A Hörmander type multiplier theorem for arbitrary discrete groups

Javier Parcet

Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM

Joint work with Marius Junge and Tao Mei

Noncommutative L_p spaces, Operator spaces and Applications

Banff International Research Station. June 28, 2010

The problem

Consider a Fourier multiplier on $(\mathbb{T}^n, \mu)$

$$T_m\Big(\sum_{k\in\mathbb{Z}^n} \widehat{f}(k)\, e^{2\pi i\,\langle k,\cdot\rangle}\Big) = \sum_{k\in\mathbb{Z}^n} m_k\, \widehat{f}(k)\, e^{2\pi i\,\langle k,\cdot\rangle}.$$

A **lifting multiplier** for m is an smooth funcion

$$\tilde{m}:\mathbb{R}^n \to \mathbb{C} \quad \text{satisfying} \quad \tilde{m}_{|\mathbb{Z}^n} = m.$$

It is well-known that L_p-boundedness is preserved, so that we have

$$\big|\partial_\xi^\beta \tilde{m}(\xi)\big| \le c_n\,|\xi|^{-|\beta|} \quad \text{for all} \quad |\beta| \le \Big[\frac{n}{2}\Big] + 1 \quad \Rightarrow \quad T_m : L_p(\mathbb{T}^n,\mu) \to L_p(\mathbb{T}^n,\mu).$$

In the case of arbitrary discrete groups

- There is no canonical differential structure to work with.
- No sufficient differentiability conditions are known for L_p-boundedness.

Our main goals in this talk is to present

- A Hörmander multiplier theorem for arbitrary discrete groups.
- A noncommutative Calderón-Zygmund theory for von Neumann algebras.

Compact duals

Let G be a discrete group and

$$f \sim \sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g) \in L_p(\widehat{\mathbb{G}}, \tau) \quad \text{such that} \quad \widehat{f}(g) = \tau(f \lambda(g^{-1}))$$

a Fourier series on its compact dual, where:

- $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$ is the left regular representation
$$\lambda(g)\delta_h = \delta_{gh} \quad \text{for} \quad \delta_g(h) = \delta_{g=h},$$
and $\widehat{\mathbb{G}} = \mathcal{L}(\mathrm{G})$ is the weak operator closure of $\mathrm{span}\lambda(\mathrm{G})$ in $\mathcal{B}(\ell_2(\mathrm{G}))$.

- We equip the group von Neumann algebra $\mathcal{L}(\mathrm{G})$ with its natural trace
$$\tau(f) = \big\langle f\delta_e, \delta_e \big\rangle_{\ell_2(\mathrm{G})} = \widehat{f}(e) \quad \text{where} \quad e = \text{Identity of } \mathrm{G}.$$
$L_p(\widehat{\mathbb{G}}, \tau)$ is the noncommutative L_p space associated to $(\mathcal{L}(\mathrm{G}), \tau)$, with norm
$$\|f\|_p = \big(\tau|f|^p\big)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_\infty = \|f\|_{\mathcal{B}(\ell_2(\mathrm{G}))}.$$

Example. If $\mathrm{G} = \mathbb{Z}$, we have an isometry

$$\ell_2(\mathrm{G}) \ni \delta_m \mapsto \exp(2\pi i m \,\cdot) \in L_2(\mathbb{T}, \mu) \;\Rightarrow\; \lambda(m) \sim \exp(2\pi i m \,\cdot).$$

We have $\tau = \int_{\mathbb{T}} \cdot \, d\mu$ for the normalized Haar measure μ and $L_p(\widehat{\mathbb{G}}, \tau) = L_p(\mathbb{T}, \mu)$.

Sketch of the talk

Given a discrete group G, the key points are:

- **Cocycles.** If $\psi : \mathrm{G} \to \mathbb{R}_+$ is such that

$$e^{-t\psi} \text{ is positive definite for all } t > 0,$$

we may find an inclusion $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ into a Hilbert space $\mathcal{H}_\psi$, which is isometric in the sense that

$$\mathrm{dist}(g,h) \equiv \sqrt{\psi(g^{-1}h)} = \big\| b_\psi(g) - b_\psi(h) \big\|_{\mathcal{H}_\psi}$$

defines a **pseudo-metric** on G, and $\mathcal{H}_\psi$ plays the role of an '**ambient space**' for G.

- **Hörmander-Mihlin condition.** Given a Fourier multiplier

$$T_m : \sum_g \widehat{f}(g)\lambda(g) \mapsto \sum_g m_g \widehat{f}(g)\lambda(g),$$

a lifting multiplier is given by $\tilde{m} : \mathcal{H}_\psi \to \mathbb{C}$ with $m = \tilde{m} \circ b_\psi$. If $\dim \mathcal{H}_\psi = n$

$$\big|\partial_\xi^\beta \tilde{m}(\xi)\big| \le c_n |\xi|^{-\beta} \text{ for all } |\beta| \le \Big[\frac{n}{2}\Big] + 1 \overset{?}{\Rightarrow} T_m : L_p(\widehat{\mathbb{G}}, \tau) \to L_p(\widehat{\mathbb{G}}, \tau)$$

- **Calderón-Zygmund theory.** This naturally leads to a form of

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{|s|>2|x|} \big| k_{\tilde{m}}(s-x) - k_{\tilde{m}}(s) \big| \, ds < \infty \quad \Rightarrow \quad T_{\tilde{m}} : L_\infty \to \mathrm{BMO}$$

the **Hörmander condition for the kernel**, valid for arbitrary von Neumann algebras.

Length functions

We will be working with functions $\psi : \mathrm{G} \to \mathbb{R}_+$ such that:

i) $\psi(e) = 0$,

ii) $\psi(g) = \psi(g^{-1})$ for all $g \in \mathrm{G}$,

iii) ψ is a conditionally negative function

$$\sum_{g \in \Lambda \subset \mathrm{G}} \gamma_g = 0 \quad \text{and} \quad |\Lambda| < \infty \quad \Rightarrow \quad \sum_{g,h \in \Lambda} \overline{\gamma}_g \gamma_h \psi(g^{-1}h) \leq 0.$$

We will call such a ψ a **length function**. By Schoenberg theorem

$$\psi \text{ length function} \quad \Leftrightarrow \quad e^{-t\psi} \text{ positive definite for all } t > 0.$$

Examples. Two standard cases:

- If $\mathrm{G} = \mathbb{Z}^n$, we may take

$$\psi_1 = |\ |^2 \quad \text{and} \quad \psi_2 = |\ |.$$

The heat and Poisson kernels are positive with Fourier transforms $e^{-t\psi_j}$, $j = 1, 2$.

- If $\mathrm{G} = \mathbb{F}_n$ is the free group with n generators, we may use

$$\psi(g) = |g| = \text{ standard length function},$$

because the associated Poisson semigroup is formed of completely positive maps.

Construction of the cocycle

Given a length function ψ

$$K_\psi(g,h) = \frac{\psi(g)+\psi(h)-\psi(g^{-1}h)}{2} \rightsquigarrow \Big\langle \sum_{g\in \mathrm{G}} \gamma_g \delta_g, \sum_{h \in \mathrm{G}} \gamma_h' \delta_h \Big\rangle_\psi = \sum_{g,h\in\mathrm{G}} \gamma_g K_\psi(g,h) \gamma_h'.$$

This is an $\mathbb{R}$-product on the group algebra $\mathbb{R}[\mathrm{G}]$ of finitely supported real functions on G. If we set N_ψ to be the null space of $\langle \cdot, \cdot \rangle_\psi$, we may define the Hilbert space $\mathcal{H}_\psi$ as the completion of $(\mathbb{R}[\mathrm{G}]/N_\psi, \langle \cdot, \ \cdot \rangle_\psi)$ and define the natural inclusion

$$b_\psi : g \in \mathrm{G} \mapsto \delta_g + N_\psi \in \mathcal{H}_\psi$$

which satisfies the isometric identity $\big\| b_\psi(g) - b_\psi(h) \big\|_{\mathcal{H}_\psi} = \sqrt{\psi(g^{-1}h)} = \mathrm{dist}(g,h)$.

There exists a natural action $\alpha_\psi : \mathrm{G} \to \mathrm{Aut}(\mathcal{H}_\psi)$

$$\alpha_{\psi,g}(b_\psi(h)) = b_\psi(gh) - b_\psi(g)$$

which is isometric in the sense that we have $\langle \alpha_{\psi,g}(\xi_1), \alpha_{\psi,g}(\xi_2) \rangle_\psi = \langle \xi_1, \xi_2 \rangle_\psi$. This allows us to construct a **semidirect product embedding** $g \mapsto b_\psi(g) \rtimes g$ which extends to the group von Neumann algebras as follows

$$\pi_\psi : \lambda(g) \in \mathcal{L}(\mathrm{G}) \mapsto \exp b_\psi(g) \rtimes \lambda(g) \in \mathcal{L}(\mathcal{H}_\psi) \rtimes_{\alpha_\psi} \mathrm{G}.$$

The key is to show $T_{\tilde{m}} : L_\infty(\mathcal{H}_\psi) \to \mathrm{BMO} \ \Rightarrow T_{\tilde{m}} \rtimes id_{\mathrm{G}}$ is still bounded (NCCZ theory).

Hörmander-Mihlin multipliers for discrete groups

Theorem [JMP]. *Let* G *be a discrete group and*

$$T_m : \sum_g \widehat{f}(g)\lambda(g) \mapsto \sum_g m_g \widehat{f}(g)\lambda(g)$$

a Fourier multiplier on its compact dual. Assume that G *is equipped with a length function* ψ*, with associated cocycle* $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ *such that* $\dim \mathcal{H}_\psi = n$*. Let* $\tilde{m} : \mathbb{R}^n \to \mathbb{C}$ *be a lifting multiplier for* m*, so that* $m = \tilde{m} \circ b_\psi$*. Then*

$$T_m : L_p(\widehat{\mathbb{G}}, \tau) \to L_p(\widehat{\mathbb{G}}, \tau) \quad \text{is cb-bounded for} \quad 1 < p < \infty$$

provided the condition below holds for some $\varepsilon > 0$

$$\left|\partial_\xi^\beta \tilde{m}(\xi)\right| \le c_n |\xi|^{-|\beta|-\varepsilon} \quad \text{for all multi-indices } \beta \text{ s.t.} \quad |\beta| \le n+2.$$

The classical hypotheses with

$$|\beta| \le \left[\frac{n}{2}\right] + 1 \quad \text{and} \quad \varepsilon = 0$$

suffice in the following particular cases:

- *If* $b_\psi(\mathrm{G})$ *is a lattice in* $\mathcal{H}_\psi$.
- *Radial Fourier multipliers* $m_g = h(\psi(g))$.

We may also prove $L_\infty \to \mathrm{BMO}$ *type inequalities and some free-dimensional estimates*.

Some comments

- If $\mathrm{G} = \mathbb{Z}^n$ and $\psi_1(k) = |k|^2$, we easily get

  $\mathcal{H}_{\psi_1} \simeq \mathbb{R}^n$ and $b_{\psi_1}(k) = k$ $\Rightarrow$ Classical Hörmander multiplier theorem.

  However, $\psi_2(k) = |k|$ gives $\dim \mathcal{H}_{\psi_2} = \infty$! **Highly non canonical** choice of cocycle.

- The are two problems to solve
  - **Interpolation problem.** Given $\psi : \mathrm{G} \to \mathbb{R}_+$, estimate

  $$\inf\Big\{ \sup_{\xi \in \mathbb{R}^n} \sup_{|\beta| \le d_n} |\xi|^{-|\beta|} \big| \partial_\beta \tilde{m}(\xi) \big| \ \text{s.t.} \ \tilde{m} \circ b_\psi(g) = m_g \Big\}.$$

    Related to Fefferman's recent work on **'smooth interpolation of data'**.
  - **Dimensional problem.** Given G, find $\inf_\psi \dim \mathcal{H}_\psi$ for $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ injective.

  Remark. We have H-dim$(\mathbb{Z}^n) = 1$!! Both problems are **'incompatible'**.

- The negative generator of the semigroup

  $$\lambda(g) \mapsto \exp(-t\psi(g))\lambda(g)$$

  is the map $A(\lambda(g)) = \psi(g)\lambda(g)$. In particular, we find

  Radial Fourier multipliers $\subset$ **McIntosh's H_∞-calculus**.

  However, our Hörmander-Mihlin type condition above is **considerably weaker**.

- If $\dim \mathcal{H}_\psi = n$ $\rightarrow$ $\mathcal{H}_\psi \simeq \mathbb{R}^n_{\mathrm{disc}}$ and $\mathcal{L}(\mathcal{H}_\psi) \simeq L_\infty(\widehat{\mathbb{R}}^n_{\mathrm{disc}}, \mu)$ $\rightarrow$ de Leeuw.

Riesz transforms

Given a discrete group G, we have seen

$$\begin{array}{ccccc} \psi : \mathrm{G} \to \mathbb{R}_+ & & (\mathcal{H}_\psi, \langle \cdot, \cdot \rangle_\psi) & & b_\psi : \mathrm{G} \to \mathcal{H}_\psi \\ \text{length function} & \Rightarrow & \text{Hilbert space} & \Rightarrow & \text{cocyle map.} \end{array}$$

Thus, we consider the η-th Riesz ψ-transform for $\eta \in \mathcal{H}_\psi$ as

$$R_\eta \Big(\sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g) \Big) = -i \sum_{g \in \mathrm{G}} \frac{\langle b_\psi(g), \eta \rangle_\psi}{\sqrt{\psi(g)}} \widehat{f}(g) \lambda(g).$$

The lifting multiplier $\tilde{m}_\eta(\xi) = -i \frac{\langle \xi, \eta \rangle_\psi}{\sqrt{\langle \xi, \xi \rangle_\psi}}$ only satisfies the classical condition

$$\big| \partial_\xi^\beta \tilde{m}_\eta(\xi) \big| \ \leq \ c_n \, |\xi|^{-|\beta|} \ \text{for any multi-index } \beta \quad \Rightarrow \quad \varepsilon = 0.$$

Theorem [JMP]. If $\dim \mathcal{H}_\psi < \infty$, any operator in

$$\mathcal{R} = \mathrm{span} \Big\{ \prod_{\eta \in \Gamma} R_\eta \, \Big| \, \Gamma \text{ finite set in } \mathcal{H}_\psi \Big\}$$

defines a cb-map $\mathcal{L}(\mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_\psi}$ and $L_p(\widehat{\mathbb{G}}, \tau) \to L_p(\widehat{\mathbb{G}}, \tau)$ for all $1 < p < \infty$.

Noncommutative tori

Given $n \geq 1$ and $\Theta = (\theta_{jk})_{n\times n}$ antisymmetric

$$\begin{aligned}\mathcal{A}_\Theta &= \Big\langle u_1, u_2, \ldots, u_n \,\Big|\, \text{unitaries with } u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j \Big\rangle \\ &= \Big\{ f \sim \sum_{k \in \mathbb{Z}^n} \widehat{f}(k) w_k \,\Big|\, w_k = u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n} \text{ with } k = (k_1, k_2, \ldots, k_n) \Big\}.\end{aligned}$$

We also need the trace $\tau(f) = \widehat{f}(0)$ and the heat semigroup $S_{\Theta,t}(f) = \sum_k \widehat{f}(k) e^{-t|k|^2} w_k$.

Theorem [JMP]. Let

$$T_m : \sum_k \widehat{f}(k) w_k \mapsto \sum_k m_k \widehat{f}(k) w_k.$$

If a lifting multiplier $\tilde{m} : \mathbb{R}^n \to \mathbb{C}$ with $\tilde{m}_{|_{\mathbb{Z}^n}} = m$ satisfies

$$\big|\partial_\xi^\beta \tilde{m}(\xi)\big| \;\leq\; c_n |\xi|^{-\beta} \quad \text{for all} \quad |\beta| \leq \Big[\frac{n}{2}\Big] + 1,$$

then we find that $T_m : L_\infty(\mathcal{A}_\Theta, \tau) \to \mathrm{BMO}_{\mathcal{S}_\Theta}$ and $L_p(\mathcal{A}_\Theta) \to L_p(\mathcal{A}_\Theta)$ for all $1 < p < \infty$.

Proof 1. Noncommutative form of Calderón's transference from $\mathbb{T}^n$.
Proof 2. We have $\mathcal{L}(\mathrm{H}_\Theta) = \int_{\mathbb{R}}^{\oplus} \mathcal{A}_{x\Theta} \, dx$ and apply our multiplier theorem to H_Θ.

Noncommutative Calderón-Zygmund theory

We are interested in a noncommutative form of

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{|s|>2|x|} \big| k(s-x) - k(s) \big| \, ds \ < \ \infty$$

$$\Updownarrow$$

$$\sup_{s>0} \Big\| \big(\chi_{\mathrm{B}_s(0)} \otimes \chi_{\mathrm{B}_s(0)} \big) \, \delta_{\mathbb{R}^n} \, T\big(f \chi_{\mathbb{R}^n \setminus \mathrm{B}_{5s}(0)} \big) \Big\|_{L_\infty(\mathbb{R}^n \times \mathbb{R}^n)} \ \le \ c_h \, \|f\|_\infty$$

for any Calderón-Zygmund T with kernel k and with $\delta_{\mathbb{R}^n}(f) = f \otimes 1_{\mathbb{R}^n} - 1_{\mathbb{R}^n} \otimes f$.

Major difficulty: Construct projections playing the role of the Euclidean balls $\mathrm{B}_s(0)$.

The key ingredients are

- Noncommutative BMO's over semigroups.
- An associated 'metric' on the von Neumann algebra.

Semigroup type BMO's

Duong and Yan recently extended BMO theory to certain semigroups on homogeneous spaces assuming certain regularity. This theory, however, *still imposes the existence of a metric in the underlying space*. We may not assume the existence of a metric.

Given a noncommutative measure space $(\mathcal{M}, \tau)$ and

$$\mathcal{S} = (S_t)_{t \geq 0} \quad \text{with} \quad S_t : f \in L_p(\mathcal{M}, \tau) \to d_t * f \in L_p(\mathcal{M}, \tau),$$

a noncommutative diffusion semigroup of convolution type, we define

$$\|f\|_{\mathrm{BMO}_{\mathcal{S}}^c} = \sup_{t \geq 0} \Big\| \Big(S_t |f|^2 - |S_t f|^2 \Big)^{\frac{1}{2}} \Big\|_\infty$$

for the *column* **semigroup type BMO**. The row analog and the general form are

$$\|f\|_{\mathrm{BMO}_{\mathcal{S}}^r} = \|f^*\|_{\mathrm{BMO}_{\mathcal{S}}^c} \quad \text{and} \quad \|f\|_{\mathrm{BMO}_{\mathcal{S}}} = \max \Big\{ \|f\|_{\mathrm{BMO}_{\mathcal{S}}^r}, \|f\|_{\mathrm{BMO}_{\mathcal{S}}^c} \Big\}.$$

Theorem [Junge/Mei]. *If $\mathcal{S}$ admits a 'nice enough' Markov dilation*

$$[\mathrm{BMO}_{\mathcal{S}}, L_p(\mathcal{M}, \tau)]_{p/q} \simeq L_q(\mathcal{M}, \tau).$$

Remark. The regularity assumed in the result above holds in our main examples.

Noncommutative 'metrics'

A **weighted spectral decomposition** for

$$\mathcal{S} = (S_t)_{t\geq 0} \quad \text{with} \quad S_t f = d_t * f$$

is a family of projections $(q_{k,t})$ in $\mathcal{M}$ —indexed by $(k,t) \in \mathbb{N} \times \mathbb{R}_+$— which are increasing in k for t fixed, together with a family of positive numbers $\beta_{k,t} \in \mathbb{R}_+$ such that the following conditions hold for absolute constants c_w, c_s, c_d

i) $\displaystyle\sum_{k\geq 1} \beta_{k,t} \tau(q_{k,t}) \ \leq \ c_s,$

ii) $\displaystyle d_t \ \leq \ c_d \sum_{k\geq 1} \beta_{k,t}(q_{k,t} - q_{k-1,t}),$

iii) $\displaystyle\sum_{k\geq 1} \beta_{k,t} w_{k,t} \tau(q_{k,t} - q_{k-1,t}) \ \leq \ c_w \quad \text{for} \quad w_{k,t} = \Big(\sum_{j\leq k} \sqrt{\tfrac{\tau(q_{j+1,t})}{\tau(q_{j,t})}} \Big)^2.$

This notion is somehow related to

- Tolsa's notion of RBMO space for nondoubling measures.
- Blunck/Kunstmann's analysis of non-integral Calderón-Zygmund operators.

We will however require a doubling property of the trace τ

$$\tau(q_{\alpha(k),t}) \ \leq c_\alpha \tau(q_{k,t}) \quad \text{for some strictly increasing function } \alpha : \mathbb{N} \to \mathbb{N}.$$

Boundedness of noncommutative CZO's

Taking $Q_{k,t}(f) = \frac{1}{\tau(q_{k,t})} q_{k,t} * f$ yields a metric type BMO, called $\mathrm{BMO}_{\mathcal{Q}}$.

Theorem [JMP]. *Let $(\mathcal{M}, \tau)$ be a noncommutative measure space and $\mathcal{S}$ a semigroup acting on it equipped with an α-doubling weighted decomposition with associated metric $\mathcal{Q} = (Q_{k,t})$. Let $T : \mathcal{A} \to \mathcal{M}$ defined on a weakly dense $*$-subalgebra of $\mathcal{M}$. If we consider the derivation $\delta_{\mathcal{M}}(f) = f \otimes \mathbf{1}_{\mathcal{M}} - \mathbf{1}_{\mathcal{M}} \otimes f$, the conditions*

- **a)** $T : L_2(\mathcal{M}, \tau) \to L_2(\mathcal{M}, \tau)$ *is bounded by* c_{22},
- **b1)** $\left\| \mathcal{R}_{q_{k,t} \otimes q_{k,t}} \delta_{\mathcal{M}} (T \otimes id_{\mathcal{M}}) \mathcal{R}_{q^{\perp}_{\alpha(k),t}} : \mathcal{M} \bar{\otimes} \mathcal{M} \to \mathcal{M} \bar{\otimes} \mathcal{M} \bar{\otimes} \mathcal{M} \right\| \leq c_h$ *for all* k, t,
- **b2)** $\left\| \mathcal{L}_{q_{k,t} \otimes q_{k,t}} \delta_{\mathcal{M}} (T \otimes id_{\mathcal{M}}) \mathcal{L}_{q^{\perp}_{\alpha(k),t}} : \mathcal{M} \bar{\otimes} \mathcal{M} \to \mathcal{M} \bar{\otimes} \mathcal{M} \bar{\otimes} \mathcal{M} \right\| \leq c_h$ *for all* k, t,

imply that $T : \mathcal{A} \to \mathrm{BMO}_{\mathcal{Q}}$. More concretely, we obtain

$$\|Tf\|_{\mathrm{BMO}_{\mathcal{Q}}} \leq (2\, c_{22} \sqrt{c_\alpha} + c_h)\, \|f\|_\infty.$$

$$\|Tf\|_{\mathrm{BMO}_{\mathcal{S}}} \leq 2\sqrt{2} \sqrt{c_d\, (c_s + c_w)}\, (2\, c_{22} \sqrt{c_\alpha} + c_h)\, \|f\|_\infty.$$

Corollary [JMP]. *Additionally, if $\mathcal{S}$ has a nice Markov dilation, we obtain L_p-boundedness.*

Remark. The heat semigroup reconstructs the classical $\mathbb{R}^n$-theory from Theorem above.

Applications and examples

- New $L_\infty \to \mathrm{BMO}$ **Schur multipliers**.
- Analysis of some **concrete groups**: $\mathbb{Z}_n, \mathcal{S}_n, \mathbb{F}_n$...
- **Burnside groups**: H-dim$(B(n,m)) = \infty$ for $n \geq 2$ and $m \geq 665$ odd.
- **Calderón's transference method** for quantum groups.
- An adapted **Littlewood-Paley theory**.

Thanks for listening!!