_0)$ as a set.

Next, consider the collection of Blaschke products of degree n ,

$$(3.10) \quad B(z) = \lambda \prod_{k=1}^n \frac{z - b_k}{1 - \bar{b}_k z}, \quad b_k \in \mathbb{D}, \quad \lambda \in \mathbb{T}.$$

Let \mathcal{M} be the Möbius group consisting of automorphisms of the unit disk

$$\phi(z) := \lambda \frac{z - b}{1 - \bar{b}z}, \quad |\lambda| = 1, \quad b \in \mathbb{D},$$

acting on Blaschke products by right composition. (Recall from the introduction that the fingerprint of a shape is only defined modulo this action on orientation preserving diffeomorphisms of \mathbb{T} .) It is readily seen that each Blaschke product can be brought to one of the form

$$(3.11) \quad B(z) = z \prod_{k=1}^{n-1} \frac{z - b_k}{1 - \bar{b}_k z}, \quad b_k \in \mathbb{D},$$

by the action of \mathcal{M} . Also, each equivalence class of Blaschke products under this action contains a finite number (n generically) of Blaschke products of this form. The Blaschke product in (3.11) is of course invariant under permutations of the roots (b_1, \dots, b_{n-1}) , and hence the set of such Blaschke products can be identified with the image \mathcal{B} of $\mathbb{D}^{n-1} := \mathbb{D} \times \dots \times \mathbb{D} \subset \mathbb{C}^{n-1}$ under the finite holomorphic mapping $\mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$

$$(3.12) \quad (b_1, \dots, b_{n-1}) \mapsto (S_1(b), S_2(b), \dots, S_{n-1}(b)),$$

where $S_j(b)$ denotes the j th symmetric function on $n - 1$ elements:

$$\prod_{j=1}^{n-1} (z - b_j) = z^{n-1} + S_1(b)z^{n-2} + \dots + S_{n-2}(b)z + S_{n-1}(b).$$

Now, let \mathcal{EB} denote the set of equivalence classes of Blaschke products under the action of \mathcal{M} , and let π denote the projection of \mathcal{B} onto \mathcal{EB} . Since the action of \mathcal{M} on $\mathcal{B} \subset \mathbb{C}^{n-1}$ is algebraic with only finitely many points in each equivalence class, \mathcal{EB} is an algebraic variety (quotient singularity) of dimension $n - 1$. Moreover, being the image under successive continuous mappings (the finite mapping (3.12) followed by π) of the connected space \mathbb{D}^{n-1} , the space \mathcal{EB} is connected.

By the discussion in Section 2, we obtain a map $\mathcal{F}: \mathcal{EL} \rightarrow \mathcal{EB}$ as follows. For an element $e = (a_0, \dots, a_{n-2}) \in \mathcal{EL}$, let P denote the corresponding polynomial (3.7), $\Gamma = \{z \in \mathbb{C}: |P(z)| = 1\}$ its proper lemniscate, $\Phi_-: \mathbb{D} \rightarrow \Omega_-$ a Riemann map, and $B := P \circ \Phi_-$ the corresponding Blaschke product (so that the fingerprint of Γ is the n th root of B by Theorem 2.2). We define $\mathcal{F}(e) := \pi(B)$. If we choose another Riemann map, we obtain another Blaschke product in the same equivalence class (and hence \mathcal{F} is well defined). Also, any Blaschke product in this equivalence class can be produced by choosing a

suitable Riemann map. Thus, to finish the existence part of Theorem 3.1 it suffices to show that \mathcal{F} is surjective. We first claim that \mathcal{F} is continuous. To see this, let $\{e_k\}$ (notation as above) be a sequence of points in \mathcal{EL} converging to e_0 . Let P_k, P_0 be the corresponding polynomials and Γ_k, Γ_0 their lemniscates with interiors Ω_k, Ω_0 . (Since we shall not need the exteriors in this argument, we shall omit the subscript “-” on the interiors and Riemann maps.) If we let K be a closed disk that contains $\bar{\Omega}_0$ in its interior, then $P_k \rightarrow P_0$ uniformly on K and, hence, $\Gamma_k \rightarrow \Gamma_0$ in the Hausdorff metric (2.6). Let us fix a $w \in \Omega_0$. Then, $w \in \Omega_k$ for k sufficiently large. Now, let $\Phi_0: \mathbb{D} \rightarrow \Omega_0, \Phi_k: \mathbb{D} \rightarrow \Omega_k$ be Riemann mappings normalized by $\Phi_0(0) = \Phi_k(0) = w$ and $\Phi_0'(0) > 0, \Phi_k'(0) > 0$. By a well-known theorem of Carathéodory (cf. [7, Ch. II, Sec. 5]) the Riemann mappings $\Phi_k: \mathbb{D} \rightarrow \Omega_k$ converge uniformly in \mathbb{D} to the Riemann mapping $\Phi_0: \mathbb{D} \rightarrow \Omega_0$. Observe that the mapping taking the n roots (ξ_1, \dots, ξ_n) of a monic polynomial to its n coefficients (a_0, \dots, a_{n-1}) is a finite holomorphic mapping (indeed, given by (3.12) above modulo notation). Hence, the roots, as a set, depend continuously on the coefficients (in the sense explained above). It follows that the roots of P_k converge to the roots of P_0 (again, as sets). We conclude that $B_k^{-1}(0) = (P_k \circ \Phi_k)^{-1}(0)$ converge to $B_0^{-1}(0) = (P_0 \circ \Phi_0)^{-1}(0)$. This means that $B_k \rightarrow B_0$ in \mathcal{B} and, hence, $\mathcal{F}(e_k) = \pi(B_k) \rightarrow \mathcal{F}(e_0) = \pi(B_0)$ in \mathcal{EB} . This proves that \mathcal{F} is continuous.

Next, we observe that the map \mathcal{F} is not injective. Indeed, the map as described above produces for each element $e \in \mathcal{EL}$ an equivalence class of a Blaschke product B such that $B = k^n$, where k is the finger print of the lemniscate Γ associated to the point in \mathcal{EL} . We proved above that the map taking the point $e \in \mathcal{EL}$ to its fingerprint is injective. Thus, two lemniscates Γ_1 and Γ_2 corresponding to two points in \mathcal{EL} will produce the same Blaschke product precisely when their fingerprints satisfy $k_2 = \epsilon k_1$, where ϵ is a root of unity: $\epsilon^n = 1$. Now, it is easy (and left to the reader) to verify that if $T(z) = \lambda z$, for some $|\lambda| = 1$, then, for any shape Γ with fingerprint k , the fingerprint of $T(\Gamma)$ is λk . Let us introduce an equivalence relation on \mathcal{EL} where two elements e_1 and e_2 are equivalent when their corresponding lemniscates Γ_1 and Γ_2 are related by $\Gamma_2 = T(\Gamma_1)$ for some $T(z) = \epsilon z$ with $\epsilon^n = 1$. If we let \mathcal{EL}' denote the set of equivalence classes of elements in \mathcal{EL} , then by the comments above the map \mathcal{F} factors as the map $\mathcal{EL} \rightarrow \mathcal{EL}'$ and an injective map $\mathcal{F}': \mathcal{EL}' \rightarrow \mathcal{EB}$. As in the case of \mathcal{EB} above, the set \mathcal{EL}' is an algebraic variety of dimension $n - 1$. The map $\mathcal{F}': \mathcal{EL}' \rightarrow \mathcal{EB}$ is continuous. To prove that \mathcal{F} is surjective, we shall employ Koebe’s continuity method based on Brouwer’s “invariance of a domain” theorem (cf. [7, Ch. 5, Sec. 6], see also [4]). Since \mathcal{F}' is continuous and injective, Brouwer’s theorem implies that \mathcal{F}' maps \mathcal{EL}' homeomorphically onto an open subset of \mathcal{EB} . Since the image $\mathcal{F}(\mathcal{EL})$ clearly equals the image $\mathcal{F}'(\mathcal{EL}')$, we conclude that $\mathcal{F}(\mathcal{EL})$ is an open subset of \mathcal{EB} . Since \mathcal{EB} is connected as explained above, to prove that \mathcal{F} is surjective it suffices to show that the image $\mathcal{F}(\mathcal{EL})$ is closed in \mathcal{EB} .

We first note that a Blaschke product B of degree n is an n -to-1 branched covering of \mathbb{D} by itself. Thus, by the Riemann-Hurwitz formula as in the proof of Proposition 2.1, B

has $n - 1$ critical points (counted with multiplicity) in \mathbb{D} . (This can also be easily seen by the argument principle, computing the change in argument of $B'(z)$ as z traverses \mathbb{T} by noting that $B(z)$ circles n times around \mathbb{T} as z traverses \mathbb{T} once.) For a Blaschke product B of the form (3.11), the critical values c_1, \dots, c_{n-1} are of course obtained by solving the equation $B'(z) = 0$ and, hence, the critical values depend continuously (as sets) on the roots b_1, \dots, b_{n-1} . Consequently, the critical values $w_1, \dots, w_{n-1} \in \mathbb{D}$ depend continuously on b_1, \dots, b_{n-1} . Also, if $B = P \circ \Phi$ for some polynomial P and conformal mapping Φ , then clearly the critical values of B and P are the same. Moreover, the critical values of any two Blaschke products in the same equivalence class in \mathcal{EB} are the same. It follows that the critical values of a polynomial corresponding to $e \in \mathcal{EL}$ and any representative of $\mathcal{F}(e)$ are the same.

Now, let $\{f_k\}$ be a sequence in $\mathcal{F}(\mathcal{EL})$ converging to $f_0 \in \mathcal{EB}$. We will show that $f_0 = \mathcal{F}(e_0)$ for some $e_0 \in \mathcal{EL}$, which will complete the proof. Let $e_k := \mathcal{F}^{-1}(f_k) \in \mathcal{EL}$ and let P_k denote the corresponding polynomials under the identification (3.7). In what follows, we shall abuse the notation and not distinguish between an element e of coefficients and its corresponding polynomial P . Now, let B_0 be a Blaschke product in the equivalence class f_0 and let $w_1^{(0)}, \dots, w_{n-1}^{(0)}$ denote the critical values (counted with multiplicity) of $B(z)$. Note that the critical values are independent of the choice of B_0 . Let us choose $B_0 \in \mathcal{B}$. Similarly, let $w_1^{(k)}, \dots, w_{n-1}^{(k)}$ denote the critical values of some (any) choice of Blaschke product $B_k \in \mathcal{B}$ in the equivalence class f_k . Since the map $\pi: \mathcal{B} \rightarrow \mathcal{EB}$ is an n -to-1 branched covering (each equivalence class in \mathcal{EB} contains at most, and generically, n distinct Blaschke products of the form (3.11) as mentioned above), we may choose the $B_k \in \mathcal{B}$ such that $B_k \rightarrow B_0$ in \mathcal{B} . Since the critical values depend continuously on the roots of the Blaschke product, we can order the critical values $w_1^{(k)}, \dots, w_{n-1}^{(k)}$ of B_k in such a way that $w_j^{(k)} \rightarrow w_j^{(0)}$ as $k \rightarrow \infty$ for each j . (For instance, for each k we can choose an ordering that minimizes the sum of the distances $|w_j^{(k)} - w_j^{(0)}|$.)

Now, recall the n -to-1 holomorphic mapping $\Lambda: \widetilde{\mathcal{EL}} \rightarrow \mathcal{EL}$, where $\widetilde{\mathcal{EL}}$ denote the polynomials of the form (3.8) whose critical values all belong to \mathbb{D} . Note that each polynomial $\tilde{P}(z)$ of the form (3.8) is uniquely determined by the conditions that $\tilde{P}(0) = 0$ and $\tilde{P}'(z)$ is monic. The map

$$(3.13) \quad (\zeta_1, \dots, \zeta_{n-1}) \mapsto \tilde{P}(z) := \int_0^z \left(\prod_{k=1}^{n-1} (z - \zeta_k) \right) dz$$

is an $(n - 1)!$ -to-1 holomorphic map of \mathbb{C}^{n-1} onto the space of polynomials of the form (3.8). By Theorem 1.2 in [2], the map sending the critical points of \tilde{P} given by (3.13) to its critical values, i.e.

$$(3.14) \quad \Psi(\zeta_1, \dots, \zeta_{n-1}) := (\tilde{P}(\zeta_1), \dots, \tilde{P}(\zeta_{n-1})),$$

is a finite n^{n-1} -to-1 holomorphic mapping. Choose $\tilde{P}_k \in \widetilde{\mathcal{EL}}$ such that $\Lambda(\tilde{P}_k) = P_k$. Since $(\mathcal{F} \circ \Lambda)(\tilde{P}_k) = \mathcal{F}(P_k) = f_k$, it follows that the critical values of \tilde{P}_k are $w_1^{(k)}, \dots, w_{n-1}^{(k)}$. Thus, we can choose a sequence $\zeta_k = (\zeta_1^{(k)}, \dots, \zeta_{n-1}^{(k)}) \in \mathbb{C}^{n-1}$ such that \tilde{P}_k is given by the map in (3.13) and $\Psi(\zeta_k) = w_k := (w_1^{(k)}, \dots, w_{n-1}^{(k)})$. Now, choose $\zeta_0 \in \mathbb{C}^{n-1}$ such that $\Psi(\zeta_0) = w_0 := (w_1^{(0)}, \dots, w_{n-1}^{(0)})$. Since $w_k \rightarrow w_0$ as $k \rightarrow \infty$, it follows that the the inverse images $\Psi^{-1}(w_k)$ converge to $\Psi^{-1}(w_0)$ as sets. Since $\zeta_k \in \Psi^{-1}(w_k)$, $\zeta_0 \in \Psi^{-1}(w_0)$, and any $\Psi^{-1}(w)$ (in particular, $\Psi^{-1}(w_0)$) contains at most n^{n-1} distinct preimages, it follows from the pigeon hole principle that there is a subsequence ζ_{k_j} that converges to ζ_0 . If we now let \tilde{P}_0 denote the image of ζ_0 in $\widetilde{\mathcal{EL}}$ under the map (3.13), then $\tilde{P}_{k_j} \rightarrow \tilde{P}_0$. Thus, if we denote by $P_0 = \Lambda(\tilde{P}_0) \in \mathcal{EL}$, then $\mathcal{F}(P_k) = (\mathcal{F} \circ \Lambda)(\tilde{P}_{k_j}) \rightarrow \mathcal{F}(P_0)$ as $j \rightarrow \infty$. Also, we have $\mathcal{F}(P_k) \rightarrow e_0$ and, hence, $\mathcal{F}(P_0) = e_0$ proving that e_0 belongs to the image of \mathcal{F} . We conclude that the image is closed. Since the image is also open and \mathcal{EB} is connected, we conclude that \mathcal{F} is surjective, which completes the proof. \square

We would like to revisit the main idea in the proof above of the existence of a lemniscate with a prescribed n th root of a Blaschke product B as its fingerprint. It is well known, and easily seen by the Riemann-Hurwitz formula or by noting that $B'(z)dz$ changes its argument by $2\pi n$ as z traverses the unit circle, that B has $n-1$ critical points z_1, \dots, z_{n-1} counted with multiplicities inside the unit disk. Let $w_j := B(z_j)$, for $j = 1, \dots, n-1$, denote the critical values of B in \mathbb{D} . The main idea in the existence proof above is to look for a candidate of a lemniscate whose fingerprint could be $k = \sqrt[n]{B}$ among those given by $\Gamma := \{z \in \mathbb{C} : |P(z)| = 1\}$, where the $P(z)$ are polynomials of degree n whose critical values are $w_1, \dots, w_{n-1} \in \mathbb{D}$. We know that the number of equivalence classes of such lemniscates is finite and, by the uniqueness part in Theorem 3.1 already proved, the map sending these equivalence classes to their fingerprints k is injective. Thus, if we could show that the number of equivalence classes of Blaschke products with a given set of critical values $w_1, \dots, w_{n-1} \in \mathbb{D}$ is the same as (or at least does not exceed) the number of equivalence classes of polynomials with this set of critical values, then the map would be a bijection, which would complete the proof of the existence. We have, however, been unable to find a direct proof of this statement and unable to find it in the existing literature. To circumvent this obstacle, we instead use Koebe's continuity method as described in the proof above. As a byproduct of this alternative completion of the proof, we are able *a posteriori* to get an accurate count of the number of equivalence class of Blaschke products with a given set of critical values in the following way.

We shall use the notation introduced in the proof of Theorem 3.1. If w_1, \dots, w_{n-1} are values in \mathbb{D} , not necessarily distinct, then by Theorem 1.2 in [2] there are n^{n-1} points $\zeta^1, \zeta^2, \dots, \zeta^{n^{n-1}} \in \mathbb{C}^{n-1}$, repeated according to multiplicity, such that $\Psi(\zeta^j) = w := (w_1, \dots, w_{n-1})$, where Ψ is defined by (3.14). Hence, if we disregard the ordering of the components of $\zeta^j = (\zeta_1^j, \dots, \zeta_{n-1}^j)$ and $w = (w_1, \dots, w_{n-1})$, we conclude that there

are n^{n-1} polynomials \tilde{P} of the form (3.8) in $\widetilde{\mathcal{EL}}$, again counted with multiplicity, whose critical values are w_1, \dots, w_{n-1} . Since the map $\Lambda: \widetilde{\mathcal{EL}} \rightarrow \mathcal{EL}$ is n -to-1, there are n^{n-2} polynomials in \mathcal{EL} with critical values w_1, \dots, w_{n-1} . Next, we observe that when $n = 2$, the set of equivalence classes \mathcal{EL}' introduced in the proof above coincides with \mathcal{EL} , but for $n \geq 3$ there are, generically, n elements of \mathcal{EL} in each equivalence class. Thus, the map $\mathcal{F}: \mathcal{EL} \rightarrow \mathcal{EB}$ is 1-to-1 when $n = 2$, but n -to-1 for $n \geq 3$. We therefore obtain the following result, which seems to be of independent interest.

Corollary 3.2. *For $n \geq 3$ and any collection w_1, \dots, w_{n-1} of values in \mathbb{D} , there are n^{n-3} equivalence classes (counted with multiplicities) of Blaschke products in \mathcal{EB} whose critical values (in \mathbb{D}) are w_1, \dots, w_{n-1} . For $n = 2$, there is one equivalence class.*

4. FURTHER REMARKS

In this section, we collect some further remarks and observations regarding lemniscates and their fingerprints. We begin by proving the following rigidity results, which can be used to give an alternative proof of the uniqueness part (in the context of lemniscates) in Theorem 3.1 but also seems to be of independent interest.

Proposition 4.1. *Let $\Omega_-^{(1)}, \Omega_-^{(2)}$ be two domains bounded by proper lemniscates of degree n and defined by the equations $|P(z)| < 1$, $|Q(z)| < 1$, respectively, where P, Q are of the polynomials of degree n . Let $\{a_j^{(1)}\}_{j=1}^n, \{a_j^{(2)}\}_{j=1}^n$ denote the respective nodes of the lemniscates, i.e., the roots of P and Q . If $F: \Omega_-^{(2)} \rightarrow \Omega_-^{(1)}$ is a conformal map that maps the nodes of $\Omega_-^{(2)}$ onto the nodes of $\Omega_-^{(1)}$, then F is an affine automorphism $F(w) = aw + b$ with $a, b \in \mathbb{C}$.*

Remark 4.2. Of course, we assume that if some nodes $a_j^{(2)}$ have non-trivial multiplicities, then F preserves multiplicities as well.

Proof of Proposition 4.1. Consider $h(w) := Q(w)/P(F(w))$ defined in $\Omega_-^{(2)}$. By our hypothesis, h is analytic and non-vanishing in $\Omega_-^{(2)}$ (all the zeros and poles are cancelled by the zeros of Q) and $|h| = 1$ on the lemniscate $\Gamma_2 := \partial\Omega_-^{(2)}$. Moreover, since we can say the same about $1/h$, we conclude that h is a unimodular constant. Hence, we have

$$(4.1) \quad P(F(w)) = cQ(w), \quad |c| = 1,$$

in $\Omega_-^{(2)}$. Equation (4.1) implies that F is an algebraic function and can therefore be continued analytically, maintaining equation (4.1), along any curve in the Riemann sphere $\hat{\mathbb{C}}$ avoiding a finite number of points $b_1, \dots, b_m \in \hat{\mathbb{C}}$. Since both P and Q have poles of order n at ∞ , the algebraic function F can only take the value ∞ at ∞ and any continuation of F will have a simple pole there. Thus, if we allow the continuation of F to have poles (i.e. we continue F as a meromorphic function or, equivalently, as an

analytic map into $\hat{\mathbb{C}}$), then the singularities b_1, \dots, b_m of F are all branch points in \mathbb{C} at which a continuation of F takes on finite values. Thus, if we continue F along a curve C from inside $\Omega_-^{(2)}$, ending at one of the branch points $b = b_j$, for some $j = 1, \dots, m$, (but avoiding the others) and F develops a singularity at b , then the value $\zeta := F(b) \in \mathbb{C}$ must be a critical point of P . (Otherwise, P would be locally biholomorphic near ζ and F would not be singular there.) If we can show that there are no true branch points, i.e. the continuation of F along C to b (for all branch points b and all curves C as described above) is analytic at b , then F extends as a holomorphic map of $\hat{\mathbb{C}}$ onto itself sending ∞ to itself with multiplicity one and no other poles, and is therefore of the form $F(w) = aw + b$, which would complete the proof of Proposition 4.1. Since F is already analytic in a neighborhood of $\overline{\Omega_-^{(2)}}$, it suffices to show that there are no branch points in $\mathbb{C} \setminus \overline{\Omega_-^{(2)}}$, so there is no loss of generality in assuming that $b \in \mathbb{C} \setminus \overline{\Omega_-^{(2)}}$. Thus, assume, in order to reach a contradiction, that the continuation of F along C is singular at b and that this continuation satisfies $\zeta = F(b)$. Then, as mentioned above, ζ is a critical point of P and, hence by Proposition 2.1, we have $\zeta \in \Omega_-^{(1)}$. This implies that $|P(F(b))| = |P(\zeta)| < 1$, which implies that $|Q(b)| < 1$. But then $b \in \Omega_-^{(2)}$, which is a contradiction since $b \notin \Omega_-^{(2)}$ by assumption. This completes the proof of Proposition 4.1. \square

We single out here the following observation, interesting in its own right, that was used in the proof of Proposition 4.1 above.

Proposition 4.3. *Let P be a polynomial of degree n and F an algebraic function satisfying*

$$(4.2) \quad P(F(w)) = Q(w),$$

where Q is a polynomial. Let $\Omega_- := \{z \in \mathbb{C} : |P(z)| < 1\}$ and assume that Ω_- is connected (i.e. $\{z \in \mathbb{C} : |P(z)| < 1\}$ is a proper lemniscate of degree n). Then, all finite branch points of F must lie inside $\{z : |Q(w)| < 1\}$.

Proof. If $b \in \mathbb{C}$ is a branch point of F and there is a branch of F that is singular at b , then the value $\zeta = F(b)$ of this branch is not ∞ (since $Q(b) \neq \infty$) and, as noted in the proof of Proposition 4.1 above, ζ is a critical point of P . By Proposition 2.1, we have $\zeta \in \Omega_-$ and, hence, $|Q(b)| = |P(F(b))| = |P(\zeta)| < 1$, which completes the proof. \square

We end this paper with two remarks. (i) Since we have not been able, so far, to find a constructive and simple way of identifying a lemniscate Γ with a given fingerprint $k = \sqrt[n]{B}$, our approach does seem to be inferior to that pioneered by Mumford and Sharon [11]. However, taking into account the extremely simple form of fingerprints of lemniscates ($\sqrt[n]{B}$, where B is a Blaschke product of degree n), perhaps, some way of efficiently combining the two approaches may be useful. More precisely: 1) Approximate a given fingerprint $k : \mathbb{T} \rightarrow \mathbb{T}$ by $\sqrt[n]{B}$ (as can be done arbitrarily well by Theorem 2.3). 2) Find an approximate shape of the lemniscate Γ corresponding to $\sqrt[n]{B}$ by using the technique

from [11] based on Schwarz–Christoffel integrals. 3) Approximate the shape obtained in [11] by a lemniscate as in the proof of Hilbert’s theorem which is quite constructive — cf. [15]. Hopefully, in the future, some numerical experiments carried out along these lines will support our envisioning of the (simple) fingerprints of lemniscates as natural “coordinates” in the space of shapes.

(ii) As a final remark in this paper, we would like to point out a direction of further study. If $R(z)$ is a rational function of degree n , then we may consider the rational lemniscate $\Gamma := \{z \in \hat{\mathbb{C}}: |R(z)| = 1\}$. Under the assumption that the interior $\Omega_- := \{z \in \hat{\mathbb{C}}: |R(z)| < 1\}$ is connected and simply connected (and there are no singularities on Γ), the rational lemniscate Γ is a shape and we can consider its fingerprint $k: \mathbb{T} \rightarrow \mathbb{T}$. If Φ_{\pm} denote the corresponding Riemann maps $\mathbb{D}_{\pm} \rightarrow \Omega_{\pm}$ as above (where we assume, say, that $\infty \in \Omega_+$, since otherwise the strategy of defining the fingerprint would need to be slightly modified), then the fingerprint would satisfy the functional equation $\Phi_+ \circ k = \Phi_-$. Composing with R to the left on both sides, we obtain an equation of the form $A \circ k = B$, where $A = R \circ \Phi_+$ and $B = R \circ \Phi_-$ are Blaschke products of degree n . In the case of polynomial lemniscates considered in this paper, we have $A(z) = z^n$. We suspect that all diffeomorphisms $k: \mathbb{T} \rightarrow \mathbb{T}$ that arise from the algebraic equation $A \circ k = B$, where A and B are Blaschke products of degree n , are fingerprints of rational lemniscates. A first obstacle in this study would be to establish an analytic criterion for when the rational lemniscate $\Gamma = \{z \in \hat{\mathbb{C}}: |R(z)| = 1\}$ has a connected and simply connected interior Ω_- . In the polynomial case, the criterion for Ω_- to be connected and simply connected is that the $n - 1$ finite critical values of the polynomial P lie in the unit disk (Proposition 2.1). In the case of a more general rational lemniscate, it is easily seen that this is also a necessary but not a sufficient condition. However, one can show that if R has $n - 1$ critical values in \mathbb{D} and Ω_- is connected or every component of Ω_- is simply connected, then Ω_- is both connected and simply connected. The authors hope to return to the case of rational lemniscates in a future paper.

REFERENCES

1. L. Ahlfors and L. Bers, *Riemann’s mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404.
2. T. K. Ng T. W. Beardon, A. F.; Carne, *The critical values of a polynomial*, Constr. Approx. **18** (2002), 343–354.
3. E. Belogay, C. Cabrelli, U. Molter, and R. Shonkwiler, *Calculating the Hausdorff distance between curves*, Inform. Process. Lett. **64** (1997), no. 1, 17–22.
4. M. S. Berger, *Nonlinearity and functional analysis*, Lectures on nonlinear problems in mathematical analysis. Pure and Applied Mathematics., Academic Press, New York - London, 1977.
5. C. J. Bishop, *Conformal welding and Koebe’s theorem*, Ann. of Math. (2) **166** (2007), no. 3, 613–656.
6. T. A. Driscoll, *Algorithm 843: improvements to the Schwarz-Christoffel toolbox for MATLAB*, ACM Trans. Math. Software **31** (2005), no. 2, 239–251.
7. G. M. Goluzin, *Geometric theory of functions of a complex variable*, Translations of Mathematical Monographs, vol. 26, American Mathematical Society, Providence, R.I., 1969.

8. B. Gustafsson, M. Sakai, and H. S. Shapiro, *On domains in which harmonic functions satisfy generalized mean value properties*, Potential Anal. **7** (1997), no. 1, 467–484.
9. A. A. Kirillov, *Kähler structure on the K -orbits of a group of diffeomorphisms of the circle*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 2, 42–45.
10. ———, *Geometric approach to discrete series of unirreps for Vir* , J. Math. Pures Appl. (9) **77** (1998), no. 8, 735–746.
11. E. Sharon and D. Mumford, *2d-shape analysis using conformal mapping*, Computer Vision and Pattern Recognition, 2004. CVPR 2004. Proceedings of the 2004 IEEE Computer Society Conference on **2** (27 June–2 July 2004), II–350–II–357 Vol.2.
12. A. Srivastava, C. Samir, S. H. Joshi, and M. Daoudi, *Elastic shape models for face analysis using curvilinear coordinates*, Preprint, 2008.
13. N. Steinmetz, *The formula of riemann-hurwitz and iteration of rational functions*, Complex Variables Theory Appl. **22** (1993), no. 3-4, 203–206.
14. L. A. Takhtajan and L.-P. Teo, *Weil-Petersson metric on the universal Teichmüller space*, Mem. Amer. Math. Soc. **183** (2006), no. 861, viii+119.
15. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Fourth edition. American Mathematical Society Colloquium Publications, Vol. XX, American Mathematical Society, Providence, R.I., 1965.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT SAN DIEGO, LA JOLLA, CA 92093-0112

E-mail address: pebenfel@math.ucsd.edu

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FL 33620-5700

E-mail address: dkhavins@cas.usf.edu

DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, STOCKHOLM, SWEDEN 100 44

E-mail address: shapiro@math.kth.se