Tensor Products of Operator Systems

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Overview

A (concrete) *operator space* is a subspace of $B(\mathcal{H})$ together with the family of induced matrix norms.

There is a very extensive theory of tensor products of operator spaces with applications to many problems in operator algebras.

A (concrete) *operator system* is a self-adjoint subspace of $B(\mathcal{H})$ containing the identity, together with the family of matrix orders.

Even though many results about nuclear and injective C*-algebras were obtained using operator systems, there is less general theory of their tensor products.

In fact, formal definitions of what we mean by a tensor product in this category hadn't been written down.

Outline

- Recall abstract characterization of operator systems
- Define tensor products in the category $\mathcal{O}$: objects= operator systems, morphisms= unital, completely positive maps
- Five tensor products and some of their properties
- A few surprises
- Operator systems for which various tensors are equal, a refinement of nuclearity
- An operator system tensor equivalence of Kirchberg's conjecture

Abstract Characterization of Operator Systems[Choi and Effros]:
An (abstract) *operator system* is a $*$-vector space V, together with a collection of sets $\{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(V)_h$, and an element $e \in V$ such that

1. For each n, C_n is a spanning cone with $C_n \cap -C_n = \{0\}$.
2. For each n and each m, whenever $X \in M_{n,m}(\mathbb{C})$ we have $X^* C_n X \subseteq C_m$.
3. The element $e_n := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix}$ is an Archimedean order unit for $M_n(V)$.

   (Order Unit: For each $u \in M_n(V)_h$ there exists $r > 0$ s.t. $re_n \geq u$.)
   (Archimedean: For each $u \in M_n(V)$, it is the case that whenever

   $$re + u \geq 0 \text{ for all } r > 0, \text{ then } u \geq 0.)$$

Properties (1) and (2) are called a *matrix ordering*.

Theorem (Choi-Effros)

If V is an abstract operator system, then there exists a Hilbert space $\mathcal{H}$ and a complete order embedding $\phi : V \to B(\mathcal{H})$ with $\phi(e) = I_{\mathcal{H}}$. In particular, $\phi(V)$ is a concrete operator subsystem of $B(\mathcal{H})$.

Such an embedding also endows V with the structure of an operator space. In fact, the matrix norms are uniquely determined by the matrix order.

For $A \in M_n(V)$ we have

$$\|A\|_n = \inf\left\{ r : \begin{pmatrix} re_n & A \\ A^* & re_n \end{pmatrix} \in C_{2n} \right\}.$$

Note: Bigger cones give smaller norms. So we will think of tensor products as "smaller" when their positive cones are bigger.

Given operator systems $(\mathcal{S}, \{P_n\}_{n=1}^{\infty}, e_1)$ and $(\mathcal{T}, \{Q_n\}_{n=1}^{\infty}, e_2)$, we let $\mathcal{S} \otimes \mathcal{T}$ denote the vector space tensor product of $\mathcal{S}$ and $\mathcal{T}$.
An **operator system structure on** $\mathcal{S} \otimes \mathcal{T}$ is a family $\{C_n\}_{n=1}^{\infty}$ of cones, where $C_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$, satisfying:

(T1) $(\mathcal{S} \otimes \mathcal{T}, \{C_n\}_{n=1}^{\infty}, e_1 \otimes e_2)$ is an operator system,

(T2) $P_n \otimes Q_m \subseteq C_{nm}$, for all $n, m \in \mathbb{N}$, and

(T3) If $\phi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are unital completely positive maps, then $\phi \otimes \psi : \mathcal{S} \otimes \mathcal{T} \to M_{mn}$ is a unital completely positive map.

(T2) is the order analogue of the cross-norm condition
(T3) is the analogue of Grothendieck's "reasonable" property.

By an **operator system tensor product,** we mean a mapping $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, such that for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, $\tau(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, denoted $\mathcal{S} \otimes_\tau \mathcal{T}$.

We call an operator system tensor product τ **functorial** if the following property is satisfied:

(T4) For any four operator systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$, and $\mathcal{T}_2$, we have that if $\phi \in \mathrm{UCP}(\mathcal{S}_1, \mathcal{S}_2)$ and $\psi \in \mathrm{UCP}(\mathcal{T}_1, \mathcal{T}_2)$, then the linear map $\phi \otimes \psi : \mathcal{S}_1 \otimes \mathcal{T}_1 \to \mathcal{S}_2 \otimes \mathcal{T}_2$ belongs to $\mathrm{UCP}(\mathcal{S}_1 \otimes_\tau \mathcal{T}_1, \mathcal{S}_2 \otimes_\tau \mathcal{T}_2)$.

Symmetric: $\mathcal{S} \otimes_\tau \mathcal{T} \cong \mathcal{T} \otimes_\tau \mathcal{S}$; $x \otimes y \mapsto y \otimes x$.
Associative: $\mathcal{R} \otimes_\tau (\mathcal{S} \otimes_\tau \mathcal{T}) \cong (\mathcal{R} \otimes_\tau \mathcal{S}) \otimes_\tau \mathcal{T}$; $x \otimes (y \otimes z) \mapsto (x \otimes y) \otimes z$.
Given two operator system structures τ_1 and τ_2 on $\mathcal{S} \otimes \mathcal{T}$, we define $\tau_1 \geq \tau_2$ to mean $\mathrm{Id} : \mathcal{S} \otimes_{\tau_1} \mathcal{T} \to \mathcal{S} \otimes_{\tau_2} \mathcal{T}$ is completely positive (or, equivalently, $M_n(\mathcal{S} \otimes_{\tau_1} \mathcal{T})^+ \subseteq M_n(\mathcal{S} \otimes_{\tau_2} \mathcal{T})^+$ for every $n \in \mathbb{N}$.)

The Minimal Tensor Product(no surprises)

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. For each $n \in \mathbb{N}$, we let

$$C_n^{min}(\mathcal{S}, \mathcal{T}) = \{(p_{i,j}) \in M_n(\mathcal{S} \otimes \mathcal{T}) : ((\phi \otimes \psi)(p_{i,j}))_{i,j} \in M_{nkm}^+ \\ \forall \text{ u.c.p. } \phi : \mathcal{S} \to M_k \text{ and } \forall \text{ u.c.p. } \psi : \mathcal{T} \to M_m\}.$$

This defines an operator system structure and we call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{C_n^{min}(\mathcal{S}, \mathcal{T})\}_{n=1}^{\infty}, 1 \otimes 1)$ the *minimal tensor product* of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{min} \mathcal{T}$.

Theorem (Min is min)

The mapping

$$min : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$$

sending $(\mathcal{S}, \mathcal{T})$ *to* $\mathcal{S} \otimes_{min} \mathcal{T}$ *is an associative, symmetric, functorial operator system tensor product.*
Moreover, if $\mathcal{S}$ *and* $\mathcal{T}$ *are operator systems and* τ *is an operator system structure on* $\mathcal{S} \otimes \mathcal{T}$, *then* $min \leq \tau$.

Theorem (Min is spatial)

Let $\mathcal{S}$ *and* $\mathcal{T}$ *be operator systems, and let* $\iota_{\mathcal{S}} : \mathcal{S} \to \mathcal{B}(H)$ *and* $\iota_{\mathcal{T}} : \mathcal{T} \to \mathcal{B}(K)$ *be embeddings that are unital complete order isomorphisms onto their ranges. The family* $\{C_n^{min}(\mathcal{S}, \mathcal{T})\}_{n=1}^{\infty}$ *is the operator system structure on* $\mathcal{S} \otimes \mathcal{T}$ *arising from the embedding* $\iota_{\mathcal{S}} \otimes \iota_{\mathcal{T}} : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(H \otimes K)$.

The Maximal Tensor Product

Recall, the maximal should have the smallest cones.
Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. For each $n \in \mathbb{N}$, we let

$$D_n^{max}(\mathcal{S},\mathcal{T}) = \{\alpha(P \otimes Q)\alpha^* : P \in M_k(\mathcal{S})^+, Q \in M_m(\mathcal{T})^+, \alpha \in M_{n,km}, k, m \in \mathbb{N}\}.$$

- $\{D_n^{max}(\mathcal{S},\mathcal{T})\}_{n=1}^{\infty}$ is a matrix ordering with order unit $1 \otimes 1$.
- However, $1 \otimes 1$ is not an Archimedean matrix order unit.

Let

$$C_n^{max}(\mathcal{S},\mathcal{T}) = \{P \in M_n(\mathcal{S}\otimes\mathcal{T}) : r(1\otimes 1)_n + P \in D_n^{max}(\mathcal{S},\mathcal{T}) \ \forall\ r > 0\}$$

Then $\{C_n^{max}(\mathcal{S},\mathcal{T})\}_{n=1}^{\infty}$ is a matrix ordering with Archimedean matrix order unit $1 \otimes 1$. Furthermore, if $\{C_n\}_{n=1}^{\infty}$ is any matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ for which $1 \otimes 1$ is an Archimedean matrix order unit, then $D_n^{max} \subseteq C_n$ implies $C_n^{max} \subseteq C_n$.

This is a special case of a process called "Archimedeanization", which was developed in detail in
"Operator system structures on ordered spaces", by V. Paulsen, I. Todorov, and M. Tomforde, preprint.

Definition

We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{C_n^{max}(\mathcal{S}, \mathcal{T})\}_{n=1}^{\infty}, 1 \otimes 1)$ the *maximal tensor product* of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{max} \mathcal{T}$.

Remark: Max can also be defined by using "jointly completely positive maps".

Theorem (Max is max)

The mapping

$$max : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$$

sending $(\mathcal{S}, \mathcal{T})$ *to* $\mathcal{S} \otimes_{max} \mathcal{T}$ *defines an associative, symmetric, functorial operator system tensor product. Moreover, if* τ *is an operator system structure on* $\mathcal{S} \otimes \mathcal{T}$*, then* $\tau \leq max$.

Unlike operator space projective....

Theorem (Max extends C*-max)

Let A and B be C-algebras. Then the operator system $A \otimes_{max} B$ is completely order isomorphic to the image of $A \otimes B$ inside the maximal C*-algebraic tensor product of $A \otimes_{\mathrm{C^*max}} B$.*

The Commuting Tensor Product

There is another operator system tensor product that agrees with the max tensor product for all pairs of C*-algebras, but does not agree with the max tensor product on all pairs of operator systems.

This gives a different extension of the maximal C*-algebraic tensor product from the category of C*-algebras to the category of operator systems and in fact is the minimal extension of C*-max!

We call this tensor product the *commuting tensor product*.

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Set

$$cp(\mathcal{S},\mathcal{T}) = \{(\phi,\psi) : \ H \text{ is a Hilbert space, } \phi \in \mathrm{CP}(\mathcal{S},\mathcal{B}(H)), \\ \psi \in \mathrm{CP}(\mathcal{T},\mathcal{B}(H)), \text{ and } \phi(\mathcal{S}) \text{ commutes with } \psi(\mathcal{T}).\}$$

Given $(\phi,\psi) \in cp(\mathcal{S},\mathcal{T})$, let $\phi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(H)$ be the map given on elementary tensors by $(\phi \cdot \psi)(x \otimes y) = \phi(x)\psi(y)$.

For each $n \in \mathbb{N}$, define a cone $P_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$ by letting

$$P_n = \{u \in M_n(\mathcal{S}\otimes\mathcal{T}) : (\phi\cdot\psi)^{(n)}(u) \geq 0, \text{ for all } (\phi,\psi) \in cp(\mathcal{S},\mathcal{T})\}.$$

Then $(\mathcal{S} \otimes \mathcal{T}, \{P_n\}_{n=1}^{\infty}, 1 \otimes 1)$ is an operator system.

Definition

We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{P_n\}_{n=1}^{\infty}, 1 \otimes 1)$ the *commuting tensor product* of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_c \mathcal{T}$.

Theorem

The mapping

$$c : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$$

sending $(\mathcal{S}, \mathcal{T})$ *to* $\mathcal{S} \otimes_c \mathcal{T}$ *defines a symmetric, functorial operator system tensor product.*

Theorem

The tensor products c and max are different, that is, there exist operator systems $\mathcal{S}$ *and* $\mathcal{T}$ *such that* $id : \mathcal{S} \otimes_c \mathcal{T} \to \mathcal{S} \otimes_{max} \mathcal{T}$, *is not completely positive.*

However . . .

Theorem (c is minimal extension of C*-max)

If A *is a unital* C^**-algebra and* $\mathcal{S}$ *is an operator system, then* $A \otimes_c \mathcal{S} = A \otimes_{\max} \mathcal{S}$. *Moreover, if* $\alpha : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ *is any functorial operator system tensor product such that* $A \otimes_\alpha B = A \otimes_{C*max} B$, *then* $c \leq \alpha$.

Some of the Surprises

- There exist operator systems $\mathcal{S}$ such that $\mathcal{S} \otimes_{\text{min}} A = \mathcal{S} \otimes_{\text{max}} A$ for every C*-algebra A, for which the identity map on $\mathcal{S}$ does not have a UCP factorization through matrices!
  Compare to the Choi-Effros theorem: If $\mathcal{S}$ was c.o.i. to a C*-algebra, then it must factor through matrices.
  In fact, we show that the operator subsystems of the matrices associated with chordal graphs all have this property.
- (K.H. Han) If $\mathcal{S} \otimes_{\text{min}} \mathcal{T} = \mathcal{S} \otimes_{\text{max}} \mathcal{T}$ for every operator system $\mathcal{T}$, then the identity map on $\mathcal{S}$ has a point-norm UCP factorization through matrices. There exist such operator systems that are not c.o.i. to a C*-algebra. Han's proof borrows ideas from Pisier's book, but is even shorter and yields a new proof of Choi-Effros.

- An operator system $\mathcal{S}$ satisfies $\mathcal{S} \otimes_{min} A = \mathcal{S} \otimes_{max} A$ for all C*-algebras iff $\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_c \mathcal{T}$ for every operator system $\mathcal{T}$ (i.e., $\mathcal{S}$ is (min, c)-nuclear). Thus, there are many (min, c)-nuclear operator systems that are not (min, max)-nuclear, i.e, which do not factor through the matrices.
- There exist operator systems $\mathcal{S}$ such that whenever A and B are unital C*-algebras with $A \subseteq B$, then every completely positive map $\phi : A \to \mathcal{S}$ has a completely positive extension $\tilde{\phi} : B \to \mathcal{S}$, but $\mathcal{S}$ is not completely order isomorphic to a C*-algebra and in particular, there is no UCP projection from $B(H)$ onto $\mathcal{S}$!

  We exhibit a family of such operator systems that are subsystems of matrix algebras.

Generalizing Nuclearity

Definition

Let α and β be functorial operator system tensor products. An operator system $\mathcal{S}$ will be called (α, β)-nuclear if the identity map between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$ is a complete order isomorphism for every operator system $\mathcal{T}$.

Remark: It could be that $\mathcal{S}$ is (α, β)-nuclear and $\mathcal{T} \otimes_\alpha \mathcal{S} \neq \mathcal{T} \otimes_\beta \mathcal{S}$, since we don't insist that the tensors are symmetric.

Two More Operator System Tensors

Definition

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. We let $\mathcal{S} \otimes_{el} \mathcal{T}$ be the operator system with underlying space $\mathcal{S} \otimes \mathcal{T}$ whose matrix ordering is induced by the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq I(\mathcal{S}) \otimes_{max} \mathcal{T}$, where $I(\mathcal{S})$ denotes the injective envelope of $\mathcal{S}$.
Similarly, we define $\mathcal{S} \otimes_{er} \mathcal{T}$ by the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{S} \otimes_{max} I(\mathcal{T})$, where $I(\mathcal{T})$ denotes the injective envelope of $\mathcal{T}$.

These tensor products are asymmetric and, clearly, if you "flip" one you get the other.

Theorem

Let $\mathcal{S}$ be an operator system. Then the following are equivalent:

- $\mathcal{S}$ *is a 1-exact operator space,*
- $\mathcal{S}$ *is a 1-exact operator system,*
- $\mathcal{S}$ *is* (min, el)*-nuclear.*

Kirchberg's Conjecture

Let F_∞ denote the free group on countably infinitely many generators and let $C^*(F_\infty)$ denote the full(not reduced) C*-algebra of this group. Kirchberg formulated a conjecture about this C*-algebra that is equivalent to Connes' embedding theorem. The work of a number of researchers has shown that these conjectures are equivalent to some questions about tensor products of operator spaces.

Theorem (KPTT)

The following are equivalent:

1. $C^*(F_\infty)$ *has WEP(Kirchberg's conjecture),*
2. *every* (min, er)*-nuclear operator system is* (el, c)*-nuclear,*
3. *for any* (min, er)*-nuclear operator system* $\mathcal{S}$*, one has that* $\mathcal{S} \otimes_{min} \mathcal{S} = \mathcal{S} \otimes_c \mathcal{S}$.

The proof of the above result relies on first defining operator system analogues of many operator space properties, such as, WEP, LLP, quotients, exactness and then obtaining characterizations of the various types of nuclearity in these terms.

Theorem

Let $\mathcal{S}$ be an operator system. The following are equivalent:

- $\mathcal{S}$ *is (min,er)-nuclear,*
- $\mathcal{S} \otimes_{min} B(H) = \mathcal{S} \otimes_{max} B(H)$ *for every* H,
- $\mathcal{S}$ *has the operator system local lifting property(OSLLP) (every UCP map into a C*-quotient has local UCP liftings).*

Theorem

Let $\mathcal{S}$ be an operator system. The following are equivalent:

- $\mathcal{S}$ *is (el,c)-nuclear,*
- $\mathcal{S}$ *has the double commutant expectation property(DCEP),*
- $\mathcal{S} \otimes_{min} C^*(F_\infty) = \mathcal{S} \otimes_{max} C^*(F_\infty)$,
- $\mathcal{S} \otimes_{max} C^*(F_\infty) \subseteq B(H) \otimes_{max} C^*(F_\infty)$ *for some inclusion,* $\mathcal{S} \subseteq B(H)$.

Thanks for your time!
Thanks to the organizers!